Weyl's integration formula for U(N) *

Angela Pasquale

A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.

H. Weyl

In the middle 1920s Hermann Weyl proved an explicit formula to integrate functions on compact simple Lie groups with respect to invariant measures. The formula, today known as Weyl's integration formula, is one of the cornerstones for the representation theory of compact Lie groups.

These notes explain Weyl's integration formula for the group $\mathrm{U}(N)$ of $N \times N$ unitary matrices. Recall that $\mathrm{U}(N)$ consists of all $N \times N$ invertible complex matrices g which satisfy the relation $g^*g = I$. Here g^* denotes the conjugate transpose of the matrix g and I is the identity matrix. Restricting ourself to this specific group, we significantly reduce the number of structural and technical details needed for writing the formula in its full generality. We refer to [Kna96, Chapter VIII. 5] and to [Sim96, IX. 3 and IX. 9] for the complete exposition.

1 Motivation

Weyl's integration formula appears as the analogue for $\mathrm{U}(N)$ of the classical formula which gives the transformation of integrals on $\mathbb{R}^3$ with respect to Cartesian coordinates to integrals in spherical coordinates. As a motivation and explanation of the procedure we will use to get Weyl's integration formula, let us start recalling how the classical formula of integration in spherical coordinates can be derived.

Consider $\mathbb{R}^3$ as an additive group, and let $H := \mathrm{SO}(3)$ denote the group of proper rotations in $\mathbb{R}^3$, i.e. $\mathrm{SO}(3) := \{A \in \mathrm{GL}(3,\mathbb{R}) : A^tA = I \text{ and } \det A = 1\}$. Then H acts on $\mathbb{R}^3$ by usual matrix multiplication: $x \in \mathbb{R}^3 \mapsto Ax$. The orbits for this action are the spheres centered at the origin 0 in $\mathbb{R}^3$. They are all 2-dimensional except for one singular orbit $\{0\}$ consisting of the single origin.

For $x \in \mathbb{R}^3 \setminus \{0\}$ we employ the notation $r = |x|$ for the radial coordinate of x and $x' = x/|x|$ for its angular coordinate. Then $r \in (0,\infty)$ parametrizes the nonsingular orbits.

Let $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ be the unit sphere in $\mathbb{R}^3$. The spherical coordinate map

$$\begin{aligned} \psi : S^2 \times (0,\infty) &\longrightarrow \mathbb{R}^3 \setminus \{0\} \\ (x', r) \qquad &\longmapsto x = rx' \end{aligned}$$

is a diffeomorphism, and the change of variables formula yields for $f \in L^1(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3\setminus\{0\}} f(x)\,dx = \int_0^\infty \left[\int_{S^2} f(rx')d\sigma(x')\right] r^2\,dr. \tag{1}$$

Here dx denotes the Lebesgue measure in $\mathbb{R}^3$ and $d\sigma(x')$ is a suitable normalization of the rotation-invariant measure on S^2. Observe that the right-hand side of (1) appears as the

*Based on an introductory lecture delivered at the DMV Seminar "The Riemann Zeta Function and Random Matrix Theory", October 15–21, 2000, Oberwolfach, Germany.

integral on the orbit of parameter r followed by the integral on the set $(0,\infty)$ parametrizing the nonsingular orbits. In fact, since we can disregard sets of zero Lesbegue measure, we can just extend the integral on the left-hand side to all of $\mathbb{R}^3$ and the exterior integral on the right-hand side to the set $[0,\infty)$ parametrizing all SO(3)-orbits. We therefore obtain the classical formula for integration in spherical coordinates:

$$\int_{\mathbb{R}^3} f(x)\, dx = \int_0^\infty \left[\int_{S^2} f(rx')d\sigma(x')\right] r^2\, dr \tag{2}$$

for all $f \in L^1(\mathbb{R}^3)$. Observe that the geometric condition of singularity of the orbit $\{0\}$ corresponds to the analytic condition that the density r^2 vanishes at $r=0$.

