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Prime lattice points in ovals

Bingrong Huang[1] · **Zeév Rudnick**[1]

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Abstract
We study the distribution of lattice points with prime coordinates lying in the dilate of a convex planar domain having smooth boundary, with nowhere vanishing curvature. Counting lattice points weighted by a von Mangoldt function gives an asymptotic formula, with the main term being the area of the dilated domain, and our goal is to study the remainder term. Assuming the Riemann Hypothesis, we give a sharp upper bound, and further assuming that the positive imaginary parts of the zeros of the Riemann zeta function are linearly independent over the rationals allows us to give a formula for the value distribution function of the properly normalized remainder term.

Keywords Prime lattice points · Ovals · Limiting distribution · Riemann Hypothesis

Mathematics Subject Classification 11P21 · 11N05 · 11K70 · 60D05 · 62E20

1 Introduction

Our goal in this note is to investigate the distribution of lattice points with prime coordinates lying in dilates of a planar convex set. Before stating our findings, we recall what is known for the classical lattice point problem.

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Dedicated to Dorian Goldfeld on the occasion of his 71st birthday.

✉ Bingrong Huang
bingronghuangsdu@gmail.com

Zeév Rudnick
rudnick@post.tau.ac.il

1 Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

1.1 Lattice points

There is a vast body of work dedicated to the question of the number of lattice points lying in the family of dilates of a planar domain. A typical context is when one takes a domain $\Omega \subset \mathbb{R}^2$, which is compact, convex, contains the origin in its interior, with smooth boundary having nowhere zero curvature. One can call such a domain an "oval". For $R > 0$, let $R\Omega$ denote the dilated domain, and let

$$N_\Omega(R) := \#\mathbb{Z}^2 \cap R\Omega$$

be the number of lattice points in the dilated domain $R\Omega$. Under our assumptions, it is known that $N_\Omega(R) \sim \text{area}(\Omega)R^2$ as $R \to \infty$, and much work has been devoted to bounding the size of the remainder term. In the beginning of the twentieth century, it was shown that

$$N_\Omega(R) = \text{area}(\Omega)R^2 + O(R^{2/3})$$

and since then the exponent $2/3$ has been improved somewhat, starting with van der Corput [23], see [13]. It is conjectured that the correct exponent is $1/2 + o(1)$. One cannot improve the exponent beyond $1/2$, and there exists arbitrarily large R such that the remainder term is, in absolute value, $\gg R^{1/2}(\log R)^{1/4}$ [15]; in the case of the circle, this is a classical result of Hardy (see [21]). Note that if one allows points of vanishing curvature, then the remainder term can in some cases be larger than $R^{2/3}$, for instance in the case of the superellipse $\{x^{2k} + y^{2k} \le 1\}$ the remainder term may be larger than $R^{1-1/(2k)}$ for arbitrarily large R [6,17].

The normalized remainder term

$$F_\Omega(R) := \frac{N_\Omega(R) - \text{area}(\Omega)R^2}{R^{1/2}}$$

has a limiting value distribution [3,4,25], that is there is a measure $\mathrm{d}\nu_\Omega$ so that for any bounded (piecewise) continuous function G,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T G\Big(F_\Omega(R)\Big)\mathrm{d}R = \int_{-\infty}^{\infty} G(u)\mathrm{d}\nu_\Omega(u).$$

When Ω is a circle [4,12], or for certain ellipses [5], the limiting distribution is absolutely continuous, that is $\mathrm{d}\nu_\Omega(u) = f_\Omega(u)\mathrm{d}u$; and the density f_Ω is real analytic, in particular supported on all of the real line, though with very rapidly decaying tails: $f_\Omega(u) \ll \exp(-u^4)$ as $|u| \to \infty$, and in particular is non-Gaussian.

Our goal here is to treat the problem of counting *prime* lattice points in the dilated domain $R\Omega$, seeking to address the analogue of the above properties of the lattice point count $N_\Omega(R)$.

1.2 Prime lattice points

We consider a convex domain Ω, which we assume is symmetric about the coordinate axes, that is under the reflections $(x, y) \mapsto (\pm x, \pm y)$ (this is natural if we want to count primes, which do not come with a definite sign). We further assume that Ω is an *oval*, meaning convex with smooth boundary $\partial\Omega$ having nowhere zero curvature. This latter assumption is made as a convenient working hypothesis, and there are interesting variants of the problem which do not satisfy this assumption.

