

LOW-LYING ZEROS OF QUADRATIC DIRICHLET L-FUNCTIONS, HYPER-ELLIPTIC CURVES AND RANDOM MATRIX THEORY

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Abstract. The statistics of low-lying zeros of quadratic Dirichlet L-functions were conjectured by Katz and Sarnak to be given by the scaling limit of eigenvalues from the unitary symplectic ensemble. The n -level densities were found to be in agreement with this in a certain neighborhood of the origin in the Fourier domain by Rubinstein in his Ph.D. thesis in 1998. An attempt to extend the neighborhood was made in the Ph.D. thesis of Peng Gao (n -level density of the low-lying zeros of quadratic Dirichlet L-functions, 2005), who under GRH gave the density as a complicated combinatorial factor, but it remained open whether it coincides with the Random Matrix Theory factor. For $n \leq 7$ this was recently confirmed by Levinson and Miller. We resolve this problem for all n , not by directly doing the combinatorics, but by passing to a function field analogue, of L-functions associated to hyper-elliptic curves of given genus g over a field of q elements. We show that the answer in this case coincides with Gao's combinatorial factor up to a controlled error. We then take the limit of large finite field size $q \rightarrow \infty$ and use the Katz–Sarnak equidistribution theorem, which identifies the monodromy of the Frobenius conjugacy classes for the hyperelliptic ensemble with the group $\mathrm{USp}(2g)$. Further taking the limit of large genus $g \rightarrow \infty$ allows us to identify Gao's combinatorial factor with the RMT answer.

1 Introduction

1.1 One-level densities for quadratic L-functions. Our goal in this paper is to study statistics of low-lying zeros of quadratic Dirichlet L-functions. To simplify the discussion, we restrict to discriminants of the form $8d$, where $d > 0$ is an odd, square-free integer. The corresponding quadratic characters χ_{8d} are then all primitive and even, and have conductor $8d$. Denote the nontrivial zeros of the corresponding L-function $L(s, \chi_{8d})$ by

$$\frac{1}{2} + i\gamma_{8d,j}, \quad j = \pm 1, \pm 2, \dots \quad (1.1)$$

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where the labeling is so that $\gamma_{8d,-j} = -\gamma_{8d,j}$. The number $N(T, 8d)$ of such zeros with $0 \leq \Re \gamma_{8d,j} \leq T$ is asymptotically, for $T > 1$,

$$N(T, 8d) = \frac{T}{2\pi} \log \frac{8dT}{2\pi} - \frac{T}{2\pi} + O(\log 8dT). \quad (1.2)$$

We wish to study statistics of the zeros of $L(s, \chi_{8d})$ for random d . To do so, set

$$\mathcal{D}(X) = \{X \leq d \leq 2X : d \text{ odd, square-free}\} \quad (1.3)$$

Then $\#\mathcal{D}(X) \sim \frac{4}{\pi^2}X$, as $X \rightarrow \infty$. To define what it means to pick a “random” discriminant from $\mathcal{D}(X)$, we take a smooth weight function $\Phi \geq 0$ supported in the interval $(1, 2)$, satisfying $\int \Phi(u) du = 1$, and define an averaging operator for functions f on $\mathcal{D}(X)$ by

$$\langle f \rangle_{\mathcal{D}(X)} := \frac{1}{\#\mathcal{D}(X)} \sum_{d \in \mathcal{D}(X)} \Phi\left(\frac{d}{X}\right) f(d). \quad (1.4)$$

Thus we obtain a probability measure on $\mathcal{D}(X)$ which endows it with the structure of a probability space (ensemble), which we call the quadratic ensemble.

To count the number of zeros on the scale of the mean spacing $\log 8d/2\pi$ between the low-lying zeros, we define the linear statistic, or one-level density, by taking an even Schwartz function $f(r)$, which is analytic in a strip $|\Im r| \leq 1/2$, and setting for $d \in \mathcal{D}(X)$

$$W_f(d) := \sum_j f(L\gamma_{8d,j}). \quad (1.5)$$

Here $L = \log X/2\pi$.

The expectation values of the one-level densities for the quadratic ensemble were studied by Katz and Sarnak [KS99, KS97] (see also [ÖS93, ÖS06]) who showed that, assuming GRH, in the “scaling limit” $X \rightarrow \infty$, their expected value coincides with the analogous quantity for the eigenphases of random matrices from unitary symplectic groups $\text{USp}(2g)$ in the limit $g \rightarrow \infty$, that is

$$\lim_{X \rightarrow \infty} \langle W_f \rangle_{\mathcal{D}(X)} = \int_{-\infty}^{\infty} f(x) \left(1 - \frac{\sin 2\pi x}{2\pi x}\right) dx \quad (1.6)$$

under the condition that the Fourier transform $\widehat{f}(u) = \int_{\mathbb{R}} f(x) e^{-2\pi i x u} dx$ is supported in the interval

$$|u| < 2. \quad (1.7)$$

The Density Conjecture [KS99] is that (1.6) holds for any test function f . See [Sto09] for numerical support for the conjecture and [Mil08] for a refined version.

1.2 Higher moments and the n -level densities. We want to study the moments of the linear statistic. The goal is to show that in the scaling limit the moments coincide with the analogous quantity for the eigenphases of random matrices from unitary symplectic groups.

The moments are determined by multi-linear statistics known as the n -level densities. To define these, one starts with a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, which is even in all variables. The n -level density for $d \in \mathcal{D}(X)$ is

$$W_f^{(n)}(d) := \sum_{\substack{j_1, \dots, j_n = \pm 1, \pm 2, \dots \\ |j_k| \text{ distinct}}} f(L\gamma_{8d, j_1}, \dots, L\gamma_{8d, j_n}), \tag{1.8}$$

where the sum is over n -tuples of indices $j_1, \dots, j_n = \pm 1, \pm 2, \dots$ with $j_r \neq \pm j_s$ for $r \neq s$, and $L = \log X/2\pi$. The density conjecture [KS99] for low lying zeros of this family of L-functions is that the scaling limit coincides with the scaling limit of the n -level densities for random matrices in the unitary symplectic group $\text{USp}(2g)$, that is

$$\lim_{X \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\mathcal{D}(X)} = \int_{\mathbb{R}^n} f(x) W_{\text{USp}}^{(n)}(x) dx, \tag{1.9}$$

where

$$\begin{aligned} W_{\text{USp}}^{(n)}(x) &= \det(K(x_i, x_j))_{i, j=1, \dots, n}, \\ K(x, y) &= \frac{\sin \pi(x - y)}{\pi(x - y)} - \frac{\sin \pi(x + y)}{\pi(x + y)}. \end{aligned} \tag{1.10}$$

The higher densities for this ensemble were investigated in the Ph.D. thesis of Mike Rubinstein [Rub98, Rub01], who assuming GRH established (1.9) under the condition that the Fourier transform $\hat{f}(u) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot u} dx$ is supported in the set

$$\sum_{j=1}^n |u_j| < 1. \tag{1.11}$$

Note that for $n = 1$, (1.11) is only half the range in (1.7).

In his Ph.D. thesis [Gao05, Gao13], Peng Gao attempted to double the range in Rubinstein’s result. He showed, assuming GRH, that if f is of the form $f(x_1, \dots, x_n) = \prod_{j=1}^n f_j(x_j)$ and each \hat{f}_j is supported in the range $|u_j| < s_j$ with $\sum s_j < 2$ so that f is supported on the range

$$\sum_{j=1}^n |u_j| < 2, \tag{1.12}$$

then

$$\left\langle W_f^{(n)} \right\rangle_{\mathcal{D}(X)} = A(f) + o(1), \quad X \rightarrow \infty, \tag{1.13}$$

where $A(f) = A(f_1, \dots, f_n)$ is a complicated combinatorial expression, taking almost a page to write down (see Theorem 7.2). In view of (1.13), proving (1.9) in this range is reduced to a purely combinatorial problem, of proving an identity

$$A(f) = \int_{\mathbb{R}^n} f(x) W_{\text{USp}}^{(n)}(x) dx \quad (1.14)$$

which Gao verified for $n = 2, 3$. More recently, Levinson and Miller [LM12] have confirmed (1.14) for $n = 4, 5, 6, 7$, aided by a machine calculation. In this paper we confirm the equality for all n .

Theorem 1.1. *Assume GRH. For test functions whose Fourier transform \widehat{f} is supported in the region $\sum_{j=1}^n |u_j| < 2$, we have*

$$\lim_{X \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\mathcal{D}(X)} = \int_{\mathbb{R}^n} f(x) W_{\text{USp}}^{(n)}(x) dx. \quad (1.15)$$

Instead of directly attacking the combinatorial problem, we approach it by comparing the densities of the zeros with a function field analogue, of zeros of L-functions for hyperelliptic curves of genus g defined over a finite field \mathbb{F}_q . We then use the equidistribution results of Deligne and Katz–Sarnak to pass to the large finite field limit $q \rightarrow \infty$ and identify the limit with RMT. This is similar in spirit to one of the ingredients in the work of Ngô on the “Fundamental Lemma”, where a complicated combinatorial identity arising from a number field setup is proved via a passage to the function field setting [Ngô10]. To explain how we do it, we first describe the RMT context and then move on to the function field setting.

1.3 Random matrix theory (RMT). For any continuous function F on the set of conjugacy classes of $\text{USp}(2g)$, we denote by $\langle F \rangle_{\text{USp}(2g)}$ its average with respect to the Haar probability measure on $\text{USp}(2g)$:

$$\langle F \rangle_{\text{USp}(2g)} = \int_{\text{USp}(2g)} F(U) dU. \quad (1.16)$$

Recall that for a unitary symplectic matrix $U \in \text{USp}(2g)$, if $e^{i\theta}$ is an eigenvalue then so is $e^{-i\theta}$. We can then label the eigenvalues of U as $e^{i\theta_{\pm j}}$, $j = 1, \dots, g$ with the eigenphases $\theta_1, \dots, \theta_g \in [0, \pi]$ and $\theta_{-j} = -\theta_j$.

To define n -level densities, one starts with a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, which is even in all variables, and sets

$$\widetilde{f}(\theta) = \sum_{m \in \mathbb{Z}^n} f\left(\frac{g}{\pi}(\theta + 2\pi m)\right), \quad (1.17)$$

which is 2π -periodic and localized on a scale of $1/g$. The n -level density is

$$W_f^{(n)}(U) = \sum_{\substack{j_1, \dots, j_n = \pm 1, \dots, \pm g \\ |j_k| \text{ distinct}}} \widetilde{f}(\theta_{j_1}, \dots, \theta_{j_n}), \quad (1.18)$$

where the sum is over n -tuples of indices $j_1, \dots, j_n = \pm 1, \dots, \pm g$ with $j_r \neq \pm j_s$ if $r \neq s$.

If we restrict the Fourier transform $\widehat{f}(u)$ to be supported in the region $|u| < \frac{1}{n}$ then the first n moments of the linear statistic $W_f^{(1)}$ in RMT are Gaussian [HR03]. This was called “mock-Gaussian” behavior in [HR03]. The higher moments are also known, but no longer have a simple expression (however see [HM07] for some nice expressions for the centered moments of orthogonal families). It is the n -level density which has a clean expression: In the scaling limit, the n -level densities are given by

$$\lim_{g \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\text{USp}(2g)} = \int_{\mathbb{R}^n} f(x) W_{\text{USp}}^{(n)}(x) dx, \tag{1.19}$$

where $W_{\text{USp}}^{(n)}$ is given by (1.10).