Suppose now that f is radial (i.e. constant on the SO(3)-orbits), and set $f(r) := f(x)$ if $|x| = r$. Then (2) implies

$$\int_{\mathbb{R}^3} f(x)\, dx = \int_0^\infty f(r) r^2\, dr. \tag{3}$$

2 The case of $\mathrm{U}(N)$

We now wish to do something similar for $G := \mathrm{U}(N)$. As group H we choose $\mathrm{U}(N)$ acting on itself by conjugation: $g \in \mathrm{U}(N)$ acts according to $x \in \mathrm{U}(N) \mapsto gxg^{-1}$. The orbits for this actions are called the *conjugacy classes*. They can be explicitely described using the diagonalization properties of unitary matrices.

Every unitary matrix x can be diagonalized: there is a diagonal matrix $\mathrm{diag}\{\lambda_1,\dots,\lambda_N\}$ (whose diagonal entries are the eigenvalues of x in some order) and a unitary matrix g so that $x = g\,\mathrm{diag}\{\lambda_1,\dots,\lambda_N\}g^{-1}$. We then say that x and $\mathrm{diag}\{\lambda_1,\dots,\lambda_N\}$ are *conjugate*. Moreover, since the eigenvalues of unitary matrices have modulus 1, we can write $\lambda_j = e^{i\theta_j}$ for some $\theta_j \in [0,2\pi)$ for all $j=1,\dots,N$.

Consider the subgroup T of $\mathrm{U}(N)$ consisting of diagonal matrices, i.e.

$$T := \{\Theta = \mathrm{diag}\{e^{i\theta_1},\dots,e^{i\theta_N}\} : \theta_j \in [0,2\pi), 1 \le j \le N\}.$$

Then $T \cong (S^1)^N$ is an N-dimensional torus, hence a compact abelian subgroup of $\mathrm{U}(N)$. The group W of permutations of $\{1,\dots,N\}$ acts on T by permuting the diagonal elements. The diagonalization properties can be then precisely stated as follows: every $g \in \mathrm{U}(N)$ is conjugate to some $\Theta \in T$; furthermore, $\Theta_1 \in T$ and $\Theta_2 \in T$ are conjugate if and only if there is $w \in W$ so that $\Theta_2 = w\Theta_1$. The conjugacy classes in $\mathrm{U}(N)$ are therefore parametrized by T/W.

For $\mathbb{R}^3$ the radial functions play a special role. Similarly, for $U(N)$ a special role is played by the functions that are constant on the conjugacy classes.

Definition 2.1 A *class function* is a function $f : \mathrm{U}(N) \to \mathbb{C}$ which is constant on the conjugacy classes: $f(gxg^{-1}) = f(x)$ for all $x, g \in G$.

The diagonalization properties imply that a class function f is uniquely determined by its restriction $f|_T$ to T, and that $f|_T$ is W-invariant. By setting

$$f(\theta_1,\dots,\theta_N) := f(\Theta) \quad \text{for} \quad \Theta = \mathrm{diag}\{e^{i\theta_1},\dots,e^{i\theta_N}\} \in T,$$

this means that $f(\theta_1,\dots,\theta_N)$ is a symmetric function in $\theta_1,\dots,\theta_N$.

2.1 Invariant measures

We want to write integral formulas for functions on $\mathrm{U}(N)$, so we need measures. To select them, let us go back to our motivating example. The Lebesgue measure on $\mathbb{R}^3$ is rather special:

(a) it is a regular Borel measure; (b) it is translation invariant, that is for every $f \in L^1(\mathbb{R}^3)$ and every $y \in \mathbb{R}^3$

$$\int_{\mathbb{R}^3} f(x+y)\, dx = \int_{\mathbb{R}^3} f(x)\, dx.$$

In fact, (a) and (b) characterize the Lebesgue measure up to a constant multiple.

$\mathrm{U}(N)$ is a compact Lie group. On every Lie group we can consider regular Borel measures, that is Borel measures which interact nicely with the locally-compact Hausdorff topology of G. For instance a regular Borel measure μ on G satisfies $\mu(K) < \infty$ for every compact subset K of G. It follows, in particular, that $\mu(G) < \infty$ when the Lie group G is compact. The regular Borel measure $\frac{1}{\mu(G)}\mu$ has then total measure 1, i.e. it is a probability measure. In the physics literature, probability measures are often referred to as *ensemble averages*. We refer to [Fol84, Chapter 7] for more information on regular Borel measures.

