Uniform Distribution

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These are the slightly expanded notes of a tutorial on the basic theory of uniformly distributed sequences I presented at the DMV - Seminar *The Riemann Zeta Function and Random Matrix Theory* which took place in Oberwolfach in October 2000.

Let $\omega = (\omega_n)_{n\geq 1}$ be a sequence in the unit interval $[0,1)$. For $0 \leq a < b \leq 1$ let $A_N\big(\omega,[a,b)\big)$ denote the number of $n \in \mathbb{N}$ with $1 \leq n \leq N$ such that $\omega_n \in [a,b)$.

Definition. The sequence ω is called uniformly distributed if (for all $[a,b) \subseteq [0,1)$)

$$\lim_{N\to\infty} \frac{A_N\big(\omega,[a,b)\big)}{N} = b-a.$$

Obviously, a uniformly distributed sequence is dense in $[0,1]$ but the converse is not true. We leave it as an exercise for the reader to construct a sequence which is dense in $[0,1]$ but not uniformly distributed.

In the following $\{x\} = x - [x]$ denotes the fractional part of $x \in \mathbb{R}$ and c_A denotes the characteristic function of a set $A \subseteq \mathbb{R}$.

Definition. A sequence $(\omega_n)_{n\geq 1}$ in $\mathbb{R}$ is called uniformly distributed modulo 1 if $\big(\{\omega_n\}\big)_{n\geq 1}$ is uniformly distributed.

For $f : [0,1] \to \mathbb{R}$ (or $\mathbb{C}$) integrable and ω a sequence in $[0,1)$ we now regard the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\omega_n) = \int_0^1 f(x)\, dx. \tag{$*$}$$

Note that we may rewrite the defintion of uniform distribution as

$$\omega \text{ is uniformly distributed} \quad \Longleftrightarrow \quad (*) \text{ holds for all } f = c_{[a,b)} \text{ (with } 0 \leq a < b \leq 1).$$

Theorem 1. *The following are equivalent:*

(1) ω is uniformly distributed,

(2) $(*)$ holds for all continuous functions $f : [0,1] \to \mathbb{R}$,

(3) $(*)$ holds for all Riemann - integrable functions $f : [0,1] \to \mathbb{R}$.

Proof. $(3 \Rightarrow 1)$ is trivial.

$(1 \Rightarrow 3)$ As $(*)$ holds for all functions of shape $c_{[a,b)}$ it also holds for all step functions.

Now let f be an arbitrary Riemann - integrable function. There are step functions $t_1, t_2 : [0,1] \to \mathbb{R}$ such that $t_1 \le f \le t_2$ and $\int_0^1 (t_2 - t_1)(x)\,dx < \varepsilon$ and therefore

$$\begin{aligned}\int_0^1 f(x)\,dx - \varepsilon \le \int_0^1 t_1(x)\,dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N t_1(\omega_n) \le \varliminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(\omega_n)\\ \le \varlimsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(\omega_n) \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N t_2(\omega_n) = \int_0^1 t_2(x)\,dx \le \int_0^1 f(x)\,dx + \varepsilon\end{aligned}$$

which implies (3).
($3 \Rightarrow 2$) is trivial.
($2 \Rightarrow 1$) For $[a,b) \subseteq [0,1)$ there are continous functions $f_1, f_2 : [0,1] \to \mathbb{R}$ such that $f_1 \le c_{[a,b)} \le f_2$ and $\int_0^1 (f_2 - f_1)(x)\,dx \le \varepsilon$ and therefore

$$b - a - \varepsilon \le \int_0^1 f_2(x)\,dx - \varepsilon \le \int_0^1 f_1(x)\,dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f_1(\omega_n) \le \varliminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N c_{[a,b)}(\omega_n)$$

$$\le \varlimsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N c_{[a,b)}(\omega_n) \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f_2(\omega_n) = \int_0^1 f_2(x)\,dx \le \int_0^1 f_1(x)\,dx + \varepsilon \le b - a + \varepsilon$$

which implies (1).

Remark. Theorem 1 cannot be extended to Lebesgue - integrable functions f. Let e.g. $\Omega = \{\omega_n \mid n \in \mathbb{N}\}$ and $f = c_\Omega$. Then $\frac{1}{N}\sum_{n=1}^N f(\omega_n) = 1$ but $\int_0^1 f(x)\,dx = 0$.

Corollary 2. *The sequence ω in $[0,1]$ is uniformly distributed if and only if $(*)$ holds for all continous functions $f : [0,1] \to \mathbb{C}$.*

Theorem 3. *(Weyl criterion) The following are equivalent:*

(1) The sequence $(\omega_n)_{n\ge1}$ is uniformly distributed modulo 1.