1.4 The hyperelliptic ensemble. For a finite field \mathbb{F}_q of odd cardinality q consider the family $\mathcal{H}(2g + 1, q)$ of all curves given in affine form by an equation

$$C_h : y^2 = h(x)$$

where

$$h(x) = x^{2g+1} + a_{2g}x^{2g} + \dots + a_0 \in \mathbb{F}_q[x]$$

is a square-free, monic polynomial of degree $2g + 1$. The curve C_h is thus nonsingular and of genus g . We consider $\mathcal{H}(2g + 1, q)$ as a probability space (ensemble) with the uniform probability measure, so that the expected value of any function F on $\mathcal{H}(2g + 1, q)$ is defined as

$$\langle F \rangle_{\mathcal{H}(2g+1,q)} := \frac{1}{\#\mathcal{H}(2g + 1, q)} \sum_{h \in \mathcal{H}(2g+1,q)} F(h). \tag{1.20}$$

The zeta function associated with the hyperelliptic curve $C_h \in \mathcal{H}(2g + 1, q)$ has the form

$$Z_h(u) = \frac{\det(I - u\sqrt{q}\Theta_h)}{(1 - u)(1 - qu)} \tag{1.21}$$

for a unique conjugacy class of $2g \times 2g$ unitary symplectic matrices $\Theta_h \in \text{USp}(2g)$ so that the eigenvalues $e^{i\theta_j}$ of Θ_h correspond to zeros $q^{-1/2}e^{-i\theta_j}$ of $Z_h(u)$. The matrix (or rather the conjugacy class) Θ_h is called the unitarized Frobenius class of C_h . Katz and Sarnak showed [KS99] that as $q \rightarrow \infty$, the Frobenius classes Θ_h become equidistributed in the unitary symplectic group $\text{USp}(2g)$: for any continuous function on the space of conjugacy classes of $\text{USp}(2g)$,

$$\lim_{q \rightarrow \infty} \langle F(\Theta_h) \rangle_{\mathcal{H}(2g+1,q)} = \langle F(U) \rangle_{\text{USp}(2g)}. \tag{1.22}$$

This implies that various statistics of the eigenvalues can, in this limit, be computed by integrating the corresponding quantities over $\mathrm{USp}(2g)$. In particular, the n -level densities for the hyper-elliptic ensemble $\mathcal{H}(2g+1, q)$ when g is fixed are given in the large finite field limit by

$$\lim_{q \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\mathcal{H}(2g+1, q)} = \left\langle W_f^{(n)} \right\rangle_{\mathrm{USp}(2g)}. \quad (1.23)$$

Therefore, on further taking the large genus limit $g \rightarrow \infty$ one gets

$$\lim_{g \rightarrow \infty} \left(\lim_{q \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\mathcal{H}_{2g+1}} \right) = \int_{\mathbb{R}^n} f(x) W_{\mathrm{USp}}^{(n)}(x) dx. \quad (1.24)$$

1.5 Comparing the hyperelliptic and quadratic ensembles. We will compute the averages of the n -level densities for the hyper-elliptic ensemble. We will show that in the range (1.12) they are asymptotically equal to a complicated combinatorial expression up to a remainder term that is negligible for large g , the same expression $A(f)$ which appears in Gao's result (1.13).

Theorem 1.2. *Assume that $f(x_1, \dots, x_n) = \prod_{j=1}^n f_j(x_j)$, with $f_j \in \mathcal{S}(\mathbb{R})$ even and each $\widehat{f}_j(u_j)$ is supported in the range $|u_j| < s_j$, with $\sum s_j < 2$. Then*

$$\left\langle W_f^{(n)} \right\rangle_{\mathcal{H}(2g+1, q)} = A(f) + O_f \left(\frac{\log g}{g} \right), \quad (1.25)$$

the implied constant independent of the finite field size q , and with $A(f) = A(f_1, \dots, f_n)$ as in Theorem 7.2.

To prove Theorem 1.2 we use a similar approach to that in [Gao05, KS97] with some simplifications and variations arising from our function field setting. In particular Poisson summation, which is used critically in [Gao05, KS97] is replaced by the functional equation of the zeta-function Z_h .

What is crucial in Theorem 1.2 is that the bound on the remainder term is uniform in q . Taking the iterated limit $\lim_{g \rightarrow \infty} (\lim_{q \rightarrow \infty})$ of (1.25) and using the Katz–Sarnak result (1.24) gives our main result on the quadratic ensemble, as well as a corresponding result for the hyper-elliptic ensemble:

COROLLARY 1.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ be even in all variables, and assume that $\widehat{f}(u)$ is supported in the region $\sum_{j=1}^n |u_j| < 2$. Then for q fixed,*

$$\lim_{g \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\mathcal{H}(2g+1, q)} = \int_{\mathbb{R}^n} f(x) W_{\mathrm{USp}}^{(n)}(x) dx, \quad (1.26)$$

and assuming GRH,

$$\lim_{X \rightarrow \infty} \left\langle W_f^{(n)} \right\rangle_{\mathcal{D}(X)} = \int_{\mathbb{R}^n} f(x) W_{\mathrm{USp}}^{(n)}(x) dx. \quad (1.27)$$

Proof. For both (1.26) and (1.27) we may assume that $f = \prod f_j(x_j)$, with each \widehat{f}_j even and supported on $|u_j| < s_j$ and $\sum s_j < 2$, since any f satisfying the conditions of the corollary can be approximated by a linear combination of functions of this form. Now it follows from Theorem 1.2 and (1.24) that (1.27) holds and

$$A(f) = A(f_1, \dots, f_n) = \int_{\mathbb{R}^n} f(x) W_{\text{USp}}^{(n)}(x) dx.$$

This is obtained by taking the limit $g \rightarrow \infty$ in Theorem 1.2 and comparing with (1.24). Now (1.26) follows from (1.13). □

1.6 Further applications. The method of this paper can in principle be used to compute statistics of zeros of other families of L-functions, provided a good function field analogue can be found. For instance, one of us (A.E.) has given an alternate proof of the result of Rudnick and Sarnak [RS96] that the n -level correlation of the Riemann zeros (that is of a *single* L-function) are given by Random Matrix Theory, by making a comparison with a family of Artin–Schreier curves [Ent13]. Very recently a different combinatorial proof of the result of [RS96] was given by Conrey and Snaith [CS12].

2 Background on Function Field Arithmetic

We review some elements of the arithmetic of $\mathbb{F}_q[x]$. A good general reference for this material is [Ros02].

2.1 Quadratic characters. Let $P \in \mathbb{F}_q[x]$ be a prime polynomial. The quadratic residue symbol $\left(\frac{f}{P}\right) \in \{\pm 1\}$ is defined for f coprime to P by

$$\left(\frac{f}{P}\right) \equiv f^{\frac{|P|-1}{2}} \pmod{P}.$$

For arbitrary monic $Q \in \mathbb{F}_q[x]$ and for f coprime to Q , the Jacobi symbol $\left(\frac{f}{Q}\right)$ is defined by writing $Q = \prod P_j$ as a product of prime polynomials and setting

$$\left(\frac{f}{Q}\right) = \prod \left(\frac{f}{P_j}\right).$$

If f, Q are not coprime we set $\left(\frac{f}{Q}\right) = 0$.

The law of quadratic reciprocity asserts that for $A, B \in \mathbb{F}_q[x]$ monic polynomials

$$\left(\frac{B}{A}\right) = (-1)^{\left(\frac{q-1}{2}\right) \deg A \deg B} \left(\frac{A}{B}\right).$$

For $D \in \mathbb{F}_q[x]$ a monic polynomial of positive degree which is not a perfect square, we define the quadratic character χ_D by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

2.2 L-functions. For the quadratic character χ_D , the corresponding L-function is defined for $|u| < \frac{1}{q}$ by

$$\mathcal{L}(u, \chi_D) := \prod_{P \text{ prime}} \left(1 - \chi_D(P)u^{\deg P}\right)^{-1} = \sum_{\beta \geq 0} A_D(\beta)u^\beta,$$

with

$$A_D(\beta) := \sum_{\substack{\deg B = \beta \\ B \text{ monic}}} \chi_D(B). \tag{2.1}$$

If D is nonsquare of positive degree, then $A_D(\beta) = 0$ for $\beta \geq \deg D$ and hence the L-function is in fact a polynomial of degree at most $\deg D - 1$.

Now, assume that D is also square-free. Then $\mathcal{L}(u, \chi_D)$ has a trivial zero at $u = 1$ if and only if $\deg D$ is even. Thus

$$\mathcal{L}(u, \chi_D) = (1 - u)^\lambda \mathcal{L}^*(u, \chi_D), \quad \lambda = \begin{cases} 1 & \deg D \text{ even,} \\ 0 & \deg D \text{ odd,} \end{cases}$$

where $\mathcal{L}^*(u, \chi_D)$ is a polynomial of even degree

$$2\delta = \deg D - 1 - \lambda$$

satisfying the functional equation

$$\mathcal{L}^*(u, \chi_D) = (qu^2)^\delta \mathcal{L}^*\left(\frac{1}{qu}, \chi_D\right). \tag{2.2}$$

We write

$$\mathcal{L}^*(u, \chi_D) = \sum_{\beta=0}^{2\delta} A_D^*(\beta)u^\beta,$$

where $A_D^*(0) = 1$, and the coefficients $A_D^*(\beta)$ satisfy

$$A_D^*(\beta) = q^{\beta-\delta} A_D^*(2\delta - \beta). \tag{2.3}$$

In particular, the leading coefficient is $A_D^*(2\delta) = q^\delta$.

2.3 The explicit formula. For h monic, square-free, and of positive degree, the zeta function of the hyperelliptic curve $y^2 = h(x)$ is

$$Z_h(u) = \frac{\mathcal{L}^*(u, \chi_h)}{(1 - u)(1 - qu)}. \tag{2.4}$$

By the Riemann Hypothesis (proved by Weil) we may write

$$\mathcal{L}^*(u, \chi_h) = \det(I - u\sqrt{q}\Theta_h) \tag{2.5}$$

for a unitary $2g \times 2g$ matrix Θ_h . Taking a logarithmic derivative of (2.5) gives

$$-\text{tr } \Theta_h^n = \frac{\lambda}{q^{n/2}} + \frac{1}{q^{n/2}} \sum_{\deg f=n} \Lambda(f)\chi_h(f). \tag{2.6}$$

2.4 The Weil bound. Assume that B is monic of positive degree and not a perfect square. Then the Riemann Hypothesis and (2.6) gives Weil’s bound for the character sum over primes:

$$\left| \sum_{\substack{\deg P=n \\ P \text{ prime}}} \left(\frac{B}{P} \right) \right| \ll \frac{\deg B}{n} q^{n/2}. \tag{2.7}$$

2.5 Averaging over \mathcal{H}_{2g+1} . The number of square-free monic polynomials of degree d in $\mathbb{F}_q[x]$ is $q^d \left(1 - \frac{1}{q}\right)$ for $d \geq 2$, and in particular we have, for $g \geq 1$,

$$\#\mathcal{H}_{2g+1} = (q - 1)q^{2g}.$$

We can execute the averaging over \mathcal{H}_{2g+1} using the Möbius function μ of $\mathbb{F}_q[x]$, by recalling that

$$\sum_{A^2|h} \mu(A) = \begin{cases} 1 & h \text{ is square-free,} \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\langle F(h) \rangle = \frac{1}{(q - 1)q^{2g}} \sum_{2\alpha+\beta=2g+1} \sum_{\deg B=\beta} \sum_{\deg A=\alpha} \mu(A)F(A^2B), \tag{2.8}$$

the sum being over all monic A, B .