Let

$$\pi_\Omega(R) := \#\Big\{(p, q) \in R\Omega \cap \mathbb{Z}^2 : |p|, |q| \quad \text{prime}\Big\}$$

be the number of lattice points in the dilated domain $R\Omega$ with both coordinates being prime. Also let

$$\psi_\Omega(R) := \sum_{(m,n)\in R\Omega\cap\mathbb{Z}^2} \Lambda(|m|)\Lambda(|n|)$$

where $\Lambda(n)$ is the von Mangoldt function, which equals $\log p$ if $n = p^k$ is a power of a prime p ($k \geq 1$), and is zero otherwise, and the sum is over all lattice points lying in the dilated domain $R\Omega$, whose coordinates are both prime powers. To start our investigation, we give a prime number theorem for lattice points in $R\Omega$:

Theorem 1.1 *Assume that Ω is a symmetric oval as above. Then*

$$\pi_\Omega(R) \sim \text{area}(\Omega)\frac{R^2}{(\log R)^2} \quad \textit{and} \quad \psi_\Omega(R) \sim \text{area}(\Omega)R^2, \quad \textit{as} \quad R \to \infty.$$

Assuming the Riemann Hypothesis (RH), we have

$$\psi_\Omega(R) = \text{area}(\Omega)R^2 + O(R^{3/2}).$$

Our main goal is to study the distribution of the normalized remainder term

$$H_\Omega(R) = \frac{\psi_\Omega(R) - \text{area}(\Omega)R^2}{R^{3/2}}.$$

The appropriate scale to use is logarithmic: We show that assuming RH, there is a probability measure $\mathrm{d}\mu_\Omega$, supported in $[-A, A]$ (where $A = \sup |H_\Omega|$) so that for any bounded continuous function G,

$$\lim_{X\to\infty} \frac{1}{\log X} \int_1^X G\Big(H_\Omega(R)\Big) \frac{\mathrm{d}R}{R} = \int_{-\infty}^{\infty} G(u)\mathrm{d}\mu_\Omega(u).$$

To proceed further, we need the Linear Independence Hypothesis (LI) for the zeros of the Riemann zeta function. Recall that the Riemann Hypothesis is the statement that the nontrivial zeros of the Riemann zeta function are of the form $\rho = \frac{1}{2} + i\gamma$, with γ real. Due to the functional equation of the Riemann zeta function, the zeros come in conjugate pairs, so that if $\rho = \frac{1}{2} + i\gamma$ is a zero, then so is $\bar{\rho} = \frac{1}{2} - i\gamma$. In addition to assuming RH, we assume:

Hypothesis LI. *The imaginary parts of all nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ with $\gamma > 0$, are linearly independent over the rationals.*

This hypothesis was used by Wintner [26], and extensively since, for instance in the study of prime number races [19]. While plausible, it seems unlikely to be provable in the foreseeable future. See [2, Table 2] for numerical checks that the first few zeros do not satisfy any linear relations with small coefficients, for instance that the first 500 zeros do not admit any nontrivial linear relations with coefficients of size at most 10^5.

From general properties of the value distribution of uniformly almost periodic functions with linearly independent frequencies [24], we deduce that

Theorem 1.2 *Assume the Linear Independence Hypothesis. Then $\mathrm{d}\mu_\Omega(u) = p_\Omega(u)\mathrm{d}u$ is absolutely continuous, with a smooth density p_Ω, which is* symmetric*: $p_\Omega(-u) = p_\Omega(u)$. It is the probability distribution function of the random function*

$$g_\Omega(\mathbf{x}) = \sum_{n=1}^{\infty} A_n \cos(x_n)$$

where $\mathbf{x} = (x_1, x_2, \dots)$ are independent random variables, uniformly distributed in $[0, \pi]$,

$$A_n = \mathcal{B}_\Omega(\gamma_n)$$

where $\{\gamma_n : n = 1, 2, \dots\}$ are the imaginary parts of the nontrivial Riemann zeros ($\gamma_n > 0$), and $\mathcal{B}_\Omega$ is a certain function depending on the domain (see (4.2)), satisfying

$$\mathcal{B}_\Omega(\gamma) \ll \gamma^{-3/2}, \quad \gamma \to \infty,$$

and is nonzero infinitely often.

Theorem 1.2 allows us to use a formula for the limiting distribution of a sum of sine waves with random phases to deduce

$$p_\Omega(u) = \frac{1}{2A} + \frac{1}{A} \sum_{k=1}^{\infty} \left(\prod_{n=1}^{\infty} J_0\left(\frac{\pi k A_n}{A}\right)\right) \cos\left(\frac{\pi k u}{A}\right), \qquad |u| < A \tag{1.1}$$

where $A = \sum_n A_n$. See Fig. 1 for the value distribution for the circle and for an ellipse, by using (1.1) with 500 zeros.

According to Theorem 1.2, the value distribution function p_Ω is *symmetric*: $p_\Omega(-u) = p_\Omega(u)$. Note that for the corresponding problem of counting all lattice points, the distribution need not be symmetric, for instance for the circle, the third moment is negative [22].

