1 Random Matrix Theory

1.1 The Circular Unitary Ensemble

The ensemble of matrices with which we will deal predominantly here is known by physicists as the circular unitary ensemble (CUE). This is the group of $N \times N$ unitary matrices $U(N)$, endowed with Haar measure. Here Haar measure is the unique left (and right) invariant probability measure on the unitary group, denoted by d_{Haar}, so effectively the matrices in the group all receive equal weighting in group averages. The eigenvalues of a unitary matrix lie on the unit circle $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N}$ and we are interested in studying the statistics of the eigenphases $\theta_1, \theta_2, \ldots, \theta_N$ of CUE matrices. As we will be working with functions depending only on the eigenvalues of the matrix U, $f(U) = \tilde{f}(\theta_1, \ldots, \theta_N)$, all we need is Weyl's integration formula

$$\int_{U(N)} f(x) d_{Haar}x = \frac{1}{N!(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{f}(\theta_1, \ldots, \theta_N) \prod_{1 \le m < n \le N} |e^{i\theta_n} - e^{i\theta_m}|^2 d\theta_1 \cdots d\theta_N. \tag{1}$$

This implies that the joint probability density function (JPDF) for the eigenphases is

$$P(\theta_1, \ldots, \theta_N) = \frac{1}{N!(2\pi)^N} \prod_{1 \le m < n \le N} |e^{i\theta_n} - e^{i\theta_m}|^2; \tag{2}$$

that is, $P(\theta_1, \ldots, \theta_N) d\theta_1 \cdots d\theta_N$ is the probability that a matrix plucked from this ensemble has eigenvalues between θ_1 and $\theta_1 + d\theta_1$, between θ_2 and $\theta_2 + d\theta_2$, etc.

1.2 Eigenvalue correlation functions and level spacing statistics

Given the JPDF for the eigenvalues of a matrix from an ensemble such as the CUE, we can examine the statistics of these eigenvalues. For more details see [Meh91]. To simplify notation, we will consider here only ensembles containing unitary matrices, so all eigenvalues lie on the unit circle and we are in fact just investigating the distribution of the eigenphases on the interval $[0, 2\pi)$.

One way to study the distribution of these eigenphases is via the correlation functions. Since the JPDF $P(\theta_1, \ldots, \theta_N)$ is the probability density for eigenvalues occurring near $\theta_1, \ldots, \theta_N$, the n-point correlation function, defined as the probability density of eigenvalues occurring around $\theta_1, \ldots, \theta_n$ (irrespective of labelling) with the positions of the other $N - n$ eigenvalues undetermined, is obtained by integrating $P(\theta_1, \ldots, \theta_N)$ over just the variables $\theta_{n+1}, \ldots, \theta_N$. A combinatorial prefactor is acquired equal to the number of ways in which the $\theta_1, \ldots, \theta_n$ can be chosen. Thus the n-point correlation function is

$$R_n(\theta_1, \ldots, \theta_n) = \frac{N!}{(N-n)!} \int_0^{2\pi} \cdots \int_0^{2\pi} P(\theta_1, \cdots \theta_N) d\theta_{n+1} \cdots d\theta_N. \tag{3}$$

We can introduce a random variable $R_n(S)$ on our matrix ensemble which is related to the above statistic, where $S \subset [0, 2\pi)^n$. For a particular matrix U, $R_n(S)(U)$ is the number of ordered n-tuples of eigenphases of U lying in S. That is

$$R_n(S)(U) = \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le N} \chi_S(\theta_{i_1},\dots,\theta_{i_n}), \tag{4}$$

where $\chi_S(\theta_{i_1},\dots,\theta_{i_n})$ is the indicator function which takes the value 1 if $(\theta_{i_1},\dots,\theta_{i_n})$ lies inside S, and zero otherwise. We then see that the expected value $\langle R_n(S)\rangle$ of this random variable averaged over the ensemble is related to the n-point correlation function in (3) by $\langle R_n(S)\rangle = \int_S R_n(\theta_1,\dots,\theta_n)d\theta_1\cdots d\theta_n$. This can be seen to be true because the expected value is just

$$\begin{aligned}\langle R_n(S)\rangle &= \int_0^{2\pi}\cdots\int_0^{2\pi} R_n(S)(U)P(\theta_1,\dots,\theta_N)d\theta_1\cdots d\theta_N \\ &= \sum_{1\le i_1\ne i_2\ne\cdots\ne i_n\le N}\int_0^{2\pi}\cdots\int_0^{2\pi}\chi_S(\theta_{i_1},\dots,\theta_{i_n})P(\theta_1,\dots,\theta_N)d\theta_1\cdots d\theta_N. \end{aligned}\tag{5}$$

However, $P(\theta_1,\dots,\theta_N)$ is symmetric, so we just obtain $N(N-1)\cdots(N-n+1)$ copies of the same integral:

$$\begin{aligned}\langle R_n(S)\rangle &= \frac{N!}{(N-n)!}\int_0^{2\pi}\cdots\int_0^{2\pi}\chi_S(\theta_1,\dots,\theta_n)P(\theta_1,\dots,\theta_N)d\theta_1\cdots d\theta_N \\ &= \int_0^{2\pi}\cdots\int_0^{2\pi}\chi_S(\theta_1,\dots,\theta_n)R_n(\theta_1,\dots,\theta_n)d\theta_1\cdots\theta_n \\ &= \int_S R_n(\theta_1,\dots,\theta_n)d\theta_1\cdots d\theta_n. \end{aligned}\tag{6}$$