For a given polynomial $f \in \mathbb{F}_q[x]$ apply (2.8) to the quadratic character $h \mapsto \chi_h(f)$ to get

$$\langle \chi_h(f) \rangle = \frac{1}{(q - 1)q^{2g}} \sum_{2\alpha+\beta=2g+1} \sum_{\substack{\deg A=\alpha \\ \gcd(A,f)=1}} \mu(A) \sum_{\deg B=\beta} \left(\frac{B}{f} \right). \tag{2.9}$$

3 A Sum of Möbius Values

Define

$$\sigma(f, \alpha) := \sum_{\substack{\deg A=\alpha \\ \gcd(A,f)=1}} \mu(A). \tag{3.1}$$

Note that $\sigma(f, \alpha)$ depends only on the degrees of the primes dividing f , hence we can write for P_1, \dots, P_n distinct primes of degrees r_1, \dots, r_n respectively: $\sigma(\prod_{i=1}^n P_i, \alpha) = \sigma(\vec{r}; \alpha)$.

LEMMA 3.1. Assume $\min(r_1, \dots, r_n) \geq 2$, then

$$\sigma(\vec{r}; \alpha) = \begin{cases} 1 & \alpha = 0, \\ -q & \alpha = 1, \\ 0 & 2 \leq \alpha < \min(r_1, \dots, r_n). \end{cases} \quad (3.2)$$

In any case we have a bound

$$|\sigma(\vec{r}, \alpha)| \leq (q+1) \frac{\alpha^n}{\prod_{j=1}^n r_j}. \quad (3.3)$$

Proof. The lemma follows from the identity

$$\sum_{\alpha=0}^{\infty} \sigma(f, \alpha) X^\alpha = \sum_{\gcd(A, f)=1} \mu(A) X^{\deg A} = \frac{1 - qX}{\prod_{P|f} (1 - X^{\deg P})}, \quad (3.4)$$

the product being over all prime divisors of f . \square

For distinct primes P_1, \dots, P_n of degrees $\deg P_j = r_j$, we define

$$\phi_\delta(\vec{r}) := \sum_{\substack{D|\prod P_j \\ \deg D \leq \delta}} \frac{\mu(D)}{q^{\deg D}}. \quad (3.5)$$

As the notation signifies, $\phi_\delta(\vec{r})$ depends only on the degrees of the primes P_j , and we can rewrite it as

$$\phi_\delta(\vec{r}) = \sum_{\substack{I \subset \mathbf{n} \\ \sigma(I) \leq \delta}} (-1)^{|I|} q^{-\sigma(I)} \quad (3.6)$$

where for a subset $I \subset \mathbf{n} = \{1, \dots, n\}$ we define

$$\sigma(I) := \sum_{i \in I} r_i$$

and denote by $|I|$ the cardinality of the index set I .

Assume now that β is odd and $\sum r_j$ is even, and $\sum r_j > \beta$. Define

$$\Phi_\beta(\vec{r}) := -q^L \phi_L(\vec{r}) + (q-1) \sum_{l=0}^{L-1} q^l \phi_l(\vec{r}), \quad (3.7)$$

where $2L = \sum r_j - 1 - \beta$.

LEMMA 3.2. Assume β is odd, $\sum r_j$ is even, and $\beta \leq \sum r_j - 2$. Then

$$\Phi_\beta(\vec{r}) = - \sum_{\substack{I \subset \mathbf{n} \\ \sigma(I) \leq L}} (-1)^{|I|}. \quad (3.8)$$

Proof. From the definition,

$$\Phi_\beta(\vec{r}) = -q^L \sum_{\sigma(I) \leq L} (-1)^{|I|} q^{-\sigma(I)} + (q-1) \sum_{l=0}^{L-1} q^l \sum_{\sigma(I) \leq l} (-1)^{|I|} q^{-\sigma(I)}. \tag{3.9}$$

Changing order of summation, we get

$$\Phi_\beta(\vec{r}) = \sum_{\substack{I \subseteq \mathbf{n} \\ \sigma(I) \leq L}} (-1)^{|I|} q^{-\sigma(I)} \left\{ -q^L + (q-1) \sum_{\sigma(I) \leq l \leq L-1} q^l \right\}. \tag{3.10}$$

Summing the geometric series gives

$$-q^L + (q-1) \sum_{\sigma(I) \leq l \leq L-1} q^l = -q^{\sigma(I)} \tag{3.11}$$

and inserting in (3.10) proves the claim. □

4 Multiple Character Sums

Define

$$S(\beta; \vec{r}) := \sum_{\substack{\deg B = \beta \\ B \text{ monic}}} \sum_{\substack{\deg P_j = r_j \\ P_i \neq P_j}} \left(\frac{B}{\prod_{j=1}^n P_j} \right). \tag{4.1}$$

These sums will play a crucial role in what follows.

By quadratic reciprocity

$$S(\beta; \vec{r}) = (-1)^{\frac{q-1}{2}\beta(\sum r_j)} \sum_{\substack{\deg P_j = r_j \\ P_i \neq P_j}} A_{\prod_{j=1}^n P_j}(\beta),$$

where the sum is over distinct primes P_j and $A_F(\beta)$ given by (2.1) is the coefficient of the L-polynomial $\mathcal{L}(u, \chi_F)$. Since the L-function is a polynomial of degree $\deg F - 1$, we have

LEMMA 4.1. *If $\beta \geq \sum_{j=1}^n r_j$ then $S(\beta; \vec{r}) = 0$.*

4.1 Duality.

4.1.1 *Duality for $\sum r_j$ odd.* Assume $\sum r_j$ is odd and $\beta \leq \sum r_j - 1$. Let P_1, \dots, P_n be distinct primes. Then $\mathcal{L}(u, \chi_{\prod_{j=1}^n P_j}) = \mathcal{L}^*(u, \chi_{\prod_{j=1}^n P_j})$, and so the coefficients

$$A_{\prod_{j=1}^n P_j}(\beta) = A_{\prod_{j=1}^n P_j}^*(\beta)$$

coincide. Therefore, from (2.3) we have

$$A_{\prod_{j=1}^n P_j}(\beta) = A_{\prod_{j=1}^n P_j} \left(\sum r_j - 1 - \beta \right) q^{\beta - \frac{\sum r_j - 1}{2}}.$$

Hence if $\sum r_j$ is odd and $\beta \leq \sum r_j - 1$ then

$$S(\beta; \vec{r}) = q^{\beta - \frac{\sum r_j - 1}{2}} S \left(\sum r_j - 1 - \beta; \vec{r} \right). \quad (4.2)$$

4.1.2 Duality for $\sum r_j$ even. Assume $\sum r_j$ is even and $\beta \leq \sum r_j - 2$. Let P_1, \dots, P_n be distinct primes. Then the equation

$$\mathcal{L}(u, \chi_{\prod_{j=1}^n P_j}) = (1 - u) \mathcal{L}^*(u, \chi_{\prod_{j=1}^n P_j})$$

implies (here we write $A(\beta)$ for $A_{\prod P_j}(\beta)$)

$$\begin{aligned} A(0) &= A^*(0) = 1, \\ A(\sum r_j - 1) &= -A^*(\sum r_j - 2), \\ A^*(\beta) &= A(\beta) + A(\beta - 1) + \dots + A(0), \end{aligned}$$

and

$$A(\beta) = A^*(\beta) - A^*(\beta - 1). \quad (4.3)$$

From (2.3) we have

$$A^*(\beta) = q^{\beta - \frac{\sum r_j - 2}{2}} A^* \left(\sum r_j - 2 - \beta \right). \quad (4.4)$$

Hence

$$A^* \left(\sum r_j - 2 \right) = q^{\frac{\sum r_j - 2}{2}},$$

and so

$$A \left(\sum r_j - 1 \right) = -q^{\frac{\sum r_j - 2}{2}}.$$

Therefore, if $\sum r_j$ is even then

$$\begin{aligned} S \left(\sum r_j - 1; \vec{r} \right) &= \sum_{\substack{\deg P_j = r_j \\ P_i \neq P_l}} -q^{\frac{\sum r_j - 2}{2}} \\ &= -q^{\frac{\sum r_j - 2}{2}} \pi(r_1) \cdots \pi(r_n) + O \left(q^{\frac{3 \sum r_j - \min r_j}{2}} \right). \end{aligned} \quad (4.5)$$

If $\beta \leq \sum r_j - 2$ then by (4.3) and (4.4) we have

$$A(\beta) = q^{\beta - \frac{\sum r_j}{2}} \left(-A \left(\sum r_j - 1 - \beta \right) + (q - 1) \sum_{l=0}^{\sum r_j - 2 - \beta} A(l) \right). \tag{4.6}$$

Hence

$$S(\beta; \vec{r}) = q^{\beta - \frac{\sum r_j}{2}} \left(-S \left(\sum r_j - 1 - \beta; \vec{r} \right) + (q - 1) \sum_{l=0}^{\sum r_j - 2 - \beta} S(l; \vec{r}) \right). \tag{4.7}$$

4.2 Estimates for $S(\beta; \vec{r})$. For the convenience of writing we assume from now on that

$$r_1 = \min(r_1, \dots, r_n).$$

LEMMA 4.2.

$$S(\beta; \vec{r}) = \eta_\beta q^{\beta/2} \phi_{\beta/2}(\vec{r}) \prod \pi(r_j) + O \left(\phi_{\beta/2}(\vec{r}) \beta^n q^{\max(\sum r_j + \frac{\beta}{2} - r_1, \sum \frac{r_j}{2} + \beta)} \right),$$

where $\phi_\delta(\vec{r})$ is as defined in (3.5).

