Received: 23 September 2023 | Revised: 7 January 2024 | Accepted: 2 February 2024

DOI: 10.1112/mtk.12244

RESEARCH ARTICLE

Mathematika

Zeros of modular forms and Faber polynomials

Zeév Rudnick

School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel

Correspondence
Zeév Rudnick, School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel.
Email: rudnick@tauex.tau.ac.il

Funding information
Horizon 2020, Grant/Award Number: 786758; Israel Science Foundation, Grant/Award Number: 1881/20

Abstract

We study the zeros of cusp forms of large weight for the modular group, which have a very large order of vanishing at infinity, so that they have a fixed number D of finite zeros in the fundamental domain. We show that for large weight the zeros of these forms cluster near D vertical lines, with the zeros of a weight k form lying at height approximately $\log k$. This is in contrast to previously known cases, such as Eisenstein series, where the zeros lie on the circular part of the boundary of the fundamental domain, or the case of cuspidal Hecke eigenforms where the zeros are uniformly distributed in the fundamental domain. Our method uses the Faber polynomials. We show that for our class of cusp forms, the associated Faber polynomials, suitably renormalized, converge to the truncated exponential polynomial of degree D.

MSC 2020
11F11 (primary)

1 | INTRODUCTION

For an even integer $k \geqslant 0$ let M_k be the space of modular forms of weight k for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. Any $f \in M_k$ has an expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H} = \{\tau : \operatorname{Im}(\tau) > 0\}$, and one defines

$$\operatorname{ord}_\infty(f) = \min(n : a_f(n) \neq 0).$$

The space of such forms M_k is finite-dimensional, spanned by the holomorphic Eisenstein series $E_k(\tau) = \frac{1}{2}\sum_{\gcd(c,d)=1}(c\tau+d)^{-k}$ ($k \geqslant 4$) and the space of cusp forms S_k, made up of forms which vanish at infinity ($\operatorname{ord}_\infty(f) \geqslant 1$). Writing

$$k = 12\ell + k', \quad k' \in \{0, 4, 6, 8, 10, 14\}$$

then $\ell = \dim S_k$ when $\ell \geqslant 1$, and for any nonzero $f \in M_k$, we have $\operatorname{ord}_\infty(f) \leqslant \ell$.

A nonzero modular form of weight k has roughly $k/12 - \operatorname{ord}_\infty(f)$ zeros (see (2.2)) in the fundamental domain

$$\mathcal{F} = \{\tau \in \mathbb{H} : \operatorname{Re}(\tau) \in [-1/2, 1/2), |\tau| \geqslant 1\},$$

where if $|\tau| = 1$ then we take $\operatorname{Re}(\tau) \leqslant 0$; see Figure 1. The question we address is how are these zeros distributed as $k \to \infty$.

There are two main types of known results on zeros of modular forms: The first of these originates in 1970, when Rankin and Swinnerton-Dyer [6] showed, by a remarkably simple argument, that all the zeros of the Eisenstein series E_k lie on the bounding arc $C = \{e^{it} : \pi/2 \leqslant t \leqslant 2\pi/3\}$, and as $k \to \infty$ they become uniformly distributed there. Several authors have used the argument of Rankin and Swinnerton-Dyer as an ingredient in proving analogous results for other distinguished forms, for instance Duke and Jenkins [1] studied certain "gap forms," see below. The second type of result concerns cuspidal Hecke eigenforms. In this case, the zeros are equidistributed in the fundamental domain with respect to the hyperbolic measure [3, 7]; see also [2, 4] for further results on this case.

Our goal is to present a different distribution result, for zeros of cusp form with a high order of vanishing at infinity. We investigate the zeros of cusp forms with very high order of vanishing at