Another type of statistic concerns the number of eigenvalues likely to be found inside an interval of a given length. To start with, imagine the probability density for n eigenvalues to lie within the range $(-\theta,\theta)$ at the points $\theta_1,\dots\theta_n$, while the remaining eigenphases lie outside this interval. This can be written as

$$A(\theta;\theta_1,\dots,\theta_n) = \frac{N!}{n!(N-n)!}\int_\theta^{2\pi-\theta}\cdots\int_\theta^{2\pi-\theta}P(\theta_1,\dots,\theta_N)d\theta_{n+1}\cdots d\theta_N. \tag{7}$$

As we are often interested in eigenvalue statistics as $N\to\infty$, we scale the eigenphases so that they have unit mean spacing. Thus, with a mean eigenvalue density of $\overline{d}$, we set

$$\phi = \theta\overline{d} \tag{8}$$

and

$$\phi_i = \theta_i\overline{d}, \qquad i = 1,\dots,N, \tag{9}$$

and we obtain the statistic

$$B(\phi;\phi_1,\dots,\phi_n) = \lim_{N\to\infty} \frac{1}{d^n} A(\theta;\theta_1,\dots,\theta_n). \tag{10}$$

If we instead don't want the n eigenvalues in the interval $(-\phi,\phi)$ to be at the specific positions $\phi_1,\dots,\phi_n$, but just want the probability that there are exactly n levels <u>somewhere</u> in the interval of length $s = 2\phi$, $(-\phi,\phi)$, then we integrate to obtain

$$E(n;s) = \int_{-\phi}^{\phi} \cdots \int_{-\phi}^{\phi} B(\phi;\phi_1,\dots,\phi_n) d\phi_1 \cdots d\phi_n. \tag{11}$$

There are many more quantities which can be derived from $B(\phi;\phi_1,\dots,\phi_n)$. For instance if we set $\phi_1 = -\phi$ and $\phi_2 = \phi$ and contain $\phi_3,\dots\phi_n$ within the interval $(-\phi,\phi)$ then we obtain $p(n-2;s)$, the probability density that the distance between two eigenphases separated by $n-2$ intervening phases is around s:

$$p(n-2;s) = n(n-1) \int_{-\phi}^{\phi} \cdots \int_{-\phi}^{\phi} B(\phi;-\phi,\phi,\phi_3,\dots,\phi_n) d\phi_3 \cdots d\phi_n, \quad n > 2. \tag{12}$$

The probability density that <u>neighbouring</u> eigenphases are separated by a distance of about s is thus

$$p(0;s) = 2B(\phi;-\phi,\phi). \tag{13}$$

This, the nearest neighbour spacing distribution, is a very commonly studied statistic. The $p(n;s)$ are related to the $E(n;s)$ statistics, as we can see simply for the case of $p(0;s)$:

$E(0;x)$ is the probability that a chosen interval of length x contains no eigenvalues. If we add a tiny increment of length on to the original interval, giving us a length $x+\delta x$, then there are two possibilities: either the extra segment δx contains at least one eigenvalue or it contains none. If the latter is true, the the entire interval $x+\delta x$ is empty, and the probability of this is $E(0;x+\delta x)$. Thus $E(0;x) - E(0;x+\delta x)$ is the probability that the interval x is empty but δx is not. If we take the limit as $\delta x \to 0$, then the probability that δx contains two or more eigenvalues will be swamped by the probability that there is just one eigenvalue in δx. Thus $-\frac{dE(0;x)}{dx}\delta x$ is the probability that an empty interval x is followed by one eigenvalue in the interval δx. Alternatively, we can think of $-\frac{dE(0;x)}{dx}$ as the chance that an eigenvalue will be followed by an empty gap of length at least x. Repeating the argument above we see that

$$\left(\frac{-\frac{dE(0;x)}{dx} + \frac{dE(0;x+\delta x)}{dx}}{\delta x} \right) \delta x = \frac{d^2 E(0;x)}{dx^2} \delta x \tag{14}$$

is the probability that two consecutive eigenvalues will be separated by a distance between x and $x+\delta x$. As we have used the notation $p(0;x)dx$ for just such a probability, we see that $p(0;x) = \frac{d^2E(0;x)}{dx^2}$.

1.3 Eigenvalue statistics of the Circular Unitary Ensemble

We now turn to consider the specific case of the circular unitary ensemble (CUE). We have already seen that the joint probability density function of the eigenvalues for this ensemble is

$$P(\theta_1, \dots, \theta_N) = \frac{1}{(2\pi)^N N!} \prod_{1 \le m < n \le N} |e^{i\theta_n} - e^{i\theta_m}|^2. \tag{15}$$

Since

$$\prod_{1 \le m < n \le N} (e^{i\theta_n} - e^{i\theta_m}) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e^{i\theta_1} & e^{i\theta_2} & \cdots & e^{i\theta_N} \\ e^{2i\theta_1} & e^{2i\theta_2} & \cdots & e^{2i\theta_N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(N-1)i\theta_1} & e^{(N-1)i\theta_2} & \cdots & e^{(N-1)i\theta_N} \end{pmatrix}, \tag{16}$$

is just the form of a Vandermond determinant, we can write

$$\prod_{1 \le m < n \le N} (e^{i\theta_m} - e^{i\theta_n})(e^{-i\theta_m} - e^{-i\theta_n}) = \det(e^{i\ell\theta_j}) \det(e^{-i\ell\theta_j}), \tag{17}$$

where the index j indicates the jth column and in the kth row, thanks to the absolute values in (15), we can choose $\ell = (2k - 1 - N)/2$ (rather than $\ell = k - 1$ as in (16)).