We can also request for regular Borel measures on Lie groups to be invariant with respect to the action of the group on itself by left (or right) translations. Here we have to distinguish between left and right because, unlike $\mathbb{R}^3$, a Lie group need not be abelian.

Definition 2.2 A *left Haar measure* on a Lie group G is a nonzero regular Borel measure μ on G which is left invariant:

$$\mu(gE) = \mu(E) \qquad \text{for every Borel set } E \subset G \text{ and all } g \in G,$$

or equivalently, for every $f \in L^1(G)$ and $g \in G$,

$$\int_G f(gx)\, d\mu(x) = \int_G f(x)\, d\mu(x).$$

Right Haar measures are similarly defined using right translations. A left Haar measure need not be also right invariant.

Definition 2.3 A left Haar measure which is also right invariant is called a *Haar measure.*

Example 2.4 The torus T in $\mathrm{U}(N)$ is abelian, so one need not distinguish between left and right translations. A Haar measure on T is given by $d\Theta := c\ d\theta_1 \dots d\theta_N$, where $\Theta = \mathrm{diag}\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$, c is a constant, and $d\theta_j$ denotes the Lebesgue measure on $[0, 2\pi)$. We will adopt the usual normalization of $d\Theta$ which fixes the constant $c = (2\pi)^{-N}$. This makes $d\Theta$ a probability measure.

It is quite easy to see for Lie groups that Haar measures *do* exist.

Theorem 2.5 *Every Lie group possesses a left Haar measure.*

Proof. Choose any inner product on the tangent space T_eG of G at the unit element e and translate it to all of G by means of the maps coming from the left translations $x \mapsto gx$. This gives a left-invariant Riemannian metric $\mathbf{g}$ on G. The corresponding volume element is also left invariant, and therefore corresponds to a left-invariant measure on G. In local coordinates $(x_1, \dots, x_n)$ on G, the volume element is $\sqrt{\det \mathbf{g}(x_1, \dots, x_n)}\, dx_1 \dots dx_n$. This gives the local description of the Haar measure. $\square$

The next (non-trivial) result states that left Haar measures are essentially uniquely determined.

Theorem 2.6 *If μ_1 and μ_2 are left Haar measures on G, then there is a constant $C > 0$ such that $\mu_2 = C\ \mu_1$.*

An important consequence is the following. Let μ be a left Haar measure on G. Fix $g \in G$ and define the measure μ_g by

$$\mu_g(E) := \mu(Eg) \quad \text{for every Borel set } E \subset G.$$

Then μ_g is again a left Haar measure because left and right translations commute (this is simply the associative law). According to Theorem 2.6, there is a constant $\Delta(g) > 0$ so that $\mu_g = \Delta(g)\mu$, i.e. $\mu(Eg) = \Delta(g)\mu(E)$ for every Borel set E.

Definition 2.7 The function $\Delta : G \to \mathbb{R}^+$, $g \mapsto \Delta(g)$, is called the *modular function.* If Δ is the constant map with value 1, then G is called *unimodular.*

Observe that $\Delta \equiv 1$ means that the left Haar measure μ is also right invariant. This proves the following corollary.

Corollary 2.8 *Every unimodular Lie group possesses a Haar measure.*

Since $\mathrm{U}(N)$ is compact, the next proposition guarantees it possesses a Haar measure.

Proposition 2.9 *Every compact Lie group is unimodular.*

Proof. Let μ be a left Haar measure on G. Then $G = Gg$ for all $g \in G$, so

$$\mu(G) = \mu(Gg) = \mu_g(G) = \Delta(g)\mu(G). \tag{4}$$

Since G is compact and μ is a nonzero regular Borel measure, we have $0 < \mu(G) < \infty$. Hence we can divide both sides of (4) by $\mu(G)$ and get $\Delta(g) = 1$. $\square$

We will write dx for the Haar measure of $\mathrm{U}(N)$ normalized so that $\mathrm{vol}(\mathrm{U}(N)) := \int_{\mathrm{U}(N)} dx = 1$.