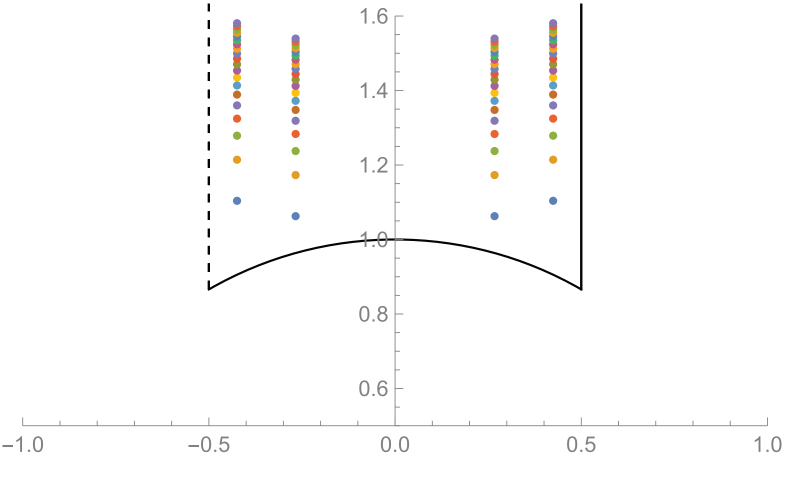

FIGURE 1 The fundamental domain $\mathcal{F}$ and the points $\frac{\sqrt{-1}}{2\pi}\log(2kz_{4,r})$, $r = 1, \dots, 4$ and $k = 1000j$, $j = 1, 20$ where $z_{4,r}$ are the inverse zeros of $\mathcal{E}_4(t) = 1 + t + t^2/2 + t^3/6 + t^4/24$.

infinity, by which we mean that we fix $D \geqslant 1$ and a bounding parameter $C > 0$, and consider cusp forms with $\mathrm{ord}_\infty(f) = \ell - D$,

$$f = q^{\ell-D}\left(1 + y_f(1)q + \cdots + y_f(D)q^D\right) + O\left(q^{\ell+1}\right) \tag{1.1}$$

with coefficients $|y_f(j)| \leqslant C$, and take $k \gg 1$. A principal example are the cofinal elements of the "Miller basis" of M_k, which are the unique elements $f_{k,m}$ of M_k with q-expansion

$$f_{k,m} = q^m + O(q^{\ell+1}), \quad m = 0, \dots, \ell.$$

If we fix D, then $f_{k,\ell-D} = q^{\ell-D} + O(q^{\ell+1})$ is of the required form.

We show that for these forms, the zeros do not lie on the arc C as is the case for the Eisenstein series [6], or for the "gap form" $f_{k,0}$ [1], nor are the zeros equidistributed as is the case for cuspidal Hecke eigenforms of large weight. Instead, we find that the zeros asymptotically lie on D lines at height $\log 2k$. Precisely, let $\mathcal{E}_D(t)$ be the truncated exponential polynomial of degree D:

$$\mathcal{E}_D(t) = 1 + t + \cdots + \frac{t^D}{D!}$$

and denote by $\{z_{D,r}\}$ the inverse zeros: $\mathcal{E}_D(t) = \prod_{r=1}^{D}\left(1 - z_{D,r}t\right)$.

Theorem 1.1. *Fix $D \geqslant 1$, $C > 0$, and let $f \in S_k$ be as in (1.1). Then the zeros $\tau_1, \dots, \tau_D$ of f in the fundamental domain, suitably labeled, satisfy*

$$\tau_r = \frac{\sqrt{-1}}{2\pi}\log\left(2kz_{D,r}\right) + O\left(\frac{1}{k}\right).$$

So in particular, the zeros of f cluster around the vertical lines

$$\mathcal{L}_r = \{\mathrm{Re}(\tau) = -\frac{\arg(z_{D,r})}{2\pi}\}, \quad r = 1, \dots, D,$$

where the argument is chosen so that $\arg(z_{D,r}) \in [-\pi, \pi)$; see Figure 1.

Our argument uses Faber polynomials: Given a nonzero $f \in M_k$, the associated Faber polynomial $F_f(t) \in \mathbb{C}[t]$ is a polynomial of degree $D = \ell - \mathrm{ord}_\infty(f)$, uniquely determined by

$$\frac{f}{\Delta^\ell E_{k'}} = F_f(j),$$

where

$$j = \frac{1}{q} + 744 + 196\,884q + \dots$$

is Klein's absolute invariant. The zeros of F_f are at $j(\tau_r)$, where $\{\tau_r\}$ are the zeros of $f/E_{k'}$. In §4 we show:

Theorem 1.2. *Let $f \in S_k$ be as in (1.1) and $F_f(t)$ its Faber polynomial. Then*

$$\frac{1}{(2k)^D} F_f(2kt) = \sum_{s=0}^{D} \frac{1}{s!}\left(1 + O\left(\frac{1}{k}\right)\right) t^{D-s}.$$

Noting that $\sum_{s=0}^{D} \frac{t^{D-s}}{s!} = t^D \mathcal{E}_D(\frac{1}{t})$, we will then obtain the limit distribution of the zeros of $F_f(t)$:

Corollary 1.3. *As $k \to \infty$, the zeros $t_1, \dots, t_D$ of $F_f(t)$ satisfy*

$$t_r = 2k \cdot z_{D,r} + O(1), \quad r = 1, \dots, D,$$

where $z_{D,1}, \dots, z_{D,D}$ are the inverse zeros of $\mathcal{E}_D$.