We now multiply the two complex conjugate matrices to obtain

$$\prod_{1 \le m < n \le N} |e^{i\theta_m} - e^{i\theta_n}|^2 = \det(\sum_{\ell} e^{i\ell(\theta_j - \theta_k)}) \tag{18}$$

where once more j and k label rows and columns and ℓ sums over the values $\frac{1}{2}(1-N), \frac{1}{2}(3-N), \dots, \frac{1}{2}(N-1)$.

We note that $\sum_\ell e^{i\ell\theta} = \frac{\sin(N\theta/2)}{\sin(\theta/2)}$, a fact which can be checked by writing the sines as pairs of exponentials, multiplying $\sin(\theta/2)$ by $\sum_\ell e^{i\ell\theta}$ and watching the vast cancellation of terms.

Using the notation of Mehta [Meh91],

$$P(\theta_1, \dots, \theta_N) = \frac{1}{N!} \det[S_N(\theta_j - \theta_k)]_{j,k=1,\dots,N} \tag{19}$$

where

$$S_N(\theta) = \frac{1}{2\pi} \sum_{\ell} e^{i\ell\theta} = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}. \tag{20}$$

To obtain the n-point correlation function, therefore, we need to evaluate the integral

$$R_n(\theta_1,\dots,\theta_n) = \frac{1}{(N-n)!}\int_0^{2\pi}\cdots\int_0^{2\pi}\det[S_N(\theta_j-\theta_k)]d\theta_{n+1}\cdots d\theta_N. \tag{21}$$

To evaluate this integral, it is convenient to call upon a theorem stated in Mehta's book.

Theorem 1 *Let $J_N = [J_{ij}]$ be an $N \times N$ Hermitian matrix such that*
i) $J_{ij} = f(x_i, x_j)$ ie. J_{ij} depends only on x_i and x_j
ii) $\int f(x,x)d\mu(x) = c$
iii) $\int f(x,y)f(y,z)d\mu(y) = f(x,z)$
where $d\mu$ is a suitable measure and c is a constant. Then

$$\int \det J_N d\mu(x_N) = (c-N+1)\det J_{N-1} \tag{22}$$

where J_{N-1} is the $(N-1)\times(N-1)$ matrix obtained from J_N by removing the row and column containing x_N.

The proof is also given by Mehta and follows simply from the definition of the determinant:

$$\begin{aligned}\det J_N &= \sum_{\pi_N}\epsilon(\pi)\prod_1^n(J_{\xi\eta}J_{\eta\zeta}\cdots J_{\theta\xi})\\ &= \sum_{\pi_N}\epsilon(\pi)\prod_1^n(f(x_\xi,x_\eta)f(x_\eta,x_\zeta)\cdots f(x_\theta,x_\xi)),\end{aligned} \tag{23}$$

where the sum is over all permutations π_N of $\{1,\dots,N\}$ and π consists of n cycles $(\xi \to \eta \to \zeta \to \cdots \to \theta \to \xi)$. Since we're integrating over x_N, we need to consider the various locations where x_N can occur in the above cycles. We'll divide the sum over permutations between those π_N^* in which N forms a 1-cycle on its own and those π_N^{**} in which it appears in a longer cycle. When we remove the 1-cycle (N) from the permutations π_N^* we are left with the set of all permutations of $\{1,2,\dots,N-1\}$, π_{N-1}. Note that the sign of a permutation $\epsilon(\pi)$ doesn't change if we remove a 1-cycle. So we have

$$\begin{aligned}\det J_N &= f(x_N,x_N)\sum_{\pi_{N-1}}\epsilon(\pi)\prod_1^m(J_{\xi\eta}J_{\eta\zeta}\cdots J_{\theta\xi})\\ &\quad +\sum_{\pi_N^{**}}\epsilon(\pi)\prod_1^n(J_{\xi\eta}J_{\eta\zeta}\cdots J_{\theta\xi}).\end{aligned} \tag{24}$$

Integration over x_N in the second sum, by part iii) of the theorem, will simply remove N from the cycle in which it occurs - diminishing the length of the cycle by one. However, removing N from $(\xi,N,\eta,\zeta,\dots)$, $(\xi,\eta,N,\zeta,\dots)$, etc, will all yield the same contribution, so the resulting sum contains $N-1$ copies of each permutation of $\{1,2,\cdots,N-1\}$. Removing

N from a cycle of length longer than just one results in a change of sign of the permutation. Thus we end up with the result (22).