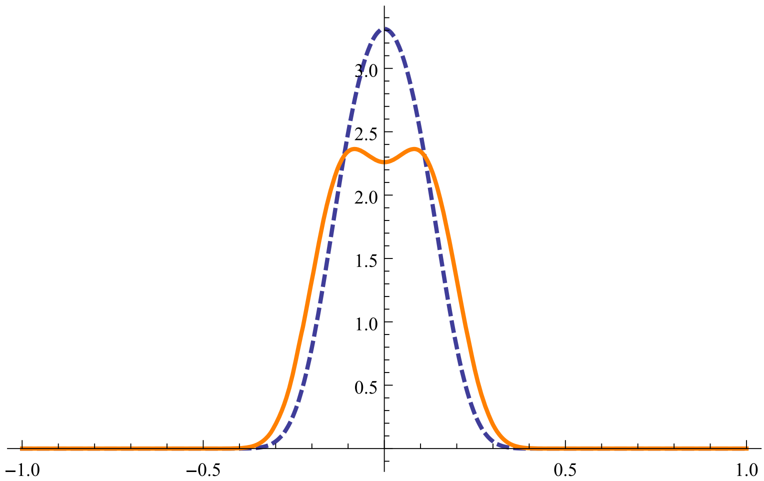

Fig. 1 The value distribution function $p_\Omega(u)$ for the circle $x^2 + y^2 \le 1$ (dashed) and the ellipse $(x/a)^2 + (y/b)^2 \le 1$ (solid) with $a = 1$, $b = 0.65$. The plots have been rescaled, replacing $p_\Omega(u)$ by $Ap_\Omega(Au)$. Note that for this particular ellipse, the distribution is bimodal

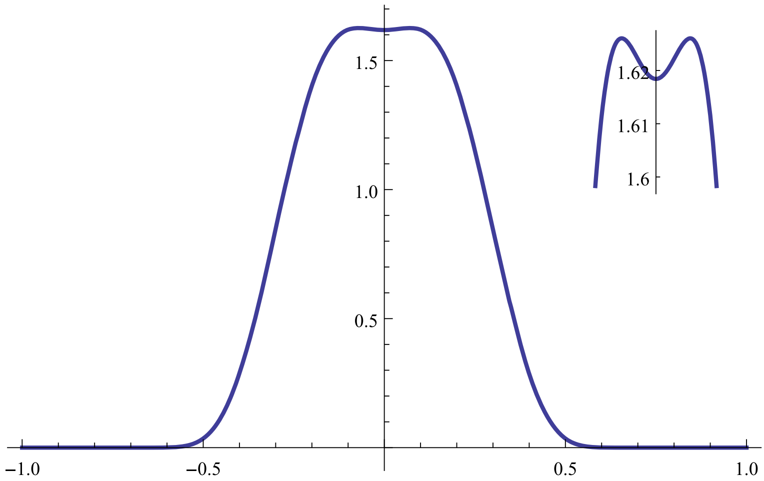

Fig. 2 The value distribution function p_T for the triangle $T = \{x + y \le 1, x, y \ge 0\}$. The inset displays the bimodal nature of the distribution

As mentioned earlier, the assumption that Ω has smooth boundary, with nowhere zero curvature, is made to get a simple set of examples. There are other natural cases one can consider, for instance when Ω is the triangle $T = \{x + y \le 1, x, y > 0\}$. Then $\psi_T(R) = \sum_{n=1}^{R} r(n)$ where $r(k) = \sum_{m+n=k} \Lambda(m)\Lambda(n)$, so that $\psi_T(R)/R$ is related to the average number of representations of an integer as a sum of two primes. In this guise, the value distribution of $(\psi_T(R) - \frac{1}{2}R^2)/R^{3/2}$ was studied by Fujii [9,10]. See Fig. 2 for a plot of the corresponding value distribution function p_T, by using (1.1) with 500 zeros.

1.3 The one-dimensional case

It is instructive to compare our findings on the remainder term for ovals in dimension two with the one-dimensional case, where we take a symmetric interval $\Omega = [-1, 1]$, and then

$$\psi_\Omega(R) = 2\psi(R) = 2\sum_{n\leq R} \Lambda(n)$$

and we are simply studying the remainder term in the Prime Number Theorem. In that case, Littlewood showed [14] (assuming RH) that the normalized remainder term $(\psi(R) - R)/R^{1/2}$ is *unbounded*, unlike what we find in the case of 2-dimensional ovals. Wintner [26] proved the existence of a limiting distribution $p(u)$ (assuming RH), which is not compactly supported. In comparison, for our symmetric ovals, the normalized remainder term $(\psi_\Omega(R) - \mathrm{area}(\Omega)R^2)/R^{3/2}$ is *bounded*, so that the limiting distribution p_Ω is compactly supported.