We also need to select a measure on $\mathrm{U}(N)/T$, the typical orbit which plays the role of S^2 on the left-hand side of (2). $\mathrm{U}(N)/T$ is the set of laterals xT for $x \in \mathrm{U}(N)$, endowed with the quotient topology. It is not a Lie group, but an analytic manifold and a homogeneous $\mathrm{U}(N)$-space on which $g \in \mathrm{U}(N)$ acts according to $xT \mapsto gxT$. So we can still ask for a $\mathrm{U}(N)$-invariant regular Borel measure $d(xT)$ on $\mathrm{U}(N)/T$. The reason for the existence of such a measure is that both $\mathrm{U}(N)$ and T are unimodular (cf. [Kna96, Theorem 8.34] for the details). Moreover, it is unique up to a constant multiple. We will fix its normalization so that the following equality holds: for all $f \in L^1(\mathrm{U}(N))$

$$\int_{\mathrm{U}(N)} f(x)\, dx = \int_{\mathrm{U}(N)/T} \left[\int_T f(x\Theta)\, d\Theta\right] d(xT). \tag{5}$$

Observe that, because of the fixed normalization of $d\Theta$ as in Example 2.4, equation (5) reduces to

$$\int_{\mathrm{U}(N)} f(x)\, dx = \int_{\mathrm{U}(N)/T} f(x)\, d(xT) \tag{6}$$

when $f \in L^1(\mathrm{U}(N))$ is constant on laterals of T in $\mathrm{U}(N)$, i.e. when $f(x\Theta) = f(x)$ for all $\Theta \in T$.

2.2 An initial guess

We are now in the position of formulating our guess for Weyl's integration formula. Let $f \in L^1(\mathrm{U}(N))$. Just by formal comparison with the formula for spherical coordinates in $\mathbb{R}^3$, we can expect our formula will be of the type

$$\int_{\mathrm{U}(N)} f(x)\, dx = \frac{1}{N!} \int_T \left[\int_{\mathrm{U}(N)\,/T} f(\psi(\Theta, xT))\, d(xT) \right] \mathrm{dens}(\Theta)\, d\Theta, \tag{7}$$

where ψ is a "coordinate map" and $\mathrm{dens}(\Theta)$ the corresponding density. On the right-hand side of (7), we first integrate on the typical orbit $\mathrm{U}(N)\,/T$, then on the set parameterizing the orbits. In fact, the set parameterizing the orbits is T/W, not T. This explains the factor $1/N!$, since $N!$ is the cardinality of W.

As in the case of spherical coordinates, we expect to derive the formula on the union of nonsingular orbits (those of maximal dimension). Proving that only these really count, we will be allowed to extend the result to all of $\mathrm{U}(N)$ and T. Moroever the geometric singularity of the orbits will correspond to the analytic singularity of the density $\mathrm{dens}(\Theta)$ occurring in the formula.

2.3 Nonsingular orbits

Let O_x denote the orbit of $x \in \mathrm{U}(N)$ under the action of $\mathrm{U}(N)$ on itself by conjugation (that is the conjugacy class containing x). Hence

$$O_x := \{gxg^{-1} : g \in \mathrm{U}(N)\} = \mathrm{U}(N)\,/\mathrm{U}(N)_x,$$

where $\mathrm{U}(N)_x := \{g \in \mathrm{U}(N) : gxg^{-1} = x\}$ is the centralizer of x in $\mathrm{U}(N)$. Since $\mathrm{U}(N)\,/\,\mathrm{U}(N)_x$ is a manifold of dimension $\dim \mathrm{U}(N) - \dim \mathrm{U}(N)_x$, the dimension of O_x is maximal precisely when $\dim \mathrm{U}(N)_x$ is minimal. Let $\lambda_1, \ldots, \lambda_r$ denote the distinct eigenvalues of x with multiplicities $m_1, \ldots, m_r$, respectively. Then $\mathrm{U}(N)_x$ consists of the block-diagonal unitary matrices with blocks of sizes respectively $m_1, m_2, \ldots, m_r$ (the k-th block corresponds to the eigenspace of x of eigenvalue λ_k). Since the dimension of the space of $n \times n$ unitary matrices is $2[(n^2 - n)/2] + n = n^2$, we have

$$\dim \mathrm{U}(N)_x = m_1^2 + m_2^2 + \cdots + m_r^2 \geq m_1 + m_2 + \cdots + m_r = N.$$

The equality holds if and only if $r = N$ and $m_k = 1$ for all k, that is when x has N distinct eigenvalues.