With this theorem under our belt it is now easy to integrate (21) because

$$\int_0^{2\pi} S_N(0)d\theta = \int_0^{2\pi} \frac{1}{2\pi}\sum_\ell 1 d\theta = N \tag{25}$$

and

$$\begin{aligned}
&\int_0^{2\pi} S_N(\theta_i-\theta_j)S_N(\theta_j-\theta_k)d\theta_j \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\sum_\ell\sum_m e^{i[\ell(\theta_i-\theta_j)+m(\theta_j-\theta_k)]}d\theta_j \\
&= \frac{1}{2\pi}\sum_\ell e^{i\ell(\theta_i-\theta_k)} = S_N(\theta_i-\theta_k).
\end{aligned} \tag{26}$$

Thus we have

$$R_n(\theta_1,\ldots,\theta_n) = \det[S_N(\theta_j-\theta_k)]_{j,k=1,\ldots,n}. \tag{27}$$

So, for example,

$$R_1(\theta) = S_N(0) = \frac{N}{2\pi} \tag{28}$$

$$R_2(\theta_1,\theta_2) = [S_N(0)]^2 - [S_N(\theta_1-\theta_2)]^2. \tag{29}$$

For the limiting 2-point statistic, as $N\to\infty$ we scale by the mean density of eigenvalues, so $\phi_i = \frac{N}{2\pi}\theta_i$, and obtain

$$\lim_{N\to\infty}\left(\frac{2\pi}{N}\right)^2 R_2(\theta_1,\theta_2) = 1 - \left(\frac{\sin[(\phi_1-\phi_2)\pi]}{\pi(\phi_1-\phi_2)}\right)^2. \tag{30}$$

To calculate the spacing distribution we remember that we must work out $A(\alpha)$, the probability that an interval of length α contains no eigenvalues. We note that the JPDF is rotationally invariant, so we may choose any interval of length 2α. Thus we write

$$\begin{aligned}
A(\alpha) &= \int_\alpha^{2\pi-\alpha}\cdots\int_\alpha^{2\pi-\alpha} P(\theta_1,\ldots,\theta_N)d\theta_1\cdots d\theta_N \\
&= \frac{1}{N!}\int_\alpha^{2\pi-\alpha}\cdots\int_\alpha^{2\pi-\alpha}\det[\frac{1}{2\pi}\sum_{j=1}^N e^{i(\ell-m)\theta_j}]d\theta_1\cdots d\theta_N.
\end{aligned} \tag{31}$$

Here the form of the JPDF comes from (17), making a different choice between the matrix and its transpose than lead to (18). ℓ and m are once more related to the jth row and kth column by $\ell = \frac{2j-1-N}{2}$ and $m = \frac{2k-1-N}{2}$.

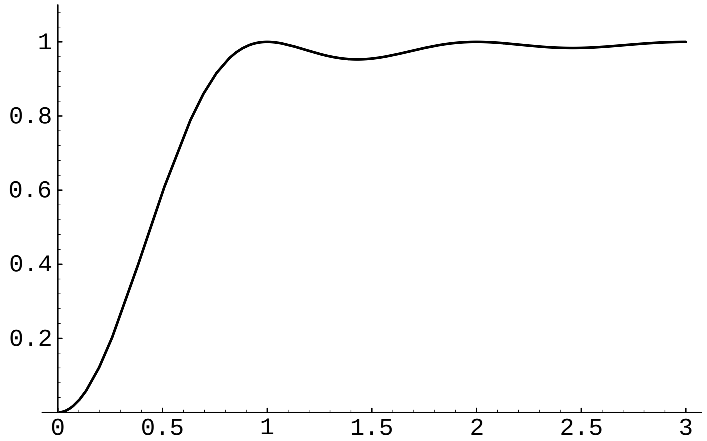

Figure 1: The two-point correlation function for the CUE: $R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$

Now we call on a lemma found in [Haa90] which can be very useful in random matrix calculations generally. It deals with the mean of an arbitrary symmetric function $f(\theta_1, \ldots, \theta_N)$ averaged over the CUE, and states

$$\begin{aligned} &\langle f(\theta_1, \ldots, \theta_N) \rangle \equiv \\ &\frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N f(\theta_1, \ldots, \theta_N) \det[\sum_j e^{i(\ell-m)\theta_j}] \\ &= \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N f(\theta_1, \ldots, \theta_N) \det(e^{i(\ell-m)\theta_\ell}). \end{aligned} \tag{32}$$

To see the validity of (32), we use basic properties of the determinant (or just imagine expanding the determinant about the first row) to obtain

$$
\begin{aligned}
&\langle f(\theta_1,\ldots,\theta_N)\rangle = \\
&\frac{1}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N f(\theta_1,\ldots,\theta_N) \\
&\times\det\begin{pmatrix}
\sum_{k=1}^N 1 & \sum_{k=1}^N e^{-i\theta_k} & \sum_{k=1}^N e^{-2i\theta_k} & \cdots & \sum_{k=1}^N e^{-(N-1)i\theta_k} \\
\sum_{k=1}^N e^{i\theta_k} & \sum_{k=1}^N e^{i\theta_k}e^{-i\theta_k} & \sum_{k=1}^N e^{i\theta_k}e^{-2i\theta_k} & \cdots & \sum_{k=1}^N e^{i\theta_k}e^{-(N-1)i\theta_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^N e^{i(N-1)\theta_k} & \sum_{k=1}^N e^{i(N-1)\theta_k}e^{-i\theta_k} & \sum_{k=1}^N e^{i(N-1)\theta_k}e^{-2i\theta_k} & \cdots & \sum_{k=1}^N e^{i(N-1)\theta_k}e^{-i(N-1)\theta_k}
\end{pmatrix} \\
&= \frac{1}{(2\pi)^N N!}\sum_{k=1}^N\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N f(\theta_1,\ldots,\theta_N) \qquad (33) \\
&\times\det\begin{pmatrix}
1 & e^{-i\theta_k} & e^{-2i\theta_k} & \cdots & e^{-(N-1)i\theta_k} \\
\sum_{j=1}^N e^{i\theta_j} & \sum_{j=1}^N e^{i\theta_j}e^{-i\theta_j} & \sum_{j=1}^N e^{i\theta_j}e^{-2i\theta_j} & \cdots & \sum_{j=1}^N e^{i\theta_j}e^{-(N-1)i\theta_j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^N e^{i(N-1)\theta_j} & \sum_{j=1}^N e^{i(N-1)\theta_j}e^{-i\theta_j} & \sum_{j=1}^N e^{i(N-1)\theta_j}e^{-2i\theta_j} & \cdots & \sum_{j=1}^N e^{i(N-1)\theta_j}e^{-i(N-1)\theta_j}
\end{pmatrix}
\end{aligned}
$$