2 Symmetric ovals

2.1 Geometric preliminaries

We take a planar domain Ω to be an *oval*, that is bounded by a smooth, convex curve, which has nowhere-vanishing curvature. We further assume that Ω is *symmetric* with respect to reflections in the coordinate axes $(x, y) \mapsto (\pm x, \pm y)$, so it necessarily contains the origin. We may then display the top half of the boundary as the graph of a function:

$$\partial\Omega \cap \{y > 0\} = \Big\{(x, f(x)) : |x| \leq a\Big\}$$

where $f(x)$ is an even function (to take into account the reflection symmetry in the y-axis), which is smooth, $f(a) = 0$, $f(x)$ is monotonically decreasing for $x > 0$ (to allow convexity), and $f''(x) < 0$ to give the nowhere vanishing curvature condition, since the curvature of Ω at $(x, f(x))$ is

$$\kappa(x, f(x)) = -\frac{f''(x)}{(1 + f'(x)^2)^{3/2}}, \quad |x| < a.$$

Likewise, we may display the right half of the boundary as a graph:

$$\partial\Omega \cap \{x > 0\} = \Big\{(g(y), y) : |y| \leq b\Big\}$$

with

$$g = f^{-1}$$

the inverse function to f.

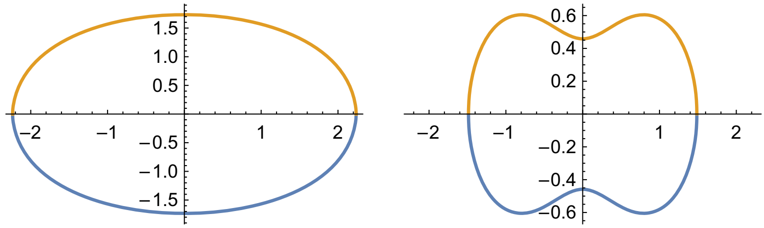

Fig. 3 The Cassini ovals $((x-\alpha)^2+y^2)((x+\alpha)^2+y^2)=\beta^4$ with $\alpha=1$, and $\beta=2$ (LHS) and $\beta=1.1$ (RHS)

For instance, if Ω is the ellipse $(x/a)^2+(y/b)^2 \le 1$, then we take $f(x) = b\sqrt{1-(x/a)^2}$, $|x|\le a$, and $g(y)=a\sqrt{1-(y/b)^2}$, $|y|\le b$.

Other examples are *Cassini ovals*, which are the locus of points such that the product of their distances from two fixed points a distance 2α apart is a constant β^2. In cartesian coordinates, if we locate the two points on the x-axis at $(\pm\alpha, 0)$, then the equation of the boundary curve is

$$\left((x-\alpha)^2+y^2\right)\left((x+\alpha)^2+y^2\right)=\beta^4, \tag{2.1}$$

which intersects the x-axis at $\pm\sqrt{\beta^2+\alpha^2}$, and the y-axis at $\pm\sqrt{\beta^2-\alpha^2}$ (assuming $\beta>\alpha$) . If $\beta>\sqrt{2}\alpha$ then we get an oval, if $\alpha<\beta<\sqrt{2}\alpha$ then we get a non-convex curve (a "dog-bone"), see Fig. 3, while for $0<\beta<\alpha$ we get two disconnected curves. For the Cassini oval (2.1) with $\beta>\sqrt{2}\alpha$, we take

$$f(x)=\sqrt{\sqrt{4\alpha^2x^2+\beta^4}-\alpha^2-x^2}, \quad |x|\le\sqrt{\beta^2+\alpha^2}$$

and

$$g(y)=\sqrt{\alpha^2-y^2+\sqrt{\beta^4-4\alpha^2y^2}}, \quad |y|\le\sqrt{\beta^2-\alpha^2}.$$

2.2 Singularities at the vertices

We note that for symmetric ovals, the intersection points $\{(\pm a, 0), (0, \pm b)\}$ of $\partial\Omega$ with the coordinate axes are *vertices*, that is local extrema of the curvature. We will need to know the nature of the singularities of $f(x)$ as $x\nearrow a$ and of the inverse function $g(y)=f^{-1}(y)$ as $y\nearrow b$:

Lemma 2.1 *Let $\kappa(x,y)$ be the curvature at a point $(x,y)\in\partial\Omega$ of the boundary. Then*

$$f(x)=\sqrt{\frac{2}{\kappa(a,0)}}\cdot\sqrt{a-x}\cdot\Big(1+O(a-x)\Big), \quad \text{as } x\nearrow a \tag{2.2}$$

and

$$g(y)=\sqrt{\frac{2}{\kappa(0,b)}}\cdot\sqrt{b-y}\cdot\Big(1+O(b-y)\Big),\quad \text{as } y\nearrow b. \tag{2.3}$$