Proof. We write

$$S(\beta; \vec{r}) = \eta_\beta \sum_{\substack{\deg B = \beta \\ B = \square}} \sum_{\substack{\deg P_j = r_j \\ P_i \neq P_l}} \left(\frac{B}{\prod_{j=1}^n P_j} \right) + \sum_{\substack{\deg B = \beta \\ B \neq \square}} \sum_{\substack{\deg P_j = r_j \\ P_i \neq P_l}} \left(\frac{B}{\prod_{j=1}^n P_j} \right),$$

where the squares only occur when β is even. We write the sum over squares $B = C^2$ as

$$\sum_{\substack{\deg P_j = r_j \\ P_i \neq P_l}} \sum_{\deg C = \frac{\beta}{2}} \left(\frac{C^2}{\prod_{j=1}^n P_j} \right).$$

The inner sum is the number of C 's coprime to $\prod_{j=1}^n P_j$, which is $q^{\frac{\beta}{2}} \phi_{\beta/2}(\vec{r})$ (this is seen by the definition (3.5) of $\phi_{\beta/2}$ and inclusion–exclusion). Summing over the distinct P_j we get that the sum over square B 's is

$$q^{\frac{\beta}{2}} \pi(r_1) \cdots \pi(r_n) \phi_{\frac{\beta}{2}}(\vec{r}) + O \left(\phi_{\beta/2}(\vec{r}) q^{\sum r_j + \frac{\beta}{2} - r_1} \right).$$

For B not a perfect square, we use Weil's theorem (2.7). Hence summing over all non-square B of degree β , of which there are at most q^β , gives

$$\sum_{\substack{\deg B = \beta \\ B \neq \square}} \sum_{\substack{\deg P_j = r_j \\ P_i \neq P_l}} \left(\frac{B}{\prod_{j=1}^n P_j} \right) \ll \beta^n q^{\sum \frac{r_j}{2} + \beta}$$

and with the contribution of square B , this concludes the lemma. □

By using duality, we can improve the estimate of the lemma when β is odd and $\sum r_j < 2\beta$.

PROPOSITION 4.3. *Assume β is odd, and $\beta \leq \sum r_j - 2$. Then*

$$S(\beta; \vec{r}) = \eta_{\sum r_j} q^\beta \Phi_\beta(\vec{r}) \prod \frac{\pi(r_j)}{q^{r_j/2}} + O\left(\left(\sum r_j\right)^n q^{\sum r_j}\right)$$

where $\Phi_\beta(\vec{r})$ is given in (3.7).

Proof. Assume $\sum r_j$ is odd. Since $\beta \leq \sum r_j - 2$ we may use (4.2) for $\sum r_j$ odd,

$$S(\beta; \vec{r}) = q^{\beta - \frac{\sum r_j - 1}{2}} S\left(\sum r_j - 1 - \beta; \vec{r}\right)$$

and inserting the inequality of Weil's theorem with β replaced by $\sum r_j - 1 - \beta$ we get

$$S\left(\sum r_j - 1 - \beta; \vec{r}\right) \ll \left(\sum r_j\right)^n q^{\frac{\sum r_j}{2} + (\sum r_j - 1 - \beta)},$$

hence

$$S(\beta; \vec{r}) \ll q^{\beta - \frac{\sum r_j - 1}{2}} \left(\sum r_j\right)^n q^{\frac{\sum r_j}{2} + (\sum r_j - 1 - \beta)} \ll \left(\sum r_j\right)^n q^{\sum r_j}$$

as claimed.

Now assume $\sum r_j$ is even. Using (4.7) and Lemma 4.2 we get

$$\begin{aligned} S(\beta; \vec{r}) &= q^{\beta - \frac{\sum r_j}{2}} \left(-S\left(\sum r_j - 1 - \beta; \vec{r}\right) + (q-1) \sum_{l=0}^{\sum r_j - \beta - 2} S(l; \vec{r}) \right) \\ &= q^{\beta - \frac{\sum r_j}{2}} \pi(r_1) \cdots \pi(r_n) \left(-\eta_{\sum r_j - 1 - \beta} q^{\frac{\sum r_j - 1 - \beta}{2}} \phi_{\frac{\sum r_j - 1 - \beta}{2}}(\vec{r}) \right. \\ &\quad \left. + (q-1) \sum_{l=0}^{\sum r_j - \beta - 2} \eta_l q^{\frac{l}{2}} \phi_{\frac{l}{2}}(\vec{r}) \right) \\ &\quad + O\left(\phi_{\beta/2}(\vec{r}) q^{\beta - \frac{\sum r_j}{2} + 1} \sum_{l=0}^{\sum r_j - \beta - 2} l^n q^{\max(\frac{\sum r_j}{2} + l, \sum r_j + \frac{l}{2} - r_1)}\right). \end{aligned}$$

The remainder term is $O((\sum r_j)^n q^{\sum r_j})$. For the main term, we note that $\sum r_j - 1 - \beta$ is even since β is odd and $\sum r_j$ is even. Denote $2L := \sum r_j - 1 - \beta$, then we can write the main term as

$$q^\beta \left(-q^L \phi_L(\vec{r}) + (q-1) \sum_{l=0}^{L-1} q^l \phi_l(\vec{r}) \right) \prod \frac{\pi(r_j)}{q^{r_j/2}} = q^\beta \Phi_\beta(\vec{r}) \prod \frac{\pi(r_j)}{q^{r_j/2}}$$

by definition (3.7) of $\Phi_\beta(\vec{r})$. □

5 The n -Level Density

In the present section we begin the calculation of the n -level density for the hyper-elliptic ensemble. First we recall the definition of n -level density. Let n be a natural number and suppose we are given n real-valued even test function $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R})$ (by $\mathcal{S}(\mathbb{R})$ we denote the Schwartz space). Let

$$\widehat{f}_k(s) = \int_{\mathbb{R}} f_k(t)e^{-2\pi ist} dt$$

be the Fourier transforms of f_k . We will assume that each \widehat{f}_j is supported on the interval $(-s_j, s_j)$ and $\sum s_j < 2$. Let $h \in \mathcal{H}(2g + 1, q)$ be a polynomial defining a curve $y^2 = h(x)$ with normalized L-zeros $e^{i\theta_j}, j = \pm 1, \dots, \pm g, \theta_{-j} = -\theta_j$. Let

$$\widetilde{f}_k(t) = \sum_{m \in \mathbb{Z}} f_k \left(2g \left(\frac{t}{2\pi} + m \right) \right)$$

be the associated periodic test functions (with period 2π). We denote

$$W_f^{(n)}(h) = \sum_{\substack{\theta_{j_1}, \dots, \theta_{j_n} \\ 1 \leq |j_k| \leq g \\ j_k \neq \pm j_l \text{ if } k \neq l}} \widetilde{f}_1(\theta_{j_1}) \dots \widetilde{f}_n(\theta_{j_n}).$$

For the rest of the section whenever we use the averaging notation we mean averaging over $h \in \mathcal{H}(2g+1, q)$ and whenever we use the asymptotic big- O notation the implicit constant may depend on n, f_1, \dots, f_n (and other test functions we introduce) but not on g, q . The aim of this section is to prove that

$$\langle W_f^{(n)}(h) \rangle = A(f_1, \dots, f_n) + O(\log g/g), \tag{5.1}$$

where $A(f_1, \dots, f_n)$ is an explicit expression in the f_i and their Fourier transforms independent of g, q .

5.1 Passage to unrestricted sums. To express $W_f^{(n)}$ in terms of unrestricted sums over zeros we use a standard combinatorial sieving method (see [Gao05, Rub01, RS96] for usage of this method in a similar context). First of all since the f_i are even we may write

$$W_f^{(n)} = 2^n \sum_{\substack{1 \leq j_1, \dots, j_n \leq g \\ \text{dist.}}} \widetilde{f}_1(\theta_{j_1}) \dots \widetilde{f}_n(\theta_{j_n})$$

(here the summation is over distinct j_1, \dots, j_n).

Denote by Π_n the set of partitions of the set $1, \dots, n$. For two partitions $\underline{F}, \underline{G} \in \Pi_n$ we say that \underline{F} refines \underline{G} and write $\underline{F} \prec \underline{G}$ if each set appearing in \underline{G} is a union of sets appearing in \underline{F} . We denote $\underline{O} = \{\{1\}, \dots, \{n\}\} \in \Pi_n$. For any finite set F we denote by $|F|$ its cardinality. For a partition $\underline{F} = \{F_1, \dots, F_\nu\}$ we denote $|\underline{F}| = \nu$.

Now suppose we have a function $R : \Pi_n \rightarrow \mathbb{R}$ and denote $C(\underline{F}) = \sum_{\underline{F} \prec \underline{G}} R(\underline{G})$. The combinatorial Möbius inversion formula states that $R(\underline{F}) = \sum_{\underline{F} \prec \underline{G}} \mu(\underline{F}, \underline{G}) C(\underline{G})$, where $\mu(\underline{F}, \underline{G})$ is the Möbius function for the partially ordered set Π_n . It is known that if $\underline{F} = \{F_1, \dots, F_\nu\}$ then

$$\mu(\underline{Q}, \underline{F}) = \prod_{l=1}^{\nu} (-1)^{|F_l|-1} (|F_l| - 1)!$$

(see [LW92, Section 12]), so we have

$$R(\underline{Q}) = \sum_{\underline{F} \in \Pi_n} \prod_{l=1}^{|\underline{F}|} (-1)^{|F_l|-1} (|F_l| - 1)! C(\underline{F})$$

(here $\underline{F} = F_1, \dots, F_\nu, \nu = |\underline{F}|$).

Now for $\underline{F} = \{F_1, \dots, F_\nu\} \in \Pi_n$ take

$$C(\underline{F}) = \sum_{1 \leq j_1, \dots, j_\nu \leq g} \prod_{l=1}^{\nu} \prod_{k \in F_l} \tilde{f}_k(\theta_{j_l}),$$

$$R(\underline{F}) = \sum_{\substack{1 \leq j_1, \dots, j_\nu \leq g \\ \text{dist.}}} \prod_{l=1}^{\nu} \prod_{k \in F_l} \tilde{f}_k(\theta_{j_l}).$$

It is easy to see that $C(\underline{F}) = \sum_{\underline{F} \prec \underline{G}} R(\underline{G})$ and so denoting

$$\tilde{U}_{\underline{F}}(\theta) = \prod_{k \in \underline{F}} \tilde{f}_k(\theta),$$

we have

$$\begin{aligned} W_f^{(n)} &= 2^n R(\underline{Q}) = 2^n \sum_{\underline{F} \in \Pi_n} \prod_{l=1}^{|\underline{F}|} (-1)^{|F_l|-1} (|F_l| - 1)! \sum_{1 \leq j_1, \dots, j_{|\underline{F}|} \leq g} \tilde{U}_{\underline{F}}(\theta_{j_l}) \\ &= 2^n \sum_{\underline{F}} \prod_{l=1}^{|\underline{F}|} (-1)^{|F_l|-1} (|F_l| - 1)! \sum_{1 \leq j \leq g} \tilde{U}_{\underline{F}}(\theta_j). \end{aligned}$$

Since the f_j and hence also the $\tilde{U}_{\underline{F}}$ are even, we may also rewrite this with a sum over all zeros:

$$W_f^{(n)} = \sum_{\underline{F}} (-2)^{n-|\underline{F}|} \prod_{l=1}^{|\underline{F}|} (|F_l| - 1)! \prod_{l=1}^{|\underline{F}|} \sum_{j=\pm 1, \dots, \pm g} \tilde{U}_{\underline{F}}(\theta_j). \quad (5.2)$$

5.2 Passage to a sum over primes. Next we will replace the sum over zeros in (5.2) with a sum over primes. For any $f \in \mathcal{S}(\mathbb{R})$ with compactly supported Fourier transform we denote

$$\mathcal{T}(f; h) := \frac{1}{g} \sum_{r=1}^{\infty} r q^{-r/2} \hat{f}\left(\frac{r}{2g}\right) \sum_{\substack{\deg P=r \\ \text{prime}}} \left(\frac{h}{P}\right). \tag{5.3}$$

PROPOSITION 5.1. *Let $f \in \mathcal{S}(\mathbb{R})$ be a real-valued even function with compactly supported Fourier transform \hat{f} and let $\tilde{f}(t) = \sum_{m \in \mathbb{Z}} f(2g(\frac{t}{2\pi} + m))$ be its associated periodic function. Then for any $h \in \mathcal{H}(2g + 1, q)$ with normalized L-zeros $e^{i\theta_j}, j = \pm 1, \dots, \pm g$ we have*

$$\sum_{1 \leq |j| \leq g} \tilde{f}(\theta_j) = \hat{f}(0) - \frac{1}{2}f(0) - \mathcal{T}(f; h) + O(\log g/g)$$

(the implicit constant may depend on f).