Since $f(\theta_1,\ldots,\theta_N)$ is symmetric in $\theta_1,\ldots\theta_N$, the integrals in the sum above are all identical, reducing the sum over k to just N copies of the integral with, say, $k=1$.

If we now subtract from the nth row $(n=2,3,\ldots,N)$ $e^{i(n-1)\theta_1}$ times the first row,

$$
\begin{aligned}
&\langle f(\theta_1,\ldots,\theta_N)\rangle = \\
&\frac{N}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N f(\theta_1,\ldots,\theta_N) \\
&\times\det\begin{pmatrix}
1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots & e^{-i(N-1)\theta_1} \\
\sum_{j=2}^N e^{i\theta_j} & \sum_{j=2}^N e^{i\theta_j}e^{-i\theta_j} & \sum_{j=2}^N e^{i\theta_j}e^{-2i\theta_j} & \cdots & \sum_{j=2}^N e^{i\theta_j}e^{-i(N-1)\theta_j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{j=2}^N e^{(N-1)i\theta_j} & \sum_{j=2}^N e^{(N-1)i\theta_j}e^{-i\theta_j} & \sum_{j=2}^N e^{(N-1)i\theta_j}e^{-2i\theta_j} & \cdots & \sum_{j=2}^N e^{(N-1)i\theta_j}e^{-(N-1)i\theta_j}
\end{pmatrix} \\
&= \frac{1}{(2\pi)^N (N-1)!}\sum_{j=2}^N\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N f(\theta_1,\ldots,\theta_N) \qquad (34) \\
&\times\det\begin{pmatrix}
1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots & e^{-i(N-1)\theta_1} \\
e^{i\theta_j} & e^{i\theta_j}e^{-i\theta_j} & e^{i\theta_j}e^{-2i\theta_j} & \cdots & e^{i\theta_j}e^{-(N-1)i\theta_j} \\
\sum_{k=2}^N e^{2i\theta_k} & \sum_{k=2}^N e^{2i\theta_k}e^{-i\theta_k} & \sum_{k=2}^N e^{2i\theta_k}e^{-2i\theta_k} & \cdots & \sum_{k=2}^N e^{2i\theta_k}e^{-i(N-1)\theta_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sum_{k=2}^N e^{(N-1)i\theta_k} & \sum_{k=2}^N e^{(N-1)i\theta_k}e^{-i\theta_k} & \sum_{k=2}^N e^{(N-1)i\theta_k}e^{-2i\theta_k} & \cdots & \sum_{k=2}^N e^{(N-1)i\theta_k}e^{-(N-1)i\theta_k}
\end{pmatrix}
\end{aligned}
$$

The sum over j becomes multiplication by $N-1$ due to the symmetry, and we set $j=2$ for simplicity. Then $e^{(n-2)i\theta_2}$ is multiplied by row 2 and subtracted from the nth row and the process is repeated, yielding eventually (32).

By precisely the same argument as above, we see that (remembering that $\ell = (2j-1-N)/2$)

$$
\begin{aligned}
A(\alpha) &= \int_{\alpha}^{2\pi-\alpha} \cdots \int_{\alpha}^{2\pi-\alpha} \det(\frac{1}{2\pi} e^{i(\ell-m)\theta_j}) d\theta_1 \cdots d\theta_N \\
&= \det[\frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} d\theta e^{i(\ell-m)\theta}] \\
&= \det[\delta_{\ell m} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \cos[(\ell-m)\theta]]. \qquad (35)
\end{aligned}
$$

To continue, we note that (here ℓ, m and n all take values in $\frac{1}{2}(1-N), \ldots, \frac{1}{2}(N-1)$)

$$
\begin{aligned}
&\sum_{\ell} [\delta_{m\ell} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \cos(m\theta)\cos(\ell\theta)][\delta_{n\ell} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \sin(n\theta)\sin(\ell\theta)] \\
&= \delta_{mn} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \cos(m\theta)\cos(n\theta) - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \sin(n\theta)\sin(m\theta) \\
&+ \sum_{\ell} \frac{1}{4\pi^2} \int_{-\alpha}^{\alpha} d\theta \cos(m\theta)\cos(\ell\theta) \int_{-\alpha}^{\alpha} d\phi \sin(n\phi)\sin(\ell\phi) \\
&= \delta_{mn} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \cos[(n-m)\theta], \qquad (36)
\end{aligned}
$$

where the final line follows because in the line before the terms in the sum where the ℓ's are equal in magnitude but opposite in sign cancel each other out.