(2) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^N e^{2\pi i h \omega_n} = 0$ for all $h \in \mathbb{Z} \setminus \{0\}$.

Proof. ($1 \Rightarrow 2$) follows from Corollary 2.
($2 \Rightarrow 1$) Let $f : [0,1] \to \mathbb{C}$ be continous. Because of the Weierstrass approximation theorem there is a trigonometric polynomial s such that $\|f - s\|_\infty \le \varepsilon$ and therefore

$$\begin{aligned}&\Big|\int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^N f(\omega_n)\Big|\\ &\le \underbrace{\Big|\int_0^1 (f-s)(x)\,dx\Big|}_{\le\varepsilon} + \underbrace{\Big|\int_0^1 s(x)\,dx - \frac{1}{N}\sum_{n=1}^N s(\omega_n)\Big|}_{\to 0 \text{ as } N\to\infty} + \underbrace{\Big|\frac{1}{N}\sum_{n=1}^N \big(s(\omega_n) - f(\omega_n)\big)\Big|}_{\le\varepsilon}\end{aligned}$$

which implies the assertion.

Corollary 4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $(n\alpha)_{n\geq 1}$ is uniformly distributed modulo 1.*

Proof. Let $h \in \mathbb{Z} \setminus \{0\}$. Then

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h\alpha n}\right| = \left|\frac{e^{2\pi i h\alpha N}-1}{N(e^{2\pi i h\alpha}-1)}\right| \leq \frac{2}{N\left|e^{2\pi i h\alpha}-1\right|} \to 0$$

as $N \to \infty$ and the assertion follows from Theorem 3.

Lemma 5. *(Van der Corput's Inequality) Let $u_1, \dots, u_N \in \mathbb{C}$ and $H \in \mathbb{N}$ with $1 \leq H \leq N$. Then*

$$H^2\Big|\sum_{n=1}^{N} u_n\Big|^2 \leq H(N+H-1)\sum_{n=1}^{N}|u_n|^2 + 2(N+H-1)\sum_{h=1}^{H-1}(H-h)\operatorname{Re}\Big(\sum_{n=1}^{N-h} u_n\overline{u}_{n+h}\Big).$$

The proof is technical and can be found as Lemma 3.1 in Chapter 1 of [1].

Theorem 6. *Let $(\omega_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$. If $(\omega_{n+h}-\omega_n)_{n\geq 1}$ is uniformly distributed modulo 1 for all $h \in \mathbb{N}$ then $(\omega_n)_{n\geq 1}$ is uniformly distributed modulo 1.*

Proof. Let $m \in \mathbb{Z} \setminus \{0\}$ and set $u_n = e^{2\pi i m\omega_n}$ for $n \in \mathbb{N}$. Lemma 5 yields

$$\begin{aligned}&\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m\omega_n}\right|^2 \\ \leq &\frac{N+H-1}{HN} + 2\sum_{h=1}^{H-1}\frac{(N+H-1)(H-h)(N-h)}{H^2N^2}\left|\frac{1}{N-h}\sum_{n=1}^{N-h} e^{2\pi i m(\omega_n-\omega_{n+h})}\right|.\end{aligned}$$

As $(\omega_{n+h}-\omega_n)_{n\geq 1}$ is uniformly distributed

$$\lim_{N\to\infty}\frac{1}{N-h}\sum_{n=1}^{N-h} e^{2\pi i m(\omega_n-\omega_{n+h})} = 0 \quad \text{for all } h \in \mathbb{N}$$

and therefore

$$\varlimsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m\omega_n}\right|^2 \leq \frac{1}{H} \quad \text{for all } H \in \mathbb{N}.$$

Altogether, we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m\omega_n} = 0 \quad \text{for all } m \in \mathbb{Z}\setminus\{0\}$$

and the assertion is implied by Weyl's criterion.

Corollary 7. *Let $p(X) = \alpha_m X^m + \alpha_{m-1} X^{m-1} + \cdots + \alpha_1 X + \alpha_0 \in \mathbb{R}[X]$ where at least one of $\alpha_1, \ldots, \alpha_m$ is irrational. Then $\big(p(n)\big)_{n\geq 1}$ is uniformly distributed modulo 1.*

Proof. If $m = 1$ this follows from Corollary 4. Suppose now that $m \geq 2$. Let $q \in \{1, \ldots, m\}$ be the maximal index such that $\alpha_q \notin \mathbb{Q}$. We prove the assertion by induction on q.