Proof. The Fourier coefficients of \tilde{f} are $\tilde{f}(r) = \frac{1}{2g} \hat{f}\left(\frac{r}{2g}\right)$, so we have

$$\tilde{f}(t) = \sum_{r \in \mathbb{Z}} \frac{1}{2g} \hat{f}\left(\frac{r}{2g}\right) e^{irt} = \frac{1}{2g} \hat{f}(0) + \frac{1}{g} \sum_{r=1}^{\infty} \hat{f}\left(\frac{r}{2g}\right) e^{irt}. \tag{5.4}$$

The explicit formula states that

$$\sum_{1 \leq |j| \leq g} e^{ir\theta_j} = -q^{-r/2} \sum_{\substack{\deg Q=r \\ \text{monic}}} \left(\frac{h}{Q}\right) \Lambda(Q), \tag{5.5}$$

where Λ is the von Mangoldt function. Combining (5.4) and (5.5) we obtain

$$\sum_{1 \leq |j| \leq g} \tilde{f}(\theta_j) = \hat{f}(0) - \frac{1}{g} \sum_{r=1}^{\infty} q^{-r/2} \hat{f}\left(\frac{r}{2g}\right) \sum_{\substack{\deg Q=r \\ \text{monic}}} \left(\frac{h}{Q}\right) \Lambda(Q). \tag{5.6}$$

The contribution to this sum from prime Q is exactly the term appearing in the statement of the proposition. Now we consider the contribution of the squares $Q = P^2$ with P prime, $\deg P = r/2$ (for r even). We use the fact that $(\frac{h}{P^2})$ is 1 unless $P|h$, in which case it is 0. We denote by $\pi(r)$ the number of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree r . Since $\pi(r) = q^r/r + O(q^{r/2}/r)$, we have

$$\begin{aligned} \sum_{\substack{\deg P=r/2 \\ \text{prime}}} \left(\frac{h}{P^2}\right) \Lambda(P^2) &= \pi(r/2) \frac{r}{2} - \frac{r}{2} \cdot \#\{P \text{ prime}, P|h\} \\ &= q^{r/2} + O(q^{r/4} + \min(g, q^{r/2})), \end{aligned}$$

since the number of prime $P|h$ of degree $r/2$ is $O(\min(g/r + q^{r/2}/r))$. We see that the contribution of these squares to the sum in (5.6) is

$$\begin{aligned} & \frac{1}{g} \sum_{r=1}^{\infty} \hat{f}(r/2g) \left(1 + O\left(q^{-r/4} + \min(gq^{-r/2}, 1)\right) \right) \\ &= 2 \int_0^{\infty} \hat{f}(t) dt + O(\log g/g) = f(0) + O(\log g/g), \end{aligned}$$

because $\sum_{r > \log g} gq^{-r/2} = O(1)$. The contribution of higher prime powers $Q = P^k, k > 3$ is $O(1/g)$ because the number of prime P with $\deg P \leq r/3$ is $O(q^{r/3}/r)$. \square

COROLLARY 5.2.

$$\begin{aligned} W_f^{(n)} &= \sum_F (-2)^{n-|E|} \prod_{l=1}^{|E|} (|F_l| - 1)! \\ &\quad \cdot \left(\hat{U}_{F_l}(0) - \frac{1}{2} U_{F_l}(0) - \mathcal{T}(\hat{U}_{F_l}; h) + O\left(\frac{\log g}{g}\right) \right), \end{aligned}$$

where $U_F(t) = \prod_{k \in F} f_k(t)$, \hat{U}_F is its Fourier transform.

Proof. This follows from (5.2) and Proposition 5.1. Note that $\tilde{U}_F(t)$ is the associated periodic function of U_F . \square

Now let $u_1, \dots, u_k \in \mathcal{S}(\mathbb{R})$ with $k \leq n$ be real-valued even functions with Fourier transforms \hat{u}_l . We denote

$$M(u_1, \dots, u_k) = \left\langle \prod_{l=1}^k \mathcal{T}(u_l; h) \right\rangle,$$

($\mathcal{T}(u_l; h)$ is defined by (5.3)). In the next subsection we will prove that if \hat{u}_l is supported in $(-\delta_l, \delta_l)$ and $\sum_{l=1}^k \delta_l < 2$ then

$$M(u_1, \dots, u_k) = B(u_1, \dots, u_k) + O(\log g/g), \quad (5.7)$$

where $B(u_1, \dots, u_k)$ is an explicit expression in the u_l and their Fourier transforms which is independent of g, q .

PROPOSITION 5.3. *Suppose that (5.7) holds under the appropriate conditions on the supports of \hat{u}_l . Then*

$$\langle W_f^{(n)} \rangle = A(f_1, \dots, f_n) + O(\log g/g)$$

holds with

$$\begin{aligned}
 A(f_1, \dots, f_n) &= \sum_{\underline{F}} (-2)^{n-|\underline{F}|} \prod_{l=1}^{|\underline{F}|} (|F_l| - 1)! \sum_{S \subset \{1, \dots, l\}} \left(\prod_{l \in S^c} \hat{U}_{F_l}(0) \right) \\
 &\quad \cdot \sum_{S_2 \subset S} (-1/2)^{|S_2^c|} \left(\prod_{l \in S_2^c} U_{F_l}(0) \right) (-1)^{|S_2|} B(U_{l_1}, \dots, U_{l_{|S_2|}}),
 \end{aligned}$$

where the first summation is over all partitions $\underline{F} = \{F_1, \dots, F_{|\underline{F}|}\} \in \Pi_n$, the second is over all subsets $S \in \{1, \dots, l\}$, S^c denotes the complement of S in $\{1, \dots, l\}$, the third summation is over all subsets $S_2 = \{l_1, \dots, l_{|S_2|}\} \subset S$, and $S_2^c = S \setminus S_2$.

Proof. First we note that if we could ignore the $O(\log g/g)$ terms in Corollary 5.2 then the Proposition would follow at once by expanding the product, averaging and using (5.7). Here we use the fact that \hat{U}_{l_j} is supported on the interval $(-\delta_j, \delta_j)$ where $\delta_j = \sum_{k \in F_{l_j}} s_k$ (recall that \hat{f}_k is supported on $(-s_k, s_k)$), because the Fourier transform takes products to convolutions, so we have $\sum_{j=1}^{|S_2|} \delta_j \leq \sum_{k=1}^n s_k < 2$, which makes (5.7) applicable.

To deal with the error terms $O(\log g/g)$ we prove by induction on m that for any even real-valued $u_1, \dots, u_m \in \mathcal{S}(\mathbb{R})$, with each \hat{u}_l supported on $(-\delta_l, \delta_l)$ and $\sum \delta_l < 2$, we have

$$\begin{aligned}
 &\left\langle \prod_{l=1}^m \left(u_l(0) - \frac{1}{2} \hat{u}_l(0) - \mathcal{T}(u; h) + O(\log g/g) \right) \right\rangle \\
 &= \left\langle \prod_{l=1}^m \left(u_l(0) - \frac{1}{2} \hat{u}_l(0) - \mathcal{T}(u; h) \right) \right\rangle + O(\log g/g)
 \end{aligned}$$

(see [Rub01, Lemma 2] for a similar argument). Assuming by induction that this holds for $m - 1$ it is enough to show that

$$\left\langle O(\log g/g) \cdot \prod_{l=1}^{m-1} \left(u_l(0) - \frac{1}{2} \hat{u}_l(0) - \mathcal{T}(u_l; h) \right) \right\rangle = O(\log g/g). \tag{5.8}$$

But

$$u_l(0) - \frac{1}{2} \hat{u}_l(0) - \mathcal{T}(u_l; h) = \sum_{1 \leq |j| \leq g} \tilde{u}_l(\theta_j) + O(\log g/g),$$

(by Proposition 5.1, here $e^{i\theta_j}$ are the normalized L-zeros corresponding to h and \tilde{u}_l is the periodic function associated with u_l), so by induction it is enough to show that

$$\left\langle O(\log g/g) \cdot \prod_{l=1}^{m-1} \sum_{1 \leq |j| \leq g} \tilde{u}_l(\theta_j) \right\rangle = O(\log g/g).$$

For this we may replace each u_l with an even real-valued function $v_l \in \mathcal{S}(\mathbb{R})$ s.t. $v_l(t) > |u_l(t)|$ for all $t \in \mathbb{R}$ and each \hat{v}_l supported on $(-\delta_l, \delta_l)$. That such functions always exist is shown in [Rub01], proof of Lemma 2. Now applying 5.7 and using the induction hypothesis we see that

$$\left\langle \prod_{l=1}^{m-1} \sum_{1 \leq |j| \leq g} \tilde{v}_l(\theta_j) \right\rangle = O(1),$$

which implies (5.8). □

5.3 Evaluation of $M(u_1, \dots, u_m)$: reduction to sums over distinct primes.

In the rest of this section we evaluate

$$M(u_1, \dots, u_m) = \left\langle \prod_{l=1}^m \mathcal{T}(u_l; h) \right\rangle$$

up to $O(\log g/g)$ for even real-valued $u_k \in \mathcal{S}(\mathbb{R})$ s.t. \hat{u}_k is supported on $(-\delta_k, \delta_k)$ with $\sum_{k=1}^m \delta_k < 2$. We want to derive a result of the form (5.7), so we assume by induction that it already holds for all $m' < m$. We denote $\mathbf{m} = \{1, \dots, m\}$.