Following identity (36), we can write

$$
A(\alpha) = A'(\alpha) A"(\alpha) \qquad (37)
$$

where

$$
\begin{aligned}
A'(\alpha) &= \det[\delta_{mn} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \cos(m\theta)\cos(n\theta)] \\
&= \det\left[\delta_{mn} - \frac{1}{2\pi}\left(\frac{\sin[(m-n)\alpha]}{m-n} + \frac{\sin[(m+n)\alpha]}{m+n}\right)\right] \qquad (38)
\end{aligned}
$$

and

$$
\begin{aligned}
A"(\alpha) &= \det[\delta_{mn} - \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\theta \sin(m\theta)\sin(n\theta)] \\
&= \det[\delta_{mn} - \frac{1}{2\pi}\left(\frac{\sin[(m-n)\alpha]}{m-n} - \frac{\sin[(m+n)\alpha]}{m+n}\right)]. \qquad (39)
\end{aligned}
$$

If we take the limit $N \to \infty$, keeping $Nm\alpha = 2\pi\xi t$ and $Nn\alpha = 2\pi\eta t$ constant then $A'(\alpha)$ is related to the eigenvalues of the integral equation

$$
\lambda f(\xi) = \int_{-1}^{1} \overline{\mathcal{Q}}(\xi, \eta) f(\eta) d\eta, \qquad (40)
$$

where

$$\overline{\mathcal{Q}}(\xi,\eta) = \frac{1}{2}(\mathcal{Q}(\xi,\eta) + \mathcal{Q}(\xi,-\eta)) \tag{41}$$

is the even part of

$$\mathcal{Q}(\xi,\eta) = \frac{\sin[(\xi-\eta)\pi t]}{(\xi-\eta)\pi}. \tag{42}$$

Thus in the limit $N \to \infty$, $A'(\alpha)$ tends to the Fredholm determinant

$$A'(\alpha) \to B'(t) = \prod_{i=0}^{\infty}(1-\lambda_{2i}), \tag{43}$$

where λ_{2i} are the eigenvalues corresponding to even solutions of the integral equation

$$\lambda f(\xi) = \int_{-1}^{1} \mathcal{Q}(\xi,\eta) f(\eta) d\eta. \tag{44}$$

Since the expression in $A"(\alpha)$ is just the odd part of $\mathcal{Q}(\xi,\eta)$, in this case in the limit $N \to \infty$ we find a Fredholm determinant involving all the eigenvalues corresponding to odd solutions of (44):

$$A"(\alpha) \to B"(t) = \prod_{i=0}^{\infty}(1-\lambda_{2i+1}). \tag{45}$$

Thus as $N \to \infty$ (with $s = 2t$),

$$A(\alpha) \to E(0;s) = \prod_{i=0}^{\infty}(1-\lambda_i). \tag{46}$$

We then recall that the nearest neighbour spacing distribution (shown in Figure 2) can be determined from $E(0;s)$ by differentiating twice. The important feature of this figure is that the distribution drops to zero as s goes to zero. This indicates that there is "repulsion" between eigenphases; it is not likely to find two eigenvalues very close together. In fact, for the CUE this repulsion is quadratic in strength (due to the exponent 2 on the factor $\prod_{m<n} |e^{i\theta_n} - e^{i\theta_m}|$ in the joint probability density function), which distinguishes it from various other random matrix ensembles.

1.4 Mean values of characteristic polynomials

The study of mean values of the Riemann zeta function can be greatly aided by the examination of the mean values of random matrix characteristic polynomials [KS00]. These latter mean values follow from a straightforward random matrix calculation which is detailed below.

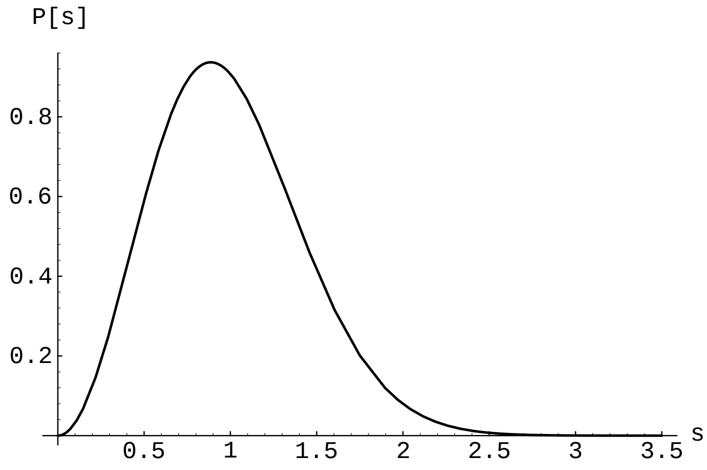

Figure 2: The nearest neighbour spacing distribution for the CUE.

We write the characteristic polynomial of an $N \times N$ matrix in the CUE (the unitary group $U(N)$ with Haar measure) as

$$Z(U,\theta) = \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right), \tag{47}$$

where the $e^{i\theta_n}$ are the matrix eigenvalues.