We call an orbit (i.e. a conjugacy class) *nonsingular* if it has maximal dimension, and *singular* otherwise.

Suppose the eigenvalues of x are counted with multiplicites and written as an N-vector $\Lambda_x = (\lambda_1, \ldots, \lambda_N)$. The above discussion shows that the orbit O_x is singular if and only if some eigenvalue of x has multiplicity > 1. Equivalently, if and only if Λ_x is a zero of the polynomial

$$p(\Lambda) := \prod_{\substack{i,j=1,\ldots,N \\ i \neq j}} (\lambda_i - \lambda_j).$$

Observe that $p(\Lambda)$ is a symmetric polynomial in $(\lambda_1, \ldots, \lambda_N)$. Hence, by Newton's theorem, it is a polynomial in the elementary symmetric functions

$$\sigma_1(\Lambda) = \sum_{j=1}^{N} \lambda_j, \quad \sigma_2(\Lambda) = \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j, \quad \ldots, \quad \sigma_N(\Lambda) = \lambda_1 \lambda_2 \ldots \lambda_N.$$

In turn these are the restriction to T of the conjugation-invariant polynomial functions $\mathrm{tr}\, x$, $\ldots$, $\det x$ on $\mathrm{U}(N)$. There is therefore a conjugation-invariant polynomial function q on $\mathrm{U}(N)$

such that O_x is singular if and only if $q(x) = 0$. It follows that the union of singular orbits is the zero set of a polynomial function and hence of measure 0 (recall the local expression of the Haar measure with respect to the Lebesgue measure given in the proof of Theorem 2.5). This allows us to work on the union of nonsingular conjugacy classes and then to extend the results to the entire space.

2.4 The coordinate map

Let $\mathrm{U}(N)'$ be the union of nonsingular conjugacy classes, which are parametrized (modulo W) by the set $T' := T \cap \mathrm{U}(N)'$.

The key step in the proof of Weyl's formula is the following lemma.

Lemma 2.10 *The map*

$$\begin{aligned} \psi : U(N)\,/T \times T' &\longrightarrow U(N)' \\ (xT, \Theta) &\longmapsto x\Theta x^{-1} \end{aligned}$$

is well defined (i.e. $x\Theta x^{-1}$ depends only on the coset xT), and an $(N!)$-to-1 analytic covering of $U(N)'$ with everywhere bijective differential $d\psi$. Moreover,

$$|\det d\psi(xT,\Theta)| = \prod_{1\le m<n\le N} |e^{i\theta_m} - e^{i\theta_n}|^2. \tag{8}$$

Proof. The fact that ψ is surjective and that every element in $\mathrm{U}(N)'$ has exactly $N!$ preimages is a consequence of the diagonalization properties of unitary matrices. It is easily checked that $x\Theta x^{-1}$ depends only on the coset xT, and ψ is analytic because so is the Lie-group product.

$d\psi(xT,\Theta)$ is a linear map from the tangent space $\mathrm{T}_{xT}(\mathrm{U}(N)\,/T) \times \mathrm{T}_\Theta T$ of $\mathrm{U}(N)\,/T \times T'$ at (xT,Θ) to the tangent space $\mathrm{T}_{x\Theta x^{-1}}\,\mathrm{U}(N)$ of $\mathrm{U}(N)'$ at $x\Theta x^{-1}$. In Lie-theoretic contexts, the tangent spaces at any point of a Lie group are often identified with its Lie algebra. We make therefore the following identifications:

$$\begin{aligned} \mathrm{T}_{x\Theta x^{-1}}\,\mathrm{U}(N) \equiv \mathfrak{u}(\mathfrak{n}) &:= \{N \times N \text{ skew-Hermitian matrices}\}, \\ \mathrm{T}_\Theta T \equiv \mathfrak{d} \quad &:= \{\mathrm{diag}\{i\theta_1,\dots,i\theta_N\} : \theta_j \in \mathbb{R}\}. \end{aligned}$$

We set furthermore

$$\mathrm{T}_{xT}(\mathrm{U}(N)/T) \equiv \mathfrak{q} := \{N \times N \text{ skew-Hermitian matrices with 0 entries on the main diagonal}\},$$

so that $\mathfrak{q} \oplus \mathfrak{d} = \mathfrak{u}(\mathfrak{n})$. We write exp for the usual exponential of matrices and for an arbitrary $N \times N$ matrix X and $x \in \mathrm{U}(N)$ we define $\mathrm{Ad}(x)X := xXx^{-1}$.