In §5 we will deduce Corollary 1.3 and Theorem 1.1 from Theorem 1.2.

2 | BACKGROUND ON MODULAR FORMS

2.1 | Basic definitions

For an even integer $k \geqslant 0$, the space of modular forms M_k consists of holomorphic functions on the upper half-plane $\mathbb{H} = \{\tau = x + iy : y > 0\}$ which transform under Möbius transformations from $\mathrm{SL}(2, \mathbb{Z})$ as $f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, and are bounded as $\mathrm{Im}(\tau) \to +\infty$. The subspace $S_k \subset M_k$ of cusp forms consists of those forms which vanish as $\mathrm{Im}(\tau) \to +\infty$. A modular form has an expansion in terms of the nome $q = e^{2\pi i \tau}$:

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n$$

and is a cusp form when $a(0) = 0$.

Some examples are the (normalized) Eisenstein series

$$E_k(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} (cz + d)^{-k}$$

which have Fourier expansion

$$E_k(\tau) = 1 - \gamma(k) \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_s(n) = \sum_{d|n} d^s$ are divisor sums, $\gamma(k) = 2k/B_k$ with B_k the Bernoulli numbers; see Table 1. For instance,

$$E_4 = 1 + 240 \sum_{n\geqslant 1} \sigma_3(n) q^n, \qquad E_6 = 1 - 504 \sum_{n\geqslant 1} \sigma_5(n) q^n.$$

TABLE 1 The numbers $\gamma(k) = 2k/B_k$.

k	4	6	8	10	12	14
$\gamma(k)$	−240	504	−480	264	−65 520/691	24

An example of a cusp form is the modular discriminant $\Delta \in S_{12}$, the unique (up to multiple) cusp form of weight 12, with Fourier expansion

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 \cdots + .$$

We will also need Klein's absolute invariant, the j-function

$$j = \frac{E_4^3}{\Delta}$$

which is a modular function (weight zero), meromorphic at infinity, with q-expansion

$$j = \frac{1}{q} + 744 + 196\,884q + 21\,493\,760q^2 + \cdots \in \mathbb{Z}[[q]]. \tag{2.1}$$

The j-function gives an isomorphism $j : \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$. Any meromorphic modular form which is entire in the finite half-plane (its only possible poles are at infinity) is a polynomial in j.

For τ in the standard fundamental domain $\mathcal{F}$, $j(\tau)$ is real if and only if τ lies on the boundary of $\mathcal{F}$ or on the imaginary axis, more precisely, j maps the arc $e^{it} : t \in [\pi/2, 2\pi/3]$ onto [0,1728], the imaginary axis $\{iy : y \geqslant 1\}$ to $[1728, \infty)$ and the left boundary segment $\{-\frac{1}{2} + iy : y > \sqrt{3}/2\}$ to the negative reals.

2.2 | Zeros

Let $f \in M_k$ be a nonzero modular form of weight $k > 0$. Then the valence formula is

$$\mathrm{ord}_\infty f + \sum_{\substack{z\in\mathcal{F}\\ z\nsim i,\rho}} \mathrm{ord}_z f + \frac{1}{2}\mathrm{ord}_i f + \frac{1}{3}\mathrm{ord}_\rho f = \frac{k}{12}, \tag{2.2}$$

where $\rho = (-1+\sqrt{-3})/2$ and the sum is over the zeros of f in the fundamental domain $\mathcal{F}$ other than ρ and i.

From the valence formula (2.2) we see that if $k' \in \{4, 6, 10\}$ then the zeros of $E_{k'}$ are all simple; if $k' = 8$ then there is a double zero at ρ and no others; and if $k' = 14$ then there is a simple zero at i and a double zero at ρ. We also see that, writing as before $k = 12\ell + k'$, and form of weight k has at least the same zeros as $E_{k'}$, so is divisible by $E_{k'}$ (for $k' = 0$ we set $E_0 = 1$). We will use the term "trivial zeros" for the zeros of $f \in M_k$ arising from these symmetries. The nontrivial zeros of f are thus the zeros of the quotient $f/E_{k'}$.