Since this is a function of the eigenvalues, an average of $|Z(U,\theta)|^s$ can be performed over the ensemble using the joint probability density function discussed earlier:

$$\begin{aligned}\langle |Z(U,\theta)|^s \rangle_{CUE} \quad &= \quad \frac{1}{(2\pi)^N N!} \times \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N \\ &\quad \times \prod_{1\le j<m\le N} \left| e^{i\theta_j} - e^{i\theta_m} \right|^2 \left| \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)}) \right|^s .\end{aligned} \tag{48}$$

Integrals similar to this (those which contain a product with the structure $\prod_{1\le j<m\le N} |x_j - x_m|$) can be dealt with by the use of a form of Selberg's integral (described at length in Chapter 17 of [Meh91]). The integral which we need for (48) is

$$
\begin{aligned}
&J(a,b,\alpha,\beta,\gamma,N)\\
&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\prod_{1\le j<\ell\le N}(x_j-x_\ell)\right|^{2\gamma}\times\prod_{j=1}^{N}(a+ix_j)^{-\alpha}(b-ix_j)^{-\beta}dx_j \qquad (49)\\
&=\frac{(2\pi)^N}{(a+b)^{(\alpha+\beta)N-\gamma N(N-1)-N}}\cdot\prod_{j=0}^{N-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(N+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)}.
\end{aligned}
$$

In the above formula, a, b, α, β and γ are complex numbers, $\mathrm{Re}\, a$, $\mathrm{Re}\, b$, $\mathrm{Re}\,\alpha$ and $\mathrm{Re}\,\beta$ are all greater than zero, $\mathrm{Re}\,(\alpha+\beta)>1$ and

$$
-\frac{1}{N}<\mathrm{Re}\,\gamma<\min\left(\frac{\mathrm{Re}\,\alpha}{N-1},\frac{\mathrm{Re}\,\beta}{N-1},\frac{\mathrm{Re}\,(\alpha+\beta-1)}{2(N-1)}\right). \qquad (50)
$$

In attempting to coerce (48) into the form of Selberg's integral, we note that

$$
\left|e^{i\theta_j}-e^{i\theta_m}\right|=2\left|\sin(\theta_j/2-\theta_m/2)\right|, \qquad (51a)
$$

and similarly

$$
\left|1-e^{i(\theta_p-\theta)}\right|=2\left|\sin(\theta_p/2-\theta/2)\right|. \qquad (51b)
$$

Therefore we can write (48) as

$$
\begin{aligned}
\langle|Z(U,\theta)|^s\rangle_{CUE} &= \frac{2^{N(N-1)}2^{sN}}{N!(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi}d\theta_1\cdots d\theta_N \qquad (52)\\
&\quad\times\prod_{1\le j<m\le N}\left|\sin(\theta_j/2-\theta_m/2)\right|^2\prod_{n=1}^{N}\left|\sin(\theta_n/2-\theta/2)\right|^s,
\end{aligned}
$$

and we note that this integral is in fact independent of θ, which we eliminate to obtain

$$
\begin{aligned}
\langle|Z|^s\rangle_{CUE} &= \frac{2^{N(N-1)}2^{sN}2^N}{N!(2\pi)^N}\int_0^{\pi}\cdots\int_0^{\pi}d\theta_1\cdots d\theta_N\prod_{1\le j<m\le N}\left|\sin(\theta_j-\theta_m)\right|^2\\
&\quad\times\prod_{n=1}^{N}\left|\sin\theta_n\right|^s. \qquad (53)
\end{aligned}
$$

If we write

$$
\begin{aligned}
\sin(\theta_j-\theta_m) &= \sin\theta_j\cos\theta_m-\cos\theta_j\sin\theta_m \qquad (54)\\
\frac{\sin(\theta_j-\theta_m)}{\sin\theta_j\sin\theta_m} &= \cot\theta_m-\cot\theta_j,
\end{aligned}
$$

then the moment can be written as

$$\begin{aligned}\langle|Z|^s\rangle_{CUE} &= \frac{2^{N^2+sN}}{N!(2\pi)^N}\int_0^\pi\cdots\int_0^\pi d\theta_1\cdots d\theta_N \prod_{1\le j<m\le N}|\cot\theta_m-\cot\theta_j|^2\\ &\quad\times\prod_{n=1}^N\left(\sin^2\theta_n\right)^{N-1}\prod_{n=1}^N|\sin\theta_n|^s.\end{aligned}\tag{55}$$

A change of variables, $x_n=\cot\theta_n$, gives us

$$\begin{aligned}\langle|Z|^s\rangle_{CUE} &= \frac{2^{N^2+sN}}{N!(2\pi)^N}\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty dx_1\cdots dx_N \prod_{1\le j<m\le N}|x_m-x_j|^2\\ &\quad\times\prod_{n=1}^N\left((1+ix_n)(1-ix_n)\right)^{-N-s/2}\\ &= \frac{2^{N^2+sN}}{N!(2\pi)^N}J(1,1,N+s/2,N+s/2,1,N)\\ &= \prod_{j=1}^N\frac{\Gamma(j)\Gamma(s+j)}{(\Gamma(j+s/2))^2}\equiv M_N(s),\end{aligned}\tag{56}$$

where J is the form of Selberg's integral defined in (49). Considering the conditions, listed in (50), on the various parameters, (56) holds if $-1<\mathrm{Re}s$.