For $Z \in \mathfrak{q}$, the map $\gamma(s) := (x\exp(sZ)T, \Theta)$ is a curve in $\mathrm{U}(N)\,/T \times T'$ with tangent vector $(Z,0)$ at (xT,Θ). We have

$$\begin{aligned} \psi(\gamma(s)) &= x\exp(sZ)\Theta\exp(-sZ)x^{-1} \\ &= x(\Theta x^{-1}x\Theta^{-1})\exp(sZ)\Theta(x^{-1}x)\exp(-sZ)x^{-1} \\ &= (x\Theta x^{-1})\exp[s(x\Theta^{-1}Z\Theta x^{-1})]\exp(-sxZx^{-1}), \end{aligned}$$

hence

$$d\psi(xT,\Theta)(Z,0) = \frac{d}{ds}\psi(\gamma(s))\bigg|_{s=0} = x\Theta^{-1}Z\Theta x^{-1} - xZx^{-1} = \mathrm{Ad}(x)(\Theta^{-1}Z\Theta - Z).$$

For $H \in \mathfrak{d}$, the map $\eta(s) := (xT, \Theta + sH)$ is a curve in $\mathrm{U}(N)\,/T \times T'$ with tangent vector $(0,H)$ at (xT,Θ). We have

$$\psi(\eta(s)) = x\Theta x^{-1} + sxHx^{-1},$$

hence

$$d\psi(xT,\Theta)(0,H) = \frac{d}{ds}\psi(\gamma(s))\bigg|_{s=0} = xHx^{-1} = \mathrm{Ad}(x)H.$$

Thus

$$d\psi(xT,\Theta)(Z,H) = d\psi(xT,\Theta)(Z,0) + d\psi(xT,\Theta)(0,H) = \mathrm{Ad}(x)\left[(\Theta^{-1}Z\Theta - Z) + H\right].$$

This shows that

$$d\psi(xT,\Theta) = \mathrm{Ad}(x) \circ \begin{array}{cc} \left[\begin{array}{c|c} A(\Theta) & 0 \\ \hline 0 & I \end{array}\right] & \begin{array}{l} \}\mathfrak{q} \\ \}\mathfrak{d} \end{array} \\ \underbrace{}_{\mathfrak{q}}\ \underbrace{}_{\mathfrak{d}} & \end{array}$$

where $A(\Theta)Z := \Theta^{-1}Z\Theta - Z$ for $Z \in \mathfrak{q}$.

The map $\mathrm{Ad}(x) : \mathfrak{u}(\mathfrak{n}) \to \mathfrak{u}(\mathfrak{n})$ is orthogonal with respect to the inner product $\langle X, Y\rangle := \Re\,\mathrm{Tr}(X^*Y)$ on $\mathfrak{u}(\mathfrak{n})$ (the star denoting conjugate transposition). Hence $|\det \mathrm{Ad}(x)| = 1$. The determinant of $A(\Theta)$ it is more conveniently computed in the complexification $\mathfrak{q}_{\mathbb{C}}$ of $\mathfrak{q}$, which consists of all complex $N \times N$ matrices with 0 entries on the main diagonal. Let E_{mn} denote the matrix with all 0 entries except for the mn-th which is equal to 1. Then the set $\{E_{mn} : m,n = 1,\ldots,N,\ m \neq n\}$ is basis of $\mathfrak{q}_{\mathbb{C}}$. With respect to this basis, $A(\Theta)$ is diagonal because

$$A(\Theta)E_{mn} = (e^{i(\theta_n - \theta_m)} - 1)E_{mn}.$$

Consequently,

$$|\det(d\psi(xT,\Theta))| = |\det A(\Theta)| = \prod_{m\neq n} |e^{i(\theta_n-\theta_m)} - 1| = \prod_{1\leq m<n\leq N} |e^{i\theta_m} - e^{i\theta_n}|^2.$$