First let $q = 1$, i.e. $\alpha_m, \ldots, \alpha_2 \in \mathbb{Q}$ and $\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$. Let D be the least common multiple of the denominators of $\alpha_2, \ldots, \alpha_m$. Let $\overline{p}(X) = \alpha_m X^m + \cdots + \alpha_2 X^2$. Then $\big\{\overline{p}(Dk + d)\big\} = \big\{\overline{p}(d)\big\}$ for all integers $k \geq 0$ and $d \geq 1$. For $h \in \mathbb{Z} \setminus \{0\}$ we get

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h p(n)} &= \frac{1}{N}\sum_{d=1}^{D}\sum_{k=0}^{[N/D]-1} e^{2\pi i h(\overline{p}(Dk+d)+\alpha_1(Dk+d)+\alpha_0)} + \frac{1}{N}\sum_{n=[N/D]D+1}^{N} e^{2\pi i h p(n)} \\ &= \Big(\sum_{d=1}^{D} e^{2\pi i h(\overline{p}(d)+\alpha_1 d+\alpha_0)}\Big)\Big(\frac{1}{N}\sum_{k=0}^{[N/D]-1} e^{2\pi i h\alpha_1 Dk}\Big) + \frac{1}{N}\sum_{n=[N/D]D+1}^{N} e^{2\pi i h p(n)}.\end{aligned}$$

From $\alpha_1 \notin \mathbb{Q}$ it follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{[N/D]-1} e^{2\pi i h\alpha_1 Dk} = 0$$

and furthermore

$$\left|\frac{1}{N}\sum_{n=[N/D]D+1}^{N} e^{2\pi i h p(n)}\right| \leq \frac{D}{N}.$$

The Weyl criterion implies that $\big(p(n)\big)_{n\geq 1}$ is uniformly distributed modulo 1.

Now let $q \in \mathbb{Z}$ with $1 < q \leq m$. Because of the induction hypothesis $\big(p(n+h) - p(n)\big)_{n\geq 1}$ is uniformly distributed modulo 1 for all $h \in \mathbb{N}$. Applying Theorem 6 concludes the proof.

Example. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $d \in \mathbb{N}$. Then $(\alpha n^d)_{n\geq 1}$ is uniformly distributed modulo 1.

Theorem 8. *(Féjer) Let $f : [1, \infty) \to \mathbb{R}$ be differentiable, $f' > 0$ and monotonous. If*

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{1}{xf'(x)} = 0$$

then $\big(f(n)\big)_{n\geq 1}$ is uniformly distributed modulo 1.

Proof. Let $u(x) = \cos(2\pi hx)$ and $v(x) = \sin(2\pi hx)$. Then

$$\begin{aligned}
\left|\int_1^N u\big(f(x)\big)\,dx\right| &= \left|\int_1^N \frac{2\pi hf'(x)}{2\pi hf'(x)} u\big(f(x)\big)\,dx\right| = \left|\int_1^N \frac{1}{2\pi hf'(x)}(v\circ f)'(x)\,dx\right| \\
&= \left|\frac{1}{2\pi hf'(1)}\int_1^{\xi_N}(v\circ f)'(x)\,dx + \frac{1}{2\pi hf'(N)}\int_{\xi_N}^N (v\circ f)'(x)\,dx\right| \\
&= \left|\frac{1}{2\pi hf'(1)}\big((v\circ f)(\xi_N) - (v\circ f)(1)\big) + \frac{1}{2\pi hf'(N)}\big((v\circ f)(N) - (v\circ f)(\xi_N)\big)\right| \\
&\le \frac{1}{\pi|h|f'(1)} + \frac{1}{\pi|h|f'(N)} \qquad (\text{where } \xi_N \in [1,N]).
\end{aligned}$$

The function f' has to be monotonously decreasing. (Otherwise $f'(x) \ge f'(1) > 0$ and thus $f(x) = f(1) + \int_1^x f'(t)\,dt \ge f(1) + f'(1)(x-1)$ which would imply $\varliminf_{x\to\infty} f(x)/x \ge f'(1) > 0$, a contradiction.) Therefore

$$\frac{1}{f'(1)} \le \frac{1}{f'(N)} \quad \text{and} \quad \left|\int_1^N u\big(f(x)\big)\,dx\right| \le \frac{2}{\pi|h|f'(N)}.$$