Proof We write $x=g(y)$ for $y\searrow 0$, when $x\nearrow a$, and expand $g(y)$ in a Taylor series around $y=0$

$$x=g(y)=g(0)+g'(0)y+\frac{1}{2}g''(0)y^2+\frac{1}{3!}g^{(3)}(0)y^3+O(y^4).$$

We use $g(0)=a$ and the vanishing of the odd derivatives at 0 since g is even: $g\prime(0)=0=g^{(3)}(0)$, and obtain

$$x=a+\frac{1}{2}g''(0)y^2+O(y^4)$$

or

$$y=f(x)=\sqrt{\frac{-2}{g''(0)}}\sqrt{a-x}\Big(1+O(x-a)\Big).$$

Now we recall that the curvature of a graph $(g(y),y)$ is given by

$$\kappa(g(y),y)=-\frac{g''(y)}{(1+g'(y)^2)^{3/2}}$$

and at $y=0$ this reduces to

$$\kappa(a,0)=-g''(0). \tag{2.4}$$

Hence we have found

$$f(x)=\sqrt{\frac{2}{\kappa(a,0)}}\sqrt{a-x}\Big(1+O(a-x)\Big),\quad x\nearrow a$$

giving (2.2). The argument for (2.3) is identical. □

Lemma 2.2 *Let*

$$A(x):=f(x)-\sqrt{\frac{2}{\kappa(a,0)}(a-x)},\quad B(y):=g(y)-\sqrt{\frac{2}{\kappa(0,b)}(b-y)}.$$

Then the derivative of A satisfies $A'(x)=O(\sqrt{a-x})$ as $x\nearrow a$, and in particular $A'(a)=0$. The second derivative of A satisfies

$$A''(x)=O\Big(\frac{1}{\sqrt{a-x}}\Big),\quad \text{as } x\nearrow a$$

and in particular A'' is integrable on $(0, a)$. *Likewise,* $B'(b) = 0$ *and* B'' *is integrable on* $(0, b)$.

Proof From $x = g(y)$ we have $1 = g'(y)y'$ or

$$y' = \frac{1}{g'(y)} \tag{2.5}$$

Hence the second derivative of $y = f(x)$ is given by

$$y'' = \left(\frac{1}{g'(y)}\right)' = -\frac{g''(y)y'}{(g'(y))^2} = -\frac{g''(y)}{(g'(y))^3}$$

after inserting (2.5).

Expanding about $y = 0$, and recalling that since g is even, all the odd derivatives vanish at $y = 0$, we obtain

$$g''(y) = g''(0) + O(y^2) = -\kappa(a, 0)\Big(1 + O(y^2)\Big)$$

after using (2.4), and

$$g'(y) = 0 + g''(0)y + O(y^3) = -\kappa(a, 0)y\Big(1 + O(y^2)\Big).$$

Hence

$$y' = \frac{1}{g'(y)} = -\frac{1}{\kappa(a, 0)y}\Big(1 + O(y^2)\Big).$$

Inserting $y = \sqrt{\frac{2}{\kappa(a,0)}(a - x)(1 + O(a - x))}$ we obtain

$$f'(x) = y' = -\frac{1}{\sqrt{2\kappa(a, 0)(a - x)}} + O\Big(\sqrt{a - x}\Big), \quad x \nearrow a$$

and so

$$A'(x) = f'(x) - \left(\sqrt{\frac{2}{\kappa(a, 0)}(a - x)}\right)' = O\Big(\sqrt{a - x}\Big)$$

and in particular, $A'(a) = 0$.

Similarly

$$y'' = -\frac{g''(y)}{(g'(y))^3} = -\frac{-\kappa(a, 0)(1 + O(y^2))}{(-\kappa(a, 0)y)^3(1 + O(y^2))} = -\frac{1}{\kappa(a, 0)^2 y^3} + O\Big(\frac{1}{y}\Big).$$

and so

$$f''(x) = -\frac{1}{\sqrt{8\kappa(a,0)}}\frac{1}{(a-x)^{3/2}} + O\Big(\frac{1}{\sqrt{a-x}}\Big)$$

which gives, after identifying the first term as the second derivative of $\sqrt{\frac{2}{\kappa(a,0)}(a-x)}$, that

$$A''(x) = f''(x) - \left(\sqrt{\frac{2}{\kappa(a,0)}(a-x)}\right)'' = O\left(\frac{1}{\sqrt{a-x}}\right)$$

as claimed. □

2.3 An oscillatory integral

Given a symmetric oval Ω as above, define

$$\mathcal{I}_1(\rho) := \int_0^a f(x)x^{\rho-1}\mathrm{d}x, \quad \mathcal{I}_2(\rho) := \int_0^b g(y)y^{\rho-1}\mathrm{d}y \tag{2.6}$$

which are the Mellin transforms of f and g. We want to asymptotically evaluate the oscillatory integrals as $|\operatorname{Im}\rho| \to +\infty$ ($|\operatorname{Re}\rho| \le 1/2$). The result is