□

2.5 Weyl's integration formula

For $f \in L^1(\mathrm{U}(N))$ and $\Theta = (e^{i\theta_1},\ldots,e^{i\theta_N}) \in T$ we set

$$g_f(\theta_1,\ldots,\theta_N) := \int_{\mathrm{U}(N)} f(x\Theta x^{-1})\,dx.$$

Theorem 2.11 (Weyl's integration formula) *For every $f \in L^1(U(N))$*

$$\int_{U(N)} f(x)\,dx = \int_{[0,2\pi)^N} g_f(\theta_1,\ldots,\theta_N)P(\theta_1,\ldots,\theta_N)\,d\theta_1\ldots d\theta_N, \tag{9}$$

where

$$P(\theta_1,\ldots,\theta_N) := \frac{1}{N!(2\pi)^N}\prod_{1\leq m<n\leq N} |e^{i\theta_n} - e^{i\theta_m}|^2.$$

In particular, if f is a class function, then $g_f(\theta_1,\ldots,\theta_N) \equiv f(\theta_1,\ldots,\theta_N)$, and the formula reduces to

$$\int_{U(N)} f(x)\,dx = \int_{[0,2\pi)^N} f(\theta_1,\ldots,\theta_N)P(\theta_1,\ldots,\theta_N)\,d\theta_1\ldots d\theta_N.$$

Proof. Since the cooordinate map ψ of Lemma 2.10 is a $(N!)$-to-1 cover, the coordinate transformation formula (cf. e.g. [Kna96, (8.59), p. 482]) gives

$$\int_{\mathrm{U}(N)'} f(x)\,dx = \frac{1}{N!}\int_{T'}\int_{\mathrm{U}(N)/T} f\big(\psi(xT,\Theta)\big)\,|\det d\psi(xT,\Theta)|\,d(xT)\,d\Theta.$$

By disregarding sets of 0 measure, we can extend the integral to all of $\mathrm{U}(N)$ and T (cf. Subsection 2.3) and get

$$\int_{\mathrm{U}(N)} f(x)\,dx = \frac{1}{N!}\int_{T}\int_{\mathrm{U}(N)/T} f\big(\psi(xT,\Theta)\big)\,|\det d\psi(xT,\Theta)|\,d(xT)\,d\Theta. \tag{10}$$

Since T is abelian, the map $\widetilde{f}_\Theta(x) := f\big(\psi(xT,\Theta)\big) = f(x\Theta x^{-1})$ satisfies

$$\widetilde{f}_\Theta(x\widetilde{\Theta}) = f(x\widetilde{\Theta}\Theta\widetilde{\Theta}^{-1}x^{-1})) = f(x\Theta x^{-1}) = \widetilde{f}_\Theta(x)$$

for all $\widetilde{\Theta} \in T$. Hence by (6)

$$\int_{\mathrm{U}(N)/T} f\big(\psi(xT,\Theta)\big)\,d(xT) = \int_{\mathrm{U}(N)} f(x\Theta x^{-1})\,dx.$$

Weyl's integration formula (9) follows then immediately from (10), (8) and the expression of $d\Theta$ in Example 2.4. □

References

[Fol84] G. B. Folland. *Real analysis. Modern techniques and their applications.* J. Wiley & Sons, Inc., 1984.

[Fol95] G. B. Folland. *A course in abstract harmonic analysis.* CRC Press, Boca Raton, 1995.

[Kna96] A. W. Knapp. *Lie groups beyond an introduction.* Birkhäuser, Boston, 1996.

[Sim96] B. Simon. *Representations of finite and compact groups.* American Mathematical Society, Providence, 1996.

[Var84] V. S. Varadarajan. *Lie groups, Lie algebras, and their representations.* Springer-Verlag, New York, 1984.

[Var99] V. S. Varadarajan. *An introduction to harmonic analysis on semisimple Lie groups.* Cambridge University Press, 1999.