3 FABER POLYNOMIALS

3.1 Definition

To any modular form $f \in M_k$ we can associate a polynomial $F_f \in \mathbb{C}[t]$ so that

$$f = \Delta^{\ell} E_{k'} \cdot F_f(j)$$

(recall $k = 12\ell + k', k' \in \{0, 4, 6, 8, 10, 14\}$). Indeed, the quotient $f/\Delta^{\ell} E_{k'}$ is a meromorphic modular form, whose only possible poles are at infinity, hence $f/\Delta^{\ell} E_{k'}$ must be a polynomial in j, of degree

$$D = \deg F_f = \ell - \mathrm{ord}_{\infty}(f).$$

By definition, for $k' \in \{0, 4, 6, 8, 10, 14\}$ we have $F_{E_{k'}}(t) = 1$ and likewise for $f = \Delta^{\ell}$. Also by definition, multiplying f by a power of Δ does not change the Faber polynomial: $F_{\Delta^m f} = F_f$.

The polynomial F_f accounts for all the "nontrivial" zeros of f (that is except for the common zeros with $E_{k'}$), in the sense that for these zeros τ, we have $f(\tau) = 0$ iff $F_f(j(\tau)) = 0$.

3.2 Computation

To compute the Faber polynomial $F_f(t)$, expand $f/\Delta^{\ell} E_{k'}$ as Laurent series in q and then match the principal part with that of a polynomial of degree D in j: With $m := \mathrm{ord}_{\infty}(f)$, expand

$$\begin{aligned}\frac{q^m}{\Delta^{\ell} E_{k'}} &= \frac{q^m}{q^{\ell} \prod_{n=1}^{\infty} (1 - q^n)^{24\ell} \left\{1 - \gamma(k') \sum_{n \geqslant 1} \sigma_{k'-1}(n) q^n\right\}} \\ &= \frac{A_k(0) + A_k(1)q + \cdots + A_k(D)q^D}{q^D} + O(q),\end{aligned}$$

with $A_k(0) = 1$, and

$$f = q^m \sum_{n=0}^{\infty} y_f(n) q^n,$$

with $y_f(0) \neq 0$, so that

$$\begin{aligned}\frac{f}{\Delta^{\ell} E_{k'}} &= \frac{A_k(0) + A_k(1)q + \cdots + A_k(D)q^D}{q^D} \sum_{n=0}^{\infty} y_f(n) q^n + O(q) \\ &= \sum_{s=0}^{D} q^{-s} \sum_{n=0}^{D-s} A_k(D - s - n) y_f(n) + O(q).\end{aligned} \tag{3.1}$$

Further, expand

$$j^r = \frac{1}{q^r} + \frac{744r}{q^{r-1}} + \cdots = \sum_{s=0}^{r} c_{r,s} q^{-s} + O(q),$$

with $0 \leqslant c_{r,s} \in \mathbb{Z}$,

$$c_{r,r} = 1, \quad c_{r,r-1} = 744 \cdot r$$

so that

$$F_f(j) = \sum_{r=0}^{D} x_{D-r} j^r = \sum_{s=0}^{D} q^{-s} \sum_{r=s}^{D} x_{D-r} c_{r,s} + O(q). \tag{3.2}$$

Comparing (3.1) with (3.2) gives a system of equations for the coefficients $x_0, x_1, \dots x_D$ of the Faber polynomial:

$$\sum_{r=s}^{D} c_{r,s} x_{D-r} = \sum_{n=0}^{D-s} A_k(D-s-n) y_f(n), \quad s = 0, 1, \dots, D. \tag{3.3}$$

For $s = D, D-1, D-2$ these are

$$\begin{aligned} x_0 &= y_f(0) \\ 744 D x_0 + x_1 &= A_k(1) y_f(0) + y_f(1) \\ c_{D,D-2} x_0 + c_{D-1,D-2} x_1 + x_2 &= A_k(2) y_f(0) + A_k(1) y_f(1) + y_f(2). \end{aligned}$$

3.3 | Examples

We determine in this way the Faber polynomials $F_{k,m}(t)$ for some examples in the Miller basis $f_{k,m} = q^m + O(q^{\ell+1})$. For instance, for the “gap form” $f_{k,0} = 1 + O(q^{\ell+1})$,

$$F_{24,0}(t) = 125\,280 - 1440\, t + t^2$$

with zeros 93.0072, 1346.99, and

$$F_{36,0} = -27\,302\,400 + 965\,520\, t - 2160\, t^2 + t^3$$

with zeros 30.3029, 582.232, 1547.46. Duke and Jenkins [1] show the gap form $f_{k,0}$ has all it zeros on the arc $\{e^{i\theta} : \frac{\pi}{2} \leqslant \theta \leqslant \frac{2\pi}{3}\}$, equivalently that the Faber polynomial has all its zeros in [0,1728].