Let F be a subset of \mathbf{m} . Denote

$$C(F) = C(F; h) = \frac{1}{g^{|F|}} \sum_{r=1}^{\infty} q^{-|F|r/2} r^{|F|} \prod_{\substack{k \in F \\ \text{prime}}} \hat{u}_k \left(\frac{r}{2g} \right) \sum_{\deg P=r} \left(\frac{h}{P^{|F|}} \right).$$

For a partition $\underline{F} = \{F_1, \dots, F_\nu\}$ of a set $S \subset \mathbf{m}$ we denote $C(\underline{F}) = \prod_{l=1}^{\nu} C(F_l)$. For two elements $i, j \in S$ we say that $i \sim_{\underline{F}} j$ if they lie in the same element of \underline{F} . We have

$$C(\underline{F}) = \frac{1}{g^{|S|}} \sum_{r_1, \dots, r_{|S|}=1}^{\infty} \left(\prod_{k \in S} \hat{u}_k \left(\frac{r_k}{2g} \right) \frac{r_k}{q^{r_k/2}} \right) \sum_{\substack{P_1, \dots, P_{|S|} \\ \text{prime} \\ P_i = P_j \text{ if } i \sim_{\underline{F}} j}} \left(\frac{h}{P_1 \dots P_{|S|}} \right). \tag{5.9}$$

Define also

$$R(\underline{F}) = \frac{1}{g^{|S|}} \sum_{r_1, \dots, r_{|S|}=1}^{\infty} \left(\prod_{k \in S} \hat{u}_k \left(\frac{r_k}{2g} \right) \frac{r_k}{q^{r_k/2}} \right) \sum_{\substack{P_1, \dots, P_{|S|} \\ \text{prime} \\ P_i = P_j \text{ iff } i \sim_{\underline{F}} j}} \left(\frac{h}{P_1 \dots P_{|S|}} \right) \tag{5.10}$$

(same expression except that the “if” is replaced with an “iff”). We have

$$C(\underline{F}) = \sum_{\underline{F} < \underline{G}} R(\underline{G}), \quad R(\underline{F}) = \sum_{\underline{F} < \underline{G}} \mu(\underline{F}, \underline{G}) C(\underline{G}). \tag{5.11}$$

PROPOSITION 5.4. *Let F be a subset of \mathbf{m} . If $F = \{a, b\}$ consists of two (distinct) elements then*

$$C(F) = 2 \int_{\mathbb{R}} \hat{u}_a(t)\hat{u}_b(t)|t| dt + O(\log g/g).$$

If $|F| > 2$ then $C(F) = O(1/g)$.

Proof. First suppose $F = \{a, b\}$. Then

$$C(F) = \frac{1}{g^2} \sum_{r=1}^{\infty} \hat{u}_a\left(\frac{r}{2g}\right) \hat{u}_b\left(\frac{r}{2g}\right) r^2 q^{-r} \sum_{\substack{\deg P=r \\ \text{prime}}} \left(\frac{h}{P^2}\right).$$

As in the proof of Proposition 5.1 we see that

$$r q^{-r} \sum_{\substack{\deg P=r \\ \text{prime}}} \left(\frac{h}{P^2}\right) = 1 + O(q^{-r/2} + \min(1, gq^{-r/2}))$$

and so

$$\begin{aligned} C(F) &= 4 \sum_{r=1}^{\infty} \hat{u}_a\left(\frac{r}{2g}\right) \hat{u}_b\left(\frac{r}{2g}\right) \frac{r}{2g} \cdot \frac{1}{2g} + O(\log g/g) \\ &= 2 \int_{\mathbb{R}} \hat{u}_a(t)\hat{u}_b(t)|t| dt + O(\log g/g). \end{aligned}$$

Now suppose that $|F| = e \geq 3$. Then

$$C(F) \ll \sum_{r=1}^{\infty} \frac{1}{g^e} q^{(1-e/2)r} r^{e-1} = O(g^{-e}). \quad \square$$

For any subset $S \subset \mathbf{m}$ denote

$$\underline{Q}_S = \{\{k\} | k \in S\}, \underline{Q} = \{\{1\}, \dots, \{m\}\}.$$

LEMMA 5.5. *For any proper subset $S \subset \mathbf{m}$ there is a function $X : \mathcal{H}(2g+1, q) \rightarrow \mathbb{R}_{\geq 0}$ s.t. $X(h) \geq |C(\underline{Q}_S; h)|$ for all h and $\langle X(h) \rangle = O(1)$.*

Proof. Since $C(\underline{Q}_S) = \prod_{i \in S} \mathcal{I}(u_i; h)$ and by Proposition 5.1 we can write

$$C(\underline{Q}_S) = \prod_{i \in S} \sum_{\theta} u_i(\theta) + \sum_{T \subsetneq S} C(\underline{Q}_T) \cdot O(1),$$

where the sum is over the normalized L-zeros corresponding to h . We may assume by induction that $C(\underline{Q}_T), T \subsetneq S$ (and therefore also $C(\underline{Q}_T) \cdot O(1)$) satisfy the assertion, so it is enough to prove it for $\prod_{i \in S} \sum_{\theta} u_i(\theta)$. For this we may replace the u_i with

$v_i \geq |u_i|$ s.t. \hat{v}_i is supported on $(-s_i, s_i)$, as we did in the proof of Proposition 5.3, so that

$$\prod_{i \in S} \sum_{\theta} v_i(\theta) \geq \left| \prod_{i \in S} \sum_{\theta} u_i(\theta) \right|$$

for all h . Now since S is proper we can apply our induction hypothesis. □

If $S = \{k_1, \dots, k_\nu\}$ we have $M(u_{k_1}, \dots, u_{k_\nu}) = \langle C(\underline{Q}_S) \rangle$. If S is a proper subset of \mathbf{m} we may assume by induction that

$$M(u_{k_1}, \dots, u_{k_\nu}) = B(u_{k_1}, \dots, u_{k_\nu}) + O(\log g/g),$$

where $B(u_{k_1}, \dots, u_{k_\nu})$ depends only on $u_{k_1}, \dots, u_{k_\nu}$.

PROPOSITION 5.6. *Let $\underline{F} \in \Pi_m$ be a partition. let $\{a_i, b_i\}, i = 1, \dots, \mu$ be the two-element sets appearing in \underline{F} and $\{c_i\}, i = 1, \dots, \kappa$, the one-element sets appearing in \underline{F} . Assume that at least one element of $F \in \underline{F}$ satisfies $|F| > 1$. If some $F \in \underline{F}$ has more than two elements then $\langle C(\underline{F}) \rangle = O(1/g)$. Otherwise*

$$\langle C(\underline{F}) \rangle = B(u_{k_1}, \dots, u_{k_\kappa}) 2^\mu \prod_{i=1}^\mu \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt + O(\log g/g).$$

Proof. Denote $S = \{c_i, 1 \leq i \leq \kappa\}$. We have $\langle C(\underline{Q}_S) \rangle = B(u_{c_1}, \dots, u_{c_\mu}) + O(\log g/g)$. Denote $\underline{G} = \underline{F} \setminus \underline{Q}_S$ (these are exactly the sets with more than one element in \underline{F}). If at least one set in \underline{F} has more than two elements then by Proposition 5.4 $C(\underline{G}) = O(1/g)$. Otherwise

$$C(\underline{G}) = 2^\mu \prod_{i=1}^\mu \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt + O(\log g/g).$$

In both cases we want to show that if we multiply the corresponding error by $C(\underline{Q}_S)$ and average we get the same order of error. This follows from Lemma 5.5, since we can bound $C(\underline{Q}_S)$ by a suitable $X(h)$. □

In the next subsection we will show that

$$\langle R(\underline{O}) \rangle = D(u_1, \dots, u_m) + O(1/g), \tag{5.12}$$

where $D(u_1, \dots, u_m)$ is an explicit expression depending only on u_1, \dots, u_m . For a subset $S = \{k_1, \dots, k_\nu\} \in \mathbf{m}$ we denote $D(S) = D(u_{k_1}, \dots, u_{k_\nu})$. Assuming (5.12) we prove the following:

PROPOSITION 5.7. $M = B + O(\log g/g)$, where

$$\begin{aligned}
 B(u_1, \dots, u_m) &= 2^{m/2} \sum_{\text{pairup } \mathbf{m}} \prod_{i=1}^{m/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \\
 &+ \sum_{S \subsetneq \mathbf{m}} 2^{|S|/2} \sum_{\text{pairup } S} \prod_{i=1}^{|S|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \cdot D(S^c).
 \end{aligned}$$

Here the first sum is over all perfect pairings of \mathbf{m} , i.e. partitions of \mathbf{m} of the form

$$\{\{a_i, b_i\}, i = 1, \dots, m/2\}, a_i \neq b_i$$

(if m is odd the sum is empty), the second sum is over the proper subsets $S \subset \mathbf{m}$, the third sum is like the first only for S and $S^c = \{1, \dots, m\} \setminus S$.

Proof. We have

$$M(u_1, \dots, u_m) = C(\underline{Q}) = \sum_{\underline{F} \in \Pi_m} R(\underline{F}). \tag{5.13}$$

First we note that if \underline{F} is a partition of $S \subset \mathbf{m}$ that has an element $F \in \underline{F}$ with $|F| > 2$ then $\langle R(\underline{F}) \rangle = O(1/g)$. This follows from (5.11) and Proposition 5.6. Next we observe that if $|F| = 2$ for all the elements $F \in \underline{F}$ then $\langle R(\underline{F}) \rangle = \langle C(\underline{F}) \rangle$, because of (5.11), the fact that $\mu(\underline{F}, \underline{F}) = 1$ and because every proper $\underline{G} \succ \underline{F}$ has $F \in \underline{G}$ with more than two elements. More generally, if $a_1, \dots, a_\mu, b_1, \dots, b_\mu, c_1, \dots, c_\kappa \in \mathbf{m}$ are distinct elements and

$$\underline{F} = \{\{a_1, b_1\}, \dots, \{a_\mu, b_\mu\}, \{c_1\}, \dots, \{c_\kappa\}\}, \quad S = \{\{c_1\}, \dots, \{c_\kappa\}\},$$

then

$$\left(\prod_{i=1}^{\mu} C(\{a_i, b_i\}) \right) R(\underline{Q}_S) = R(\underline{F}) + \sum_{\underline{G} \succ \underline{F}} R(\underline{G}),$$

where the sum is only over those proper $\underline{G} \succ \underline{F}$ which leave the elements of S in different sets. In particular each \underline{G} contains a set with more than two elements. We conclude that

$$\langle R(\underline{F}) \rangle = \left\langle \left(\prod_{i=1}^{\mu} C(\{a_i, b_i\}) \right) R(\underline{Q}_S) \right\rangle + O(1/g).$$

Now the proof of Proposition 5.6 can be imitated to show that

$$\langle R(\underline{F}) \rangle = D(S) \cdot 2^\mu \prod_{i=1}^{\mu} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt + O(\log g/g)$$

(the required bound $|R(\underline{Q}_S)| \leq X(h)$ with $\langle X(h) \rangle = O(1)$ follows from (5.11), Lemma 5.5 and Proposition 5.6). Combining this with (5.13) gives the assertion.

It remains for us to evaluate $\langle R(\underline{Q}) \rangle$ and show that

$$\langle R(\underline{Q}) \rangle = D(u_1, \dots, u_m) + O(\log g/g),$$

where $D(u_1, \dots, u_m)$ is an explicit expression depending on u_1, \dots, u_m (and find this expression). We recall that (compare (5.10))

$$R(\underline{Q}) = \frac{1}{g^m} \sum_{r_1, \dots, r_m=1}^{\infty} \left(\prod_{i=1}^m \hat{u}_i \left(\frac{r_i}{2g} \right) \frac{r_i}{q^{r_i/2}} \right) \sum_{\substack{\deg P_i=r_i \\ \text{distinct primes}}} \left(\frac{h}{P_1 \cdots P_m} \right). \quad (5.14)$$

To evaluate the average of this expression we need to know, for a particular tuple (r_1, \dots, r_m) , the average of

$$\mathcal{P}(r_1, \dots, r_m) := \left(\prod_{i=1}^m \frac{r_i}{q^{r_i/2}} \right) \sum_{\substack{\deg P_i=r_i \\ \text{distinct primes}}} \left(\frac{h}{P_1 \cdots P_m} \right). \quad (5.15)$$

We will compute this average in the following section.