As N becomes large, we can look at the asymptotic form of $M_N(s)$ and we see that for an integer $k\le N-1$,

$$\begin{aligned}\langle|Z|^{2k}\rangle_{CUE} &= \prod_{j=1}^N\frac{(j-1)!(2k+j-1)!}{(j+k-1)!^2}\\ &= \frac{0!1!\cdots(N-1)!(2k)!(2k+1)!\cdots(2k+N-1)!}{k!(k+1)!\cdots(k+N-1)!k!(k+1)!\cdots(k+N-1)!}\\ &= \frac{0!1!\cdots(k-1)!(k+N)!(k+N+1)!\cdots(2k+N-1)!}{N!(N+1)!\cdots(k+N-1)!k!(k+1)!\cdots(2k-1)!}\\ &= \frac{0!1!\cdots(k-1)!}{k!(k+1)!\cdots(2k-1)!}[(N+1)\cdots(k+N)]\,[(N+2)\cdots\\ &\qquad(k+N+1)]\cdots\cdots[(k+N)\cdots(2k+N-1)]\\ &= \prod_{j=0}^{k-1}\frac{j!}{(k+j)!}N^{k^2}+O(N^{k^2-1}).\end{aligned}\tag{57}$$

For non-integer exponent the asymptotics are also known and can be written as

$$M_N(s)\sim\frac{(G(1+s/2))^2}{G(1+s)}N^{(s/2)^2},\tag{58}$$

where $G(z)$ is the Barnes G-function [Bar00, Vor87, Vig79] defined by

$$G(1+z) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2+z]/2} \prod_{n=1}^{\infty} \left[(1+z/n)^n e^{-z+z^2/(2n)} \right]. \tag{59}$$

This function has zeros at the negative integers, $-n$, with multiplicity n ($n = 1, 2, 3\ldots$), and is related to the gamma function, as can be seen by

$$\begin{aligned} G(1) &= 1, \\ G(z+1) &= \Gamma(z)\, G(z). \end{aligned} \tag{60}$$

Having determined the mean values of $|Z|^s$ in (56), it is straightforward to obtain both the integer moments and the value distribution of the logarithm of $|Z|$. These log moments are generated by $M_N(s)$, so we have

$$M_N(s) = \sum_{j=0}^{\infty} \frac{\langle (\log|Z|)^j \rangle_{CUE}}{j!} s^j, \tag{61}$$

and, defining Q_j as the cumulants of the real part of the log,

$$\log M_N(s) = \sum_{j=1}^{\infty} \frac{Q_j}{j!} s^j. \tag{62}$$

Thus

$$\begin{aligned} Q_n &= \frac{d^n}{ds^n} \log M_N(s) \big|_{s=0} \\ &= \frac{d^{n-1}}{ds^{n-1}} \sum_{j=1}^{N} \left(\psi(j+s) - \psi(j+s/2) \right) \big|_{s=0} \qquad (63) \\ &= \sum_{j=1}^{N} \left(\psi^{(n-1)}(j+s) - \frac{1}{2^{n-1}} \psi^{(n-1)}(j+s/2) \big|_{s=0} \right) \\ &= \frac{2^{n-1}-1}{2^{n-1}} \sum_{j=1}^{N} \psi^{n-1}(j), \qquad (64) \end{aligned}$$

where we have introduced the polygamma functions $\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z)$.

If we consider the cumulants when N is large then we can employ the asymptotic expansion for the polygamma functions [AS65]

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right] \tag{65}$$

for $|z| \to \infty$ with $|\arg z| < \pi$. Here the B_{2k} are Bernoulli numbers.

From (65) we see that the second cumulant is of order $\log N$ as N becomes large:

$$
\begin{aligned}
Q_2(N,0) &= \frac{1}{2}\sum_{j=1}^{N} \psi^{(1)}(j) \\
&= \frac{1}{2}\sum_{j=1}^{N} \left(\frac{1}{j} + \frac{1}{2j^2} + O\left(\frac{1}{j^3}\right)\right) \\
&\sim \frac{1}{2}\log N. \qquad (66)
\end{aligned}
$$

An identical argument shows that the leading order term of all cumulants Q_n with $n \geq 3$ is a constant, and it is clear that $Q_1 = 0$.

This knowledge of the cumulants is alone enough to show that the value distribution of $(\mathrm{Re} \log Z)/\sqrt{Q_2}$ is Gaussian in the limit $N \to \infty$. To see this we start with

$$
\rho_N(x) = \langle \delta(\log |Z|/\sqrt{Q_2} - x)\rangle_{CUE}. \qquad (67)
$$

which is the probability density function for the values taken on by $\log |Z|$.
Taking the Fourier transform of this,

$$
\begin{aligned}
\widehat{\rho_N}(s) &= \left\langle \int_{-\infty}^{\infty} e^{isx}\delta(\log |Z|/\sqrt{Q_2} - x)dx \right\rangle_{CUE} \\
&= \left\langle e^{is\log|Z|/\sqrt{Q_2}} \right\rangle_{CUE}, \qquad (68)
\end{aligned}
$$

we obtain the moments of $|Z|$ itself, $M_N(is/\sqrt{Q_2})$. Therefore, the distribution of $\log |Z|$ is

$$
\begin{aligned}
\rho_N(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} M_N(is/\sqrt{Q_2})ds \qquad (69) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-isx - s^2/2 - iQ_3 s^3/(3!Q_2^{3/2}) + Q_4 s^4/(4!Q_2^2) + \cdots)ds.
\end{aligned}
$$

Since

$$
\frac{Q_j}{Q_2^{j/2}} \to 0 \quad \text{as} \quad N \to \infty, \quad \text{for} \quad j \geq 3, \qquad (70)
$$

as $N \to \infty$,

$$
\begin{aligned}
\rho_N(x) &\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx}e^{-y^2/2}dy \\
&\sim \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right). \qquad (71)
\end{aligned}
$$

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