Here are some examples of $F_{k,\ell-D}$ when $k = 12\ell$ and $D = \ell - m$ is small:

$$F_{12\ell,\ell-1}(t) = t + (2k - 744).$$

$$\begin{aligned} F_{12\ell,\ell-2}(t) &= t^2 + 24(\ell - 62)t + 36(8\ell^2 - 495\ell + 4438) \\ &= t^2 + (2k - 1488)t + \left(\frac{(2k)^2}{2} - 1485k + 159\,768\right). \end{aligned}$$

$$\begin{aligned} F_{12\ell,\ell-3}(t) &= t^3 + 24(-93 + \ell)t^2 + 36(29\,721 - 991\ell + 8\ell^2)t \\ &\quad + 32(-1\,152\,093 + 118\,990\ell - 6669\ell^2 + 72\ell^3) \end{aligned}$$

$$= t^3 + (2k - 2232)t^2 + \left(\frac{(2k)^2}{2} - 2973k + 1\,069\,956\right)t$$
$$+ \left(\frac{(2k)^3}{6} - 1482k^2 + \frac{951\,920}{3}k - 36\,866\,976\right).$$

4 | PROOF OF THEOREM 1.2

Recall that we fix $D = \ell - m$, fix $C > 0$, and consider cusp forms of the shape

$$f = q^m\left(1 + y_f(1)q + \cdots + y_f(D)q^D\right) + O(q^{\ell+1})$$

with bounded coefficients:

$$|y_f(1)|, \dots |y_f(D)| \leqslant C.$$

We will prove Theorem 1.2, which states that

$$\frac{1}{(2k)^D} F_f(2kt) = \sum_{s=0}^{D} \frac{1}{s!}\left(1 + O\left(\frac{1}{k}\right)\right) t^{D-s}.$$

Proof. Using

$$\frac{1}{(1-x)^N} = 1 + Nx + \cdots = \sum_{r=0}^{\infty} \binom{N-1+r}{r} x^r$$

we obtain

$$\frac{1}{\prod_{n=1}^{D}(1-q^n)^{24\ell}} = \prod_{n=1}^{D} \sum_{r_n=0}^{D} \binom{24\ell - 1 + r_n}{r_n} q^{nr_n} + O(q^{D+1})$$
$$= \sum_{r=0}^{D} q^r \sum_{r_1+2r_2+\cdots+Dr_D=r} \prod_{n=1}^{D} \binom{24\ell - 1 + r_n}{r_n} + O(q^{D+1}).$$

As $\ell \to \infty$, the coefficient $B_{24\ell}(r)$ of q^r in the above expansion is dominated by the contribution of the d-tuple $(r_1, \dots, r_D) = (r, 0, \dots, 0)$:

$$B_{24\ell}(r) = \sum_{r_1+2r_2+\cdots+Dr_D=r} \prod_{n=1}^{D} \binom{24\ell - 1 + r_n}{r_n}$$
$$= \sum_{r_1+2r_2+\cdots+Dr_D=r} \frac{(24\ell)^{r_1+r_2+\cdots+r_D}}{r_1!r_2!\dots r_D!}\left(1 + O\left(\frac{1}{\ell}\right)\right)$$
$$= \frac{(24\ell)^r}{r!}\left(1 + O\left(\frac{1}{\ell}\right)\right) = \frac{(2k)^r}{r!}\left(1 + O\left(\frac{1}{k}\right)\right).$$

We next expand $f/(q^m E_{k'})$ up to $O(q^{D+1})$:

$$\frac{f}{q^m E_{k'}} = \frac{1 + y_f(1)q + \cdots + y_f(D)q^D}{1 - \gamma(k') \sum_{n=1}^{D} \sigma_{k'-1}(n) q^n} + O(q^{D+1}) = \sum_{s=0}^{D} \alpha_s q^s + O\left(q^{D+1}\right)$$

with $\alpha_0 = 1$ and

$$\alpha_s = O(1)$$

because we assume that the coefficients $y_f(1), \ldots, y_f(D)$ are uniformly bounded as $k \to \infty$.