6 Estimation of $\langle \mathcal{P}(\vec{r}) \rangle$

In this section we focus on the contribution $\mathcal{P}(\vec{r})$ of different primes defined by (5.15). We use (2.9) and the explicit formula of (2.6) for the mean value of $\mathcal{P}(\vec{r})$:

$$\begin{aligned} \langle \mathcal{P}(\vec{r}) \rangle &= \frac{\prod_{j=1}^m r_j}{q^{\frac{\sum r_j}{2} + 2g} (q-1)} \sum_{\substack{\deg P_j=r_j \\ P_i \neq P_i}} \sum_{2\alpha+\beta=2g+1} \sum_{\substack{\deg A=\alpha \\ \gcd(A, P_j)=1}} \mu(A) \sum_{\deg B=\beta} \left(\frac{B}{\prod_{j=1}^m P_j} \right) \\ &= \frac{\prod_{j=1}^m r_j}{q^{\frac{\sum r_j}{2} + 2g} (q-1)} \sum_{0 \leq \alpha \leq g} \sigma(\vec{r}; \alpha) S(2g+1-2\alpha; \vec{r}). \end{aligned} \quad (6.1)$$

PROPOSITION 6.1. Assume that $\sum r_j < (1-\delta)4g$. Then

$$\langle \mathcal{P}(\vec{r}) \rangle = \frac{\prod_{j=1}^m r_j}{q^{\frac{\sum r_j}{2} + 2g} (q-1)} (S(2g+1; \vec{r}) - qS(2g-1; \vec{r})) + O(q^{-\delta g} + q^{-r_1/2}).$$

Proof. It suffices to show that the terms with $\alpha \geq 2$ contributes $O(q^{-\delta g} + q^{-r_1/2})$. Note that $\sigma(\vec{r}; \alpha) = 0$ unless $\alpha \geq r_1 := \min r_j$ by Lemma 3.1. Thus it suffices to take $\alpha \geq r_1$. Recall that in any case,

$$|\sigma(\vec{r}; \alpha)| \leq (q+1) \frac{\alpha^m}{\prod r_j}. \quad (6.2)$$

If $\sum r_j \leq 2g-3$ then $S(2g+1-2\alpha, \vec{r}) = 0$ for $\alpha \geq 2$ by Lemma 4.1. Thus we may assume that $\sum r_j \geq 2g-2$.

We first assume that $\sum r_j \geq 2g - 1$ so that for $\alpha \geq 2$, we have $\beta \leq \sum r_j - 2$. Using duality, we obtained a bound for the sums $S(\beta; \vec{r})$ in Proposition 4.3 which implies that if $\beta \leq \sum r_j - 2$,

$$|S(\beta; \vec{r})| \ll q^{\beta - \frac{1}{2} \sum r_j} \prod \pi(r_j) + (\sum r_j)^m q^{\sum r_j}. \tag{6.3}$$

We insert (6.3) into (6.1) and first bound the contributions of the second term on the RHS of (6.3), call it II , namely of $(\sum r_j)^m q^{\sum r_j}$. Inserting (6.2) and using $\sum r_j < (1 - \delta)4g$ we get

$$\begin{aligned} II &\ll \frac{1}{q^{2g+1}} \prod \frac{r_j}{q^{r_j/2}} \sum_{r_1 \leq \alpha \leq g} q \frac{\alpha^m}{\prod r_j} (\sum r_j)^m q^{\sum r_j} \\ &\ll q^{\frac{1}{2} \sum r_j - 2g} g^{2m+1} \ll q^{-\delta g}. \end{aligned} \tag{6.4}$$

Now for the contribution of the first term on the RHS of (6.3), call it I , which we can bound by

$$\begin{aligned} I &\ll \frac{1}{q^{2g+1}} \prod \frac{r_j}{q^{r_j/2}} \sum_{r_1 \leq \alpha \leq g} q \frac{\alpha^m}{\prod r_j} q^{2g+1-2\alpha-\frac{1}{2} \sum r_j} \prod \pi(r_j) \\ &\ll \frac{q}{\prod r_j} \sum_{\alpha \geq r_1} \frac{\alpha^m}{q^{2\alpha}} \ll \frac{q}{\prod r_j} \frac{r_1^m}{q^{2r_1}} \ll q^{-r_1/2} \end{aligned} \tag{6.5}$$

on using the bound $\sum_{\alpha \geq r} \alpha^m z^\alpha \ll_m r^m z^r$, ($|z| \leq \frac{1}{4}$, $r \geq 1$, $m \geq 1$), giving our claim when $\sum r_j \geq 2g - 1$.

It remains to deal with the case $\sum r_j = 2g - 2$ and $\alpha = r_1 = 2$, where we need to bound the contribution to $\langle \mathcal{P}(\vec{r}) \rangle$ of

$$\frac{1}{(q-1)q^{2g}} \prod \frac{r_j}{q^{r_j/2}} \sigma(\vec{r}, 2) S(2g-3, \vec{r}) \ll \frac{1}{q^{2g+\frac{1}{2} \sum r_j}} |S(2g-3, \vec{r})|. \tag{6.6}$$

By (4.5), if $\sum r_j - 1 = 2g - 3$ then

$$|S(2g-3, \vec{r})| \ll q^{3g-3} \left(\frac{1}{q \prod r_j} + \frac{1}{q^{r_1}} \right)$$

and hence

$$(6.6) \ll \frac{1}{q^2} \left(\frac{1}{q \prod r_j} + \frac{1}{q^{r_1}} \right),$$

and since $\prod r_j \geq \max r_j \geq \sum r_j / m \geq g / m$, we recover the proposition in this case as well. \square

We now compute $\langle \mathcal{P}(\vec{r}) \rangle$. For a subset of indices $I \subseteq \mathbf{m} = \{1, \dots, m\}$ we denote its complement by I^c . Each subset $I \subseteq \mathbf{m}$ defines a hyperplane

$$\sigma(I^c) - \sigma(I) = 2g. \tag{6.7}$$

We will call these 2^m hyperplanes ‘‘exceptional’’.

PROPOSITION 6.2. Assume $\sum r_j < (1 - \delta)4g$.

(i) If $\sum_{j=1}^m r_j > 2g + 2$ and $\sum_{j=1}^m r_j$ is even, then away from the exceptional hyperplanes (6.7) we have

$$\langle \mathcal{P}(\vec{r}) \rangle = - \sum_{\sigma(I) < \sigma(I^c) - 2g} (-1)^{|I|} + O(q^{-\delta g} + q^{-r_1/2}). \quad (6.8)$$

(ii) If $\sum_{j=1}^m r_j = 2g, 2g + 2$ or if $\sum_{j=1}^m r_j > 2g + 2$ and (6.7) holds for some $I \subset \mathbf{m}$, then

$$|\langle \mathcal{P}(\vec{r}) \rangle| = O(1). \quad (6.9)$$

(iii) If $\sum_{j=1}^m r_j < 2g$ or if $\sum_{j=1}^m r_j > 2g$ and $\sum_{j=1}^m r_j$ is odd, then

$$|\langle \mathcal{P}(\vec{r}) \rangle| \ll q^{-\delta g} + q^{-r_1/2}.$$

Proof. **The case $\sum_{j=1}^m r_j < 2g$:** We use Proposition 6.1 and note that in this case $S(2g \pm 1; \vec{r}) = 0$ by Lemma 4.1. Hence

$$\langle \mathcal{P}(\vec{r}) \rangle = O(q^{-\delta g} + q^{-r_1/2}).$$

The case $\sum_{j=1}^m r_j = 2g$: For $\sum r_j = 2g$ we have $S(2g + 1; \vec{r}) = 0$ by Lemma 4.1. Thus by Proposition 6.1

$$\langle \mathcal{P}(\vec{r}) \rangle = - \frac{\prod_{j=1}^m r_j}{q^{\sum_{j=1}^m r_j/2 + 2g} (q - 1)} q S(2g - 1; \vec{r}) + O(q^{-\delta g} + q^{-r_1/2}).$$

By (4.5) and using $\sum r_j = 2g$, we have

$$\begin{aligned} \langle \mathcal{P}(\vec{r}) \rangle &= \frac{\prod r_j}{q^{\frac{1}{2} \sum r_j + 2g} (q - 1)} q \cdot q^{\frac{1}{2} \sum r_j - 1} \#\{\deg P_j = r_j, P_i \neq P_j\} \\ &\quad + O(q^{-\delta g} + q^{-r_1/2}) \\ &= \frac{1}{q - 1} + O(q^{-\delta g} + q^{-r_1/2}) = O(1). \end{aligned}$$

The case $\sum_{j=1}^m r_j = 2g + 1$: We have $S(2g + 1; \vec{r}) = 0$ by Lemma 4.1. Thus by Proposition 6.1

$$\langle \mathcal{P}(\vec{r}) \rangle = - \frac{\prod_{j=1}^m r_j}{q^{\sum_{j=1}^m r_j/2 + 2g} (q - 1)} q S(2g - 1; \vec{r}) + O(q^{-\delta g} + q^{-r_1/2}).$$

By Proposition 4.3, and using $\sum r_j = 2g + 1$ in (4.5), we have

$$\langle \mathcal{P}(\vec{r}) \rangle \ll \frac{g^{2m}}{q^{g - \frac{1}{2}}} + q^{-\delta g} + q^{-r_1/2} = O(q^{-\delta g} + q^{-r_1/2}).$$

The case $\sum_{j=1}^m r_j = 2g + 2$: By Proposition 6.1

$$\langle \mathcal{P}(\vec{r}) \rangle = \frac{\prod_{j=1}^m r_j}{q^{3g+1}(q-1)} (S(2g+1; \vec{r}) - qS(2g-1; \vec{r})) + O(q^{-\delta g} + q^{-r_1/2}).$$

Using (4.5) we have

$$S(2g+1; \vec{r}) = \frac{-q^{3g+2}}{\prod r_j} \left(1 + O(q^{-r_1/2}) \right),$$

and by Proposition 4.3,

$$S(2g-1; \vec{r}) = O\left(\frac{q^{3g}}{\prod r_j}\right).$$

Hence $\langle \mathcal{P}(\vec{r}) \rangle = O(1)$.

The case $\sum_{j=1}^m r_j > 2g+2$: In this case $\beta = 2g \pm 1$ satisfies $\beta \leq \sum_{j=1}^m r_j - 2$, hence we may use Proposition 4.3 which gives that for $\sum r_j$ even, β odd, and $\sum r_j - 2 \geq \beta$,

$$S(\beta, \vec{r}) = q^\beta \Phi_\beta(\vec{r}) \prod \frac{\pi(r_j)}{q^{r_j/2}} + O(q^{-\delta g} + q^{-r_1/2}). \tag{6.10}$$

If $\sum r_j$ is odd then there is no main term.