Lemma 2.3 *Let $\kappa(x, y)$ be the curvature of the boundary $\partial\Omega$ at the point (x, y). Then*

$$\mathcal{I}_1(\rho) = \sqrt{\frac{\pi}{2\kappa(a,0)}}\frac{a^{\frac{1}{2}+\rho}}{\rho^{3/2}} + O\Big(\frac{1}{|\rho|^2}\Big), \quad \mathcal{I}_2(\rho) = \sqrt{\frac{\pi}{2\kappa(0,b)}}\frac{b^{\frac{1}{2}+\rho}}{\rho^{3/2}} + O\Big(\frac{1}{|\rho|^2}\Big)$$

as $|\rho| \to \infty$ ($|\operatorname{Re}\rho| \le 1/2$).

Proof We use Lemma 2.2 to write

$$f(x) = \sqrt{\frac{2}{\kappa(a,0)}} \cdot \sqrt{a-x} + A(x)$$

with $A'' \in L^1(0, a)$, and insert this into the integral $\mathcal{I}_1$ to obtain

$$\mathcal{I}_1(\rho) = \sqrt{\frac{2}{\kappa(a,0)}}\int_0^a \sqrt{a-x}\cdot x^{\rho-1}\mathrm{d}x + \int_0^a A(x)x^{\rho-1}\mathrm{d}x. \tag{2.7}$$

We have

$$\sqrt{\frac{2}{\kappa(a,0)}}\int_0^a \sqrt{a-x}\cdot x^{\rho-1}\mathrm{d}x = \sqrt{\frac{2}{\kappa(a,0)}}\, a^{\frac{1}{2}+\rho}\int_0^1 (1-z)^{1/2}z^{\rho-1}\mathrm{d}z$$
$$= \sqrt{\frac{2}{\kappa(a,0)}}\, a^{\frac{1}{2}+\rho} B(\tfrac{3}{2},\rho)$$

where $B(x,y)$ is the Euler Beta function. By Stirling's formula,

$$B(\tfrac{3}{2},\rho) = \frac{\frac{1}{2}\sqrt{\pi}}{\rho^{3/2}}\left(1+O\left(\frac{1}{|\rho|}\right)\right).$$

For the second term in (2.7), we can integrate by parts twice, using $A(a) = A'(a) = 0$, to find

$$\int_0^a A(x)x^{\rho-1}\mathrm{d}x = \frac{1}{\rho(\rho+1)}\int_0^a A''(x)x^{\rho+1}dx = O\Big(\frac{1}{|\rho|^2}\Big)$$

since A'' is integrable by Lemma 2.2. Thus we find

$$\mathcal{I}_1(\rho) = \sqrt{\frac{\pi}{2\kappa(a,0)}}\,\frac{a^{\frac{1}{2}+\rho}}{\rho^{3/2}} + O\Big(\frac{1}{|\rho|^2}\Big)$$

as claimed. The integral $\mathcal{I}_2$ can be treated identically. □

Lemma 2.3 gives an upper bound for $\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)$. In Sect. 4, we will also need a non-vanishing result for $\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)$. The following lemma will suffice:

Lemma 2.4 *For infinitely many (in fact a positive proportion) of the zeros $\rho = \frac{1}{2}+i\gamma$, we have*

$$|\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)| \gg \frac{1}{\gamma^{3/2}}$$

and in particular $\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)$ is nonzero infinitely often.

Proof According to Lemma 2.3, we have

$$|\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)| = \frac{\sqrt{\pi}}{\sqrt{2}|\rho|^{3/2}}\left|\frac{a}{\sqrt{\kappa(a,0)}}a^{i\gamma} + \frac{b}{\sqrt{\kappa(0,b)}}b^{i\gamma}\right| + O\Big(\frac{1}{|\rho|^2}\Big)$$
$$= \frac{C}{|\rho|^{3/2}}\left|e^{i\gamma\log(b/a)}+c\right| + O\Big(\frac{1}{|\rho|^2}\Big)$$

where $C>0$, $c>0$ are independent of γ.

Now if $a = b$ then we get

$$|\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)| \sim \frac{C(1+c)}{\gamma^{3/2}} \gg \frac{1}{\gamma^{3/2}}$$

for all $\gamma \gg 1$ as we claim.