Multiplying by $\prod_{n=1}^{D}(1-q^n)^{-24\ell}$ gives that the Taylor polynomial of degree D in the expansion of $1/(E_{k'} \prod_{n=1}^{D}(1-q^n)^{24\ell})$ is

$$\sum_{r=0}^{D} \frac{(24\ell)^r}{r!} q^r \left(1 + O\left(\frac{1}{k}\right)\right) \sum_{s=0}^{D} \alpha_s q^s = \sum_{i=0}^{D} q^i \sum_{\substack{r+s=i \\ r,s \geqslant 0}} \frac{(24\ell)^r}{r!} \alpha_s \left(1 + O\left(\frac{1}{k}\right)\right)$$

$$= \sum_{i=0}^{D} \frac{(24\ell)^i}{i!} \left(1 + O\left(\frac{1}{k}\right)\right) q^i$$

since $\alpha_0 = 1$ and $\alpha_s = O(1)$. Therefore, the terms up to $O(q)$ of $f/(\Delta^\ell E_{k'})$ are

$$\frac{1}{q^D} \sum_{r=0}^{D} \frac{(24\ell)^r}{r!} \left(1 + O\left(\frac{1}{k}\right)\right) q^r.$$

Finally, replacing 24ℓ by $2k = 24\ell + O(1)$ we obtain

$$\frac{f}{\Delta^\ell E'_k} = \frac{1}{q^D} \sum_{r=0}^{D} A_k(r) q^r + O(q)$$

with

$$A_k(r) = \frac{(2k)^r}{r!} \left(1 + O\left(\frac{1}{k}\right)\right), \quad r = 0, \ldots, D.$$

Now compare with $F_f(j) = j^D + x_1 j^{D-1} + \cdots + x_D$: Using the equations (3.3) gives

$$744D + x_1 = 2k\left(1 + O\left(\frac{1}{k}\right)\right)$$

which says that

$$x_1 = 2k + O(1).$$

Next, we have

$$c_{D,D-2} + c_{D-1,D-2} x_1 + x_2 = \frac{(2k)^2}{2!} + O(k)$$

and since $x_1 = O(k)$ we get

$$x_2 = \frac{(2k)^2}{2!} + O(k).$$

Continuing, we assume by induction that

$$x_1 = O(k), \dots, x_{s-1} = O(k^{s-1})$$

and then obtain

$$c_{D,D-s} + \sum_{i=1}^{s-1} c_{D-i,D-s} x_i + x_s = \frac{(2k)^s}{s!} + O(k^{s-1})$$

which gives

$$x_s = \frac{(2k)^s}{s!} + O(k^{s-1}).$$

Thus

$$\begin{aligned}\frac{1}{(2k)^D} F_f(2kt) &= \frac{1}{(2k)^D} \sum_{s=0}^{D} \frac{(2k)^s}{s!} (2kt)^{D-s} \left(1 + O\left(\frac{1}{k}\right)\right) \\ &= \sum_{s=0}^{D} \frac{1}{s!} \left(1 + O\left(\frac{1}{k}\right)\right) t^{D-s}.\end{aligned}$$

as claimed. □

5 | BACK TO ZEROS OF MODULAR FORMS

Having at hand the convergence of the coefficients of the renormalized Faber polynomials $F_f(2kt)/(2k)^D$ to those of $t^D \mathcal{E}_D(1/t)$, we can deduce convergence of zeros.

5.1 | Proof of Corollary 1.3

We set

$$g_k(z) = \frac{F_f(2k \cdot z)}{(2k)^D}, \quad g(z) = z^d \mathcal{E}\left(\frac{1}{z}\right) = \sum_{r=0}^{D} \frac{z^{D-r}}{r!}.$$

The zeros of $g(z)$ are simple, as follows from the corresponding fact for $\mathcal{E}_D(z)$, which is in fact irreducible; see [8] for a survey.