We can exchange the meaning of u and v in the proof presented so far, i.e. set $u(x) = \sin(2\pi hx)$ and $v(x) = \cos(2\pi hx)$, and it remains valid. This yields

$$\left|\int_1^N e^{2\pi ihf(x)}\,dx\right| \le \frac{4}{\pi|h|f'(N)}$$

and using Euler summation we can estimate

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^N e^{2\pi ihf(n)}\right| \\
&\quad= \frac{1}{N}\left|\frac{1}{2}\Big(e^{2\pi ihf(1)} + e^{2\pi ihf(N)}\Big) + \int_1^N e^{2\pi ihf(x)}\,dx + \int_1^N \big(\{x\} - \tfrac{1}{2}\big)2\pi ihf'(x)e^{2\pi ihf(x)}\,dx\right| \\
&\quad\le \frac{1}{N}\Big(1 + \frac{4}{\pi|h|f'(N)} + \pi|h|\int_1^N f'(x)\,dx\Big) \\
&\quad= \frac{1}{N}\Big(1 + \frac{4}{\pi|h|f'(N)} + \pi|h|\big(f(N) - f(1)\big)\Big).
\end{aligned}$$

As $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty}\big(xf'(x)\big)^{-1} = 0$ it follows that $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N e^{2\pi ihf(n)} = 0$ which implies the assertion.

Corollary 9. *(1) Let $\alpha \in \mathbb{R}\setminus\{0\}$ and $\sigma \in (0,1)$. Then $(\alpha n^\sigma)_{n\ge1}$ is uniformly distributed modulo 1.*
(2) Let $\sigma > 1$. Then $\big((\log n)^\sigma\big)_{n\ge1}$ is uniformly distributed modulo 1.

Proof. Apply Theorem 8 to $f(x) = \alpha x^\sigma$ and to $f(x) = (\log x)^\sigma$.

Remark. It can be shown that the sequence $\big(\{\log n\}\big)_{n\geq 1}$ is dense in $[0,1]$ but not uniformly distributed modulo 1.

We finish by presenting the theory of normal numbers as an application of uniformly distributed sequences. Let $b \in \mathbb{N} \setminus \{1\}$ and $\alpha \in \mathbb{R}$ with b - adic expansion

$$\alpha = [\alpha] + \sum_{n=1}^{\infty} \frac{a_n}{b^n}$$

where $0 \leq a_n < b$ for all $n \in \mathbb{N}$ and $a_n < b-1$ for infinitely many $n \in \mathbb{N}$. For given $a \in \{0,1,\dots,b-1\}$ and $N \in \mathbb{N}$ let $\bar{A}_b(a,N,\alpha)$ denote the number of $n \in \{1,\dots,N\}$ such that $a_n = a$. For a given block $B_k (= b_1 \dots b_k)$ of digits let $\bar{A}_b(B_k,N,\alpha)$ denote the number of occurrences of this block. (More precisely, $\bar{A}_b(B_k,N,\alpha)$ denotes the number of $n \in \{1,\dots,N-k+1\}$ such that $a_{n+j-1} = b_j$ for $1 \leq j \leq k$.)

Definition. A number $\alpha \in \mathbb{R}$ is called simply normal to the base b if

$$\lim_{N\to\infty} \frac{\bar{A}_b(a,N,\alpha)}{N} = \frac{1}{b} \quad \text{for all } a \in \{0,1,\dots,b-1\}.$$

The number α is called normal to the base b if

$$\lim_{N\to\infty} \frac{\bar{A}_b(B_k,N,\alpha)}{N} = \frac{1}{b^k} \quad \text{for all Blocks } B_k \text{ (and all } k \in \mathbb{N}).$$

Finally, α is called absolutely normal if it is normal to all bases $b \geq 2$.

Remarks. (1) If α is normal to the base b it is simply normal to the base b but the converse is not true: $\alpha = 0,010101\dots$ is simply normal to the base 2 but not normal to the base 2 as the block 11 does not occur.
(2) The number $\alpha = 0,1234567891011121314151617181920\dots$ is normal to the base 10 (Champernowne) but it is not known whether e.g. $\sqrt{2}, \log 2, e$ and π are normal to the base 10.
(3) It can be shown that almost all numbers are normal to a base $b \geq 2$. This implies that almost all numbers are absolutely normal. Nevertheless, it is hard to construct an absolutely normal number.

Theorem 10. *The number $\alpha \in \mathbb{R}$ is normal to the base b if and only if the sequence $\omega = (b^n\alpha)_{n\geq 1}$ is uniformly distributed modulo 1.*

Proof. The digits $a_m a_{m+1} \dots a_{m+k-1}$ equal a block $B_k = b_1 \dots b_k$ of length k if and only if

$$\begin{aligned}
& \alpha = [\alpha] + \sum_{n=1}^{m-1} \frac{a_n}{b^n} + \frac{b_1}{b^m} + \cdots + \frac{b_k}{b^{m+k-1}} + \sum_{n=m+k}^{\infty} \frac{a_n}{b^n} \\
\iff \quad & \{b^{m-1}\alpha\} = \frac{b_1 b^{k-1} + \cdots + b_k}{b^k} + \sum_{n=k+1}^{\infty} \frac{a_{n+m-1}}{b^n} \\
\iff \quad & \{b^{m-1}\alpha\} \in \left[\frac{b_1 b^{k-1} + \cdots + b_k}{b^k}, \frac{b_1 b^{k-1} + \cdots + b_k + 1}{b^k}\right) =: J(B_k).
\end{aligned}$$

This implies that $\bar{A}_b(B_k, N, \alpha) = A_{N-k+1}\big(\omega, J(B_k)\big)$.