We now insert (6.10) and Lemma 3.2 in the computation of $\langle \mathcal{P}(\vec{r}) \rangle$ to get that, up to a remainder term of $O(q^{-\delta g} + q^{-r_1/2})$, we have that if $\sum r_j > 2g$, $\sum r_j$ even then

$$\begin{aligned} \langle \mathcal{P}(\vec{r}) \rangle &\sim \frac{1}{q^{2g}(q-1)} \prod_j \frac{r_j}{q^{r_j/2}} (S(2g+1, \vec{r}) - qS(2g-1, \vec{r})) \\ &\sim \Phi_{2g+1}(\vec{r}) + \frac{\Phi_{2g+1}(\vec{r}) - \Phi_{2g-1}(\vec{r})}{q-1} \\ &= - \sum_{\sigma(I) \leq L_+} (-1)^{|I|} + \frac{\Phi_{2g+1}(\vec{r}) - \Phi_{2g-1}(\vec{r})}{q-1}, \end{aligned} \tag{6.11}$$

where $2L_+ = \sum r_j - 1 - (2g + 1)$.

We have $\sigma(I) + \sigma(I^c) = \sum r_j$ and hence the condition $\sigma(I) \leq L_+$ becomes $\sigma(I) - \sigma(I^c) \leq -(2g + 2)$, and since $\sum r_j = \sigma(I) + \sigma(I^c)$ is even, so is $\sigma(I) - \sigma(I^c)$ and thus this condition is equivalent to

$$\sigma(I) - \sigma(I^c) < -2g.$$

Moreover,

$$\Phi_{2g+1}(\vec{r}) - \Phi_{2g-1}(\vec{r}) = \sum_{\sigma(I^c) - \sigma(I) = 2g} (-1)^{|I|}$$

and so the second term in (6.11) vanishes off the exceptional hyperplanes (6.7). Thus we have shown (6.8) and (6.9). □

7 Conclusion

Now we are ready to prove

PROPOSITION 7.1. *The mean value of $R(\underline{Q})$ is*

$$\langle R(\underline{Q}) \rangle = -2^{m-1} \sum_{I \subset \mathbf{m}} (-1)^{|I|} \int_{\substack{t_1, \dots, t_m \geq 0 \\ \sum t_i \geq 1 \\ \sum_{i \in I} t_i \leq \sum_{i \in I^c} t_i - 1}} \prod_{i=1}^m (\hat{u}_i(t_i) dt_i) + O(1/g).$$

Proof. We average (5.14) over $\mathcal{H}(2g+1, q)$ substituting the values provided by Proposition 6.2. First let us ignore the errors and examine the contribution of the main terms. We get that the main term in $\langle R(\underline{Q}) \rangle$ is

$$\begin{aligned} & -2^m \sum_{I \subset \mathbf{m}} (-1)^{|I|} \sum_{\substack{r_1, \dots, r_m \geq 1 \\ \sum r_i > 2g+2 \text{ even} \\ \sum_{i \in I} r_i < \sum_{i \in I^c} r_i - 2g}} \prod_{i=1}^m \left(\hat{u}_i \left(\frac{r_i}{2g} \right) \cdot \frac{1}{2g} \right) \\ & = -2^{m-1} \sum_{I \subset \mathbf{m}} (-1)^{|I|} \int_{\substack{\mathbb{R}_{\geq 0}^m \\ \sum t_i \geq 1 \\ \sum_{i \in I} t_i \leq \sum_{i \in I^c} t_i - 1}} \prod_{i=1}^m (\hat{u}_i(t_i) dt_i) + O(1/g) \end{aligned}$$

using an approximation of the integral by a Riemann sum with step $1/2g$, with the restriction that $\sum r_j$ is even providing a factor of $1/2$. This is the main term in the assertion.

Now we consider the various error terms. Due to the condition on the supports of \hat{u}_i we only need to consider $\sum r_i < (1 - \delta)4g$ for some fixed $\delta > 0$. For the error term of the form $O(q^{-\delta g})$, we use that the number of suitable tuples r_i is $O(g^m)$, so the total contribution of these errors is $O(q^{-\delta g} g^m) \ll O(1/g)$. For error terms of the form $O(q^{-\min r_j/2})$, note that for any r the number of suitable r_1, \dots, r_m s.t. $\min(r_i) = r$ is $O(g^{m-1})$, each contributing an error term of $g^{-m} q^{-r/2}$, so the total contribution of these errors is $O(1/g)$. Finally, the number of r_1, \dots, r_m on exceptional hyperplanes is also $O(g^{m-1})$, so the total contribution of the additional errors is $O(1/g)$. \square

Putting together Propositions 5.3, 5.7, 7.1 we obtain

Theorem 7.2. *Assume that $f_j \in \mathcal{S}(\mathbb{R})$ are even and each $\hat{f}_j(u_j)$ is supported in the range $|u_j| < s_j$, with $\sum s_j < 2$. Then*

$$\langle W_f^{(n)} \rangle = A(f_1, \dots, f_n) + O(\log g/g),$$

where

$$\begin{aligned}
 A(f_1, \dots, f_n) &= \sum_{\underline{F} \in \Pi_n} (-2)^{n-|\underline{F}|} \prod_{l=1}^{|\underline{F}|} (|F_l| - 1)! \sum_{S \subset \{1, \dots, |\underline{F}|\}} \left(\prod_{l \in S^c} \hat{U}_{F_l}(0) \right) \\
 &\cdot \sum_{S_2 \subset S} (-1/2)^{|S_2^c|} \left(\prod_{l \in S_2^c} U_{F_l}(0) \right) \left(2^{|S_2|/2} \sum_{\text{pair up } S_2} \prod_{i=1}^{|S_2|/2} \int_{\mathbb{R}} \hat{U}_{a_i}(t) \hat{U}_{b_i}(t) |t| dt \right. \\
 &- \frac{1}{2} \sum_{S_3 \subsetneq S_2} 2^{|S_3|/2} \sum_{\text{pair up } S_3} \left. \left(\prod_{i=1}^{|S_3|/2} \int_{\mathbb{R}} \hat{U}_{a_i}(t) \hat{U}_{b_i}(t) |t| dt \right) \right) \\
 &\cdot (-2)^{|S_3^c|} \sum_{I \subset S_3^c} (-1)^{|I|} \int_{\substack{\mathbb{R}_{\geq 0}^{|S_3^c|} \\ \sum_{i \in I} t_i \leq \sum_{i \in I^c} t_i - 1}} \prod_{i \in S_3^c} (\hat{U}_i(t) dt_i).
 \end{aligned}$$

Here $\underline{F} = \{F_1, \dots, F_{|\underline{F}|}\}$ ranges over the partitions of $\{1, \dots, n\}$, S over the subsets of $\{1, \dots, |\underline{F}|\}$, $U_{F_l}(t) = \prod_{k \in F_l} f_k(t)$, S_2 ranges over the subsets of S , a pair up sum ranges over partitions $\{\{a_1, b_1\}, \dots, \{a_{|T|/2}, b_{|T|/2}\}\}$ of a set T (it is empty if $|T|$ is odd), S_3 ranges over the proper subsets of S_2 and I ranges over the subsets of S_3 .

This coincides with the expression obtained in [Gao05] for the n -level density statistics of the family of quadratic L -functions.

Proof. We go through the verification. From Proposition 5.3,

$$\begin{aligned}
 \langle W_f^{(n)} \rangle &\sim \sum_{\underline{F}} (-2)^{n-|\underline{F}|} \prod_{l=1}^{|\underline{F}|} (|F_l| - 1)! \sum_{S \subset \{1, \dots, l\}} \left(\prod_{l \in S^c} \hat{U}_{F_l}(0) \right) \\
 &\cdot \sum_{S_2 \subset S} (-1/2)^{|S_2^c|} \left(\prod_{l \in S_2^c} U_{F_l}(0) \right) (-1)^{|S_2|} B(U_{l_1}, \dots, U_{l_{|S_2|}}).
 \end{aligned}$$

By Proposition 5.7,

$$\begin{aligned}
 B(U_{l_1}, \dots, U_{l_{|S_2|}}) &\sim 2^{|S_2|/2} \sum_{\text{pair up } S_2} \prod_{i=1}^{|S_2|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \\
 &+ \sum_{S_3 \subsetneq S_2} 2^{|S_3|/2} \sum_{\text{pair up } S_3} \prod_{i=1}^{|S_3|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \cdot D(S_3^c).
 \end{aligned}$$

Taking into account that the first term above only occurs if $|S_2|$ is even, so that $(-1)^{|S_2|} = 1$, gives

$$\begin{aligned} \langle W_f^{(n)} \rangle &\sim \sum_{\underline{F}} (-2)^{n-|\underline{F}|} \prod_{l=1}^{|\underline{F}|} (|F_l| - 1)! \sum_{S \subset \{1, \dots, l\}} \left(\prod_{l \in S^c} \hat{U}_{F_l}(0) \right) \\ &\cdot \sum_{S_2 \subset S} (-1/2)^{|S_2^c|} \left(\prod_{l \in S_2^c} U_{F_l}(0) \right) \\ &\times \left\{ 2^{|S_2|/2} \sum_{\text{pair up } S_2} \prod_{i=1}^{|S_2|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \right. \\ &\left. + (-1)^{|S_2|} \sum_{S_3 \subsetneq S_2} 2^{|S_3|/2} \sum_{\text{pair up } S_3} \prod_{i=1}^{|S_3|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \cdot D(S_3^c) \right\}. \end{aligned}$$

We also note that the term with S_3 only occur if $|S_3|$ is even, so that we may replace $(-1)^{|S_2|} = (-1)^{|S_3^c|}$. Inserting Proposition 7.1 (which says $\langle R \rangle \sim D$) gives

$$\begin{aligned} \langle W_f^{(n)} \rangle &\sim \sum_{\underline{F}} (-2)^{n-|\underline{F}|} \prod_{l=1}^{|\underline{F}|} (|F_l| - 1)! \sum_{S \subset \{1, \dots, l\}} \left(\prod_{l \in S^c} \hat{U}_{F_l}(0) \right) \\ &\cdot \sum_{S_2 \subset S} \left(-\frac{1}{2}\right)^{|S_2^c|} \left(\prod_{l \in S_2^c} U_{F_l}(0) \right) \\ &\times \left\{ 2^{|S_2|/2} \sum_{\text{pair up } S_2} \prod_{i=1}^{|S_2|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \right. \\ &- \frac{1}{2} \sum_{S_3 \subsetneq S_2} 2^{|S_3|/2} \sum_{\text{pair up } S_3} \prod_{i=1}^{|S_3|/2} \int_{\mathbb{R}} \hat{u}_{a_i}(t) \hat{u}_{b_i}(t) |t| dt \\ &\left. \cdot (-2)^{|S_3^c|} \sum_{I \subset S_3^c} (-1)^{|I|} \int_{\substack{t_1, \dots, t_{|S_3^c|} \geq 0 \\ \sum t_i \geq 1 \\ \sum_{i \in I} t_i \leq \sum_{i \in I^c} t_i - 1}} \prod_{i=1}^{|S_3^c|} \hat{u}_i(t_i) dt_i \right\}, \end{aligned}$$

with a remainder of $O(\log g/g)$. This is exactly the expression derived in Gao's thesis (see [Gao05, Theorem II.1] or [Gao13, Theorem 2.1]). \square

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