Moreover,

$$g_k(z) = \sum_{r=0}^{D} \frac{(2k)^r}{r!}\left(1+O\left(\frac{1}{k}\right)\right)\frac{(2kz)^{D-r}}{(2k)^D}$$
$$= \sum_{r=0}^{D} \frac{z^{D-r}}{r!}\left(1+O\left(\frac{1}{k}\right)\right).$$

Therefore, we deduce that for $k \gg 1$, the zeros $z_{k,1}, \dots, z_{k,D}$ of g_k are simple and converge to the zeros $z_1, \dots, z_D$ of g, with a rate

$$z_{k,r} = z_r + O\left(\frac{1}{k}\right). \tag{5.1}$$

This follows for instance from [5, Appendix A, Theorem] which states that given monic polynomials $f(t) = t^D + \sum_{\nu=1}^{D} a_\nu t^{D-\nu}$ with zeros $x_1, \dots, x_D$ and $g(t) = t^D + \sum_{\nu=1}^{D} b_\nu t^{D-\nu}$ with zeros $y_1, \dots, y_D$, then possibly after relabeling the zeros, we have a bound on their differences

$$\max_{\nu=1,\dots,D} |x_\nu - y_\nu| \leqslant 2D\left(\sum_{\nu=1}^{D} |a_\nu - b_\nu|\, \Gamma^{D-\nu}\right)^{1/D},$$

where

$$\Gamma = \max_{\nu=1,\dots,D}(|a_\nu|^{1/\nu}, |b_\nu|^{1/\nu}).$$

In our case, taking $f = g_k$ for $k \gg 1$, and $g(z) = \sum_{\nu=0}^{D} z^{D-\nu}/\nu!$, we clearly have $|a_\nu - b_\nu| \ll 1/k$ and $\Gamma = O(1)$ and so we obtain (5.1)

The zeros of F_f are $t_{k,r} = 2kz_{k,r}$, hence (5.1) implies that they satisfy

$$t_{k,r} = 2kz_r = 2kz_r + O(1)$$

which proves Corollary 1.3.

5.2 | Proof of Theorem 1.1

The nontrivial zeros $\tau_1, \dots, \tau_D$ of f, are the zeros of $F_f\,(j\,(\tau))$, so their j-values satisfy

$$j(\tau_r) = 2kz_{D,r} + O(1), \quad r = 1, \dots, D.$$

Therefore these j-values tend to infinity, and in terms of the nome $q_r = e^{2\pi i \tau_r}$,

$$\frac{1}{q_r} + O(1) = j(\tau_r) = 2kz_{D,r} + O(1).$$

Hence

$$e^{-2\pi i \tau_r} = 2kz_{D,r} + O(1)$$

giving

$$\tau_r = \frac{i}{2\pi i}\log\left(2kz_{D,r}\right) + O\left(\frac{1}{k}\right)$$

as claimed in Theorem 1.1.

ACKNOWLEDGMENTS

We thank Mikhail Sodin for a helpful discussion. This research was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 786758) and by the Israel Science Foundation (grant no. 1881/20).

JOURNAL INFORMATION

Mathematika is owned by University College London and published by the London Mathematical Society. All surplus income from the publication of *Mathematika* is returned to mathematicians and mathematics research via the Society's research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Zeév Rudnick https://orcid.org/0000-0003-0813-2238

REFERENCES

1. W. Duke and P. M. Jenkins, *On the zeros and coefficients of certain weakly holomorphic modular forms*, Pure Appl. Math. Q. **4** (2008), no. 4, Part 1, 1327–1340.
2. A. Ghosh and P. Sarnak, *Real zeros of holomorphic Hecke cusp forms*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 465–487.
3. R. Holowinsky and K. Soundararajan, *Mass equidistribution for Hecke eigenforms*, Ann. of Math. (2) **172** (2010), no. 2, 1517–1528.
4. S. Lester, K. Matomäki, and M. Radziwiłł, *Small scale distribution of zeros and mass of modular forms*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 7, 1595–1627.
5. A. M. Ostrowski, *Solution of equations and systems of equations*, Pure and Applied Mathematics, vol. IX, Academic Press, New York, 1960.
6. F. K. C. Rankin and H. P. F. Swinnerton-Dyer, *On the zeros of Eisenstein series*, Bull. London Math. Soc. **2** (1970), 169–170.
7. Z. Rudnick, *On the asymptotic distribution of zeros of modular forms*, IMRN **34** (2005), 2059–2076.
8. S. M. Zemyan, *On the zeroes of the N -th partial sum of the exponential series*, Amer. Math. Monthly **112** (2005), no. 10, 891–909.