Assume that ω is uniformly distributed modulo 1. Then

$$\lim_{N\to\infty} \frac{\bar{A}_b(B_k, N, \alpha)}{N} = \lim_{N\to\infty} \frac{A_{N-k+1}\big(\omega, J(B_k)\big)}{N-k+1} \cdot \frac{N-k+1}{N} = \frac{1}{b^k},$$

i.e. α is normal to the base b.

Assume now that α is normal to the base b. Then

$$\lim_{N\to\infty} \frac{A_N\big(\omega, J(B_k)\big)}{N} = \lim_{N\to\infty} \frac{\bar{A}_b(B_k, N+k-1, \alpha)}{N+k-1} \cdot \frac{N+k-1}{N} = \frac{1}{b^k}$$

for all blocks B_k. Now let $J \subseteq [0,1)$ be an arbitrary half-open interval. Choose intervals $J_1, J_2 \subseteq [0,1)$ of shape $J_1 = J(B_{k,1})$, $J_2 = J(B_{k,2})$ such that $J_1 \subseteq J \subseteq J_2$ and $\lambda(J) - \varepsilon < \lambda(J_1) \le \lambda(J_2) < \lambda(J) + \varepsilon$ where $\lambda(I)$ denotes the length of an interval I. Then

$$\lambda(J) - 2\varepsilon < \lambda(J_1) - \varepsilon < \frac{A_N(\omega, J_1)}{N} \le \frac{A_N(\omega, J)}{N} \le \frac{A_N(\omega, J_2)}{N} < \lambda(J_2) + \varepsilon < \lambda(J) + 2\varepsilon$$

for large N and therefore

$$\lim_{N\to\infty} \frac{A_N(\omega, J)}{N} = \lambda(J),$$

i.e. ω is uniformly distributed modulo 1.

Remarks. (1) There is a metric theory of uniform distribution we have not presented. A typical result is the following: The sequence $(\alpha^n)_{n\ge1}$ is uniformly distributed modulo 1 for almost all $\alpha \ge 1$. But it is not known whether e.g. $(e^n)_{n\ge1}$ is uniformly distributed modulo 1.

(2) For many of the results we have presented there are stronger or more general versions. For example, in the Weyl criterion the family of functions $x \mapsto e^{2\pi i h x}$ (with $h \in \mathbb{Z} \setminus \{0\}$) can be replaced by any family of functions whose span is dense in the space of continous functions on $[0,1]$ endowed with the supremum norm.

(3) The sequence of imaginary parts of the nontrivial zeros of the Riemann Zeta Function is uniformly distributed modulo 1. A proof can be found in [3].
(4) Uniform distribution has been studied in more general settings. The simplest generalization is the following: Let $s \in \mathbb{N}$, ω a sequence in $[0,1)^s$, $a = (a_1, \dots, a_s), b = (b_1, \dots, b_s) \in [0,1)^s$ with $a_i < b_i$ for $1 \le i \le s$. Let $A_N\big(\omega, [a,b)\big)$ denote the number of $n \in \{1, \dots, N\}$ with $1 \le n \le N$ such that $\omega_n \in [a,b) := [a_1, b_1) \times \cdots \times [a_s, b_s)$. The sequence ω is called uniformly distributed if

$$\lim_{N\to\infty} \frac{A_N\big(\omega, [a,b)\big)}{N} = \prod_{i=1}^{s} (b_i - a_i)$$

for all $[a,b) \subseteq [0,1)^s$. For many of the results we presented there are multidimensional versions. For example, the s - dimensional Weyl criterion says that ω is uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i < h, \omega_n >} = 0 \quad \text{for all } h \in \mathbb{Z}^s \setminus \{0\}$$

where $<\ ,\ >$ denotes the standard inner product.
Uniformly distributed sequences can also be studied in more abstract settings, e.g. in compact spaces with a Borel measure.

References

The references are listed in the order of appearance, i.e. chronological. An English translation of [3] was published in 1984 under the name *The Theory of Uniform Distribution.*

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