If $a \neq b$, that is $\log(b/a) \neq 0$, then we use a result of Hlawka [11] (see also Rademacher [16] for a proof assuming RH), for any $\alpha \neq 0$, the sequence $\{\alpha\gamma : \gamma > 0\}$ is uniformly distributed modulo one. Hence for a positive proportion of γ's, we have $\operatorname{Re} e^{i\gamma\log(b/a)} > \frac{1}{2}$, and hence for these we have

$$\left|e^{i\gamma\log(b/a)} + c\right| \geq \frac{1}{2} + c \geq \frac{1}{2}$$

so that for a positive proportion of γ's,

$$|\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)| \gg \frac{1}{\gamma^{3/2}}$$

as claimed. □

3 Counting prime points

Now we want to consider prime points in a symmetric oval Ω as above.

Let $y = f(x)$ be the function which gives the boundary of Ω in the first quadrant. By our assumptions, we know $f(x)$ satisfies that

$$f(0) = b > 0,\ f(a) = 0,\ f'(0) = 0 \quad \text{and} \quad f'(x) \searrow -\infty \text{ as } x \nearrow a.$$

Since the curvature of the boundary is non-vanishing, we know that $f'(x) < 0$ for all $x \in (0, a)$. Let $R > 0$ be a large parameter.

3.1 The main term

We first give the main terms in Theorem 1.1, as a simple consequence of the Prime Number Theorem:

Proposition 3.1 *Let Ω be a symmetric oval. Then*

$$\psi_\Omega(R) \sim \operatorname{area}(\Omega)R^2 \quad \textit{and} \quad \pi_\Omega(R) \sim \frac{\operatorname{area}(\Omega)R^2}{(\log R)^2}, \qquad R \to \infty.$$

Proof Using the symmetry of Ω, it suffices to perform the analysis in the positive quadrant, where we sum over lattice points with prime power coordinates lying under the graph of $y = f(x)$:

$$\psi_\Omega(R) = 4 \sum_{m \le aR} \Lambda(m) \sum_{n \le Rf(m/R)} \Lambda(n).$$

By the Prime Number Theorem, the inner sum is

$$\sum_{n \le Rf(m/R)} \Lambda(n) = Rf\left(\frac{m}{R}\right) + o(R)$$

and so

$$\psi_\Omega(R) = 4R \sum_{m \le aR} \Lambda(m) f\left(\frac{m}{R}\right) + o(R^2).$$

Applying summation by parts, using the Prime Number Theorem again, gives

$$\psi_\Omega(R) = 4R^2 \int_0^a f(v)\mathrm{d}v + o(R^2) = \mathrm{area}(\Omega)R^2 + o(R^2).$$

To prove the claim about π_Ω, we first bound the contribution to $\psi_\Omega(R)$ of pairs (m, n) where at least one of them is less than $R/(\log R)^{10}$ by

$$\begin{aligned}
&\sum_{\substack{m,n \ll R \\ \min(m,n) < R/(\log R)^{10}}} \Lambda(m)\Lambda(n) \\
&\ll (\log R)^2 \# \left\{ (m, n) : m, n \ll R, \min(m, n) < \frac{R}{(\log R)^{10}} \right\} \\
&\ll (\log R)^2 \frac{R^2}{(\log R)^{10}}
\end{aligned}$$

which is negligible for our purposes.

Moreover, the contribution of (m, n) for which at least one is not a prime, is bounded by

$$\ll \log R \sum_{p \ll R^{1/2}} \log p \sum_{q \ll R} \log q \ll R^{3/2} (\log R)^3$$

which is again negligible. Thus

$$\frac{1}{4}\psi_\Omega(R) \sim \sum_{R/(\log R)^{10} < p < aR} \log p \sum_{R/(\log R)^{10} < q < Rf(p/R)} \log q$$

the sum over primes.

For $p \in (R/(\log R)^{10}, R)$, we have $\log p \sim \log R$ and likewise for the sum over q. Hence we find

$$\frac{1}{4}\psi_\Omega(R) \sim (\log R)^2 \sum_{R/(\log R)^{10}<p<aR} \sum_{R/(\log R)^{10}<q<Rf(p/R)} 1.$$

Arguing as above, we find

$$\sum_{R/(\log R)^{10}<p<aR} \sum_{R/(\log R)^{10}<q<Rf(p/R)} 1 = \frac{1}{4}\pi_\Omega(R) + O\Big(\frac{R^2}{(\log R)^{10}}\Big).$$

Therefore we find

$$\psi_\Omega(R) \sim (\log R)^2 \pi_\Omega(R)$$

and hence

$$\pi_\Omega(R) \sim \frac{\operatorname{area}(\Omega) R^2}{(\log R)^2}$$

as claimed. □

3.2 Using RH

In this section, we give a formula for $\psi_\Omega(R)$ in terms of a sum over zeros of the Riemann zeta function: Define

$$\tilde{H}_\Omega(R) := -4 \sum_\rho R^{\rho-1/2}\Big(\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)\Big) \tag{3.1}$$

where the Mellin transforms $\mathcal{I}_j$ are given in (2.6). Then we show that up to a negligible error, $\tilde{H}_\Omega(R)$ coincides with the normalized remainder term $H_\Omega(R) = (\psi_\Omega(R) - \operatorname{area}(\Omega)R^2)/R^{3/2}$:

Proposition 3.2 *Assume RH. Then*

$$\psi_\Omega(R) = \operatorname{area}(\Omega) R^2 + R^{3/2}\tilde{H}_\Omega(R) + O\left(R^{4/3}(\log R)^{7/2}\right).$$

Proof By the approximate explicit formula (see e.g. Davenport [7, §17, eq. (9) and (10)]), for $x \geq 2$ and $T > 1$,

$$\sum_{n\leq x} \Lambda(n) = x - \sum_{\substack{\rho \\ |\gamma|\leq T}} \frac{x^\rho}{\rho} + O\left(\frac{x(\log xT)^2}{T} + \log x\right).$$

Together with the symmetry of Ω, we have

$$\begin{aligned}\frac{1}{4}\psi_\Omega(R) &= \sum_{m\le aR}\Lambda(m)\sum_{n\le Rf(m/R)}\Lambda(n)\\ &= R\sum_{m\le aR}\Lambda(m)f(m/R)\\ &\quad -\sum_{m\le aR}\Lambda(m)\sum_{\substack{\rho\\|\gamma|\le T}}\frac{(Rf(m/R))^\rho}{\rho}+O\left(\frac{R^2(\log R)^2}{T}\right)\\ &=:\ \mathrm{I}+\mathrm{II}+O\left(\frac{R^2(\log R)^2}{T}\right),\end{aligned} \tag{3.2}$$

say, where we assume $1<T\ll R$.

By the partial summation, for $1<T'\ll R$, we have

$$\begin{aligned}\sum_{m\le aR}\Lambda(m)f(m/R)^\rho &= \int_2^{aR} f(u/R)^\rho \mathrm{d}\sum_{m\le u}\Lambda(m)\\ &= \int_2^{aR} f(u/R)^\rho\mathrm{d}\left(u-\sum_{\substack{\rho'\\|\gamma|\le T'}}\frac{u^{\rho'}}{\rho'}\right)+O\left(\frac{R(\log R)^2}{T'}\int_2^{aR}|\mathrm{d}f(u/R)^\rho|\right)\\ &= \int_2^{aR} f(u/R)^\rho\mathrm{d}\left(u-\sum_{\substack{\rho'\\|\gamma|\le T'}}\frac{u^{\rho'}}{\rho'}\right)\\ &\quad + O\left(\frac{R(\log R)^2}{T'}\frac{|\rho|}{R}\int_2^{aR} f\left(\frac{u}{R}\right)^{-1/2}\left|f'\left(\frac{u}{R}\right)\right|\mathrm{d}u\right)\\ &= \int_0^{aR} f(u/R)^\rho\mathrm{d}u-\sum_{\substack{\rho'\\|\gamma'|\le T'}}\frac{1}{\rho'}\int_2^{aR} f(u/R)^\rho\mathrm{d}u^{\rho'}+O\left(\frac{|\rho|R(\log R)^2}{T'}\right).\end{aligned}$$

We have again

$$\begin{aligned}\sum_{\substack{\rho'\\|\gamma'|\le T'}}\frac{1}{\rho'}\int_0^2 f(u/R)^\rho\mathrm{d}u^{\rho'} &= \sum_{\substack{\rho'\\|\gamma'|\le T'}}\frac{1}{\rho'}\left(f(u/R)^\rho u^{\rho'}\Big|_0^2-\int_0^2 u^{\rho'}\mathrm{d}f(u/R)^\rho\right)\\ &= O\left((\log R)^2+|\rho|R^{-1}(\log R)^2\right).\end{aligned}$$

Hence

$$\sum_{m\le aR}\Lambda(m)f(m/R)^{\rho}=R\int_0^a f(v)^{\rho}\mathrm{d}u - R^{\rho'}\sum_{\substack{\rho'\\|\gamma'|\le T'}}\int_0^a f(v)^{\rho}v^{\rho'-1}\mathrm{d}v+O\left(\frac{|\rho|R(\log R)^2}{T'}\right). \tag{3.3}$$

The above asymptotic formula holds for $\rho=1$ by a similar argument.

Note that $R^2\int_0^a f(v)dv=\frac{1}{4}\,\mathrm{area}(\Omega)R^2$. By (3.3) with $\rho=1$, we get

$$\mathrm{I}=\frac{1}{4}\,\mathrm{area}(\Omega)R^2-R\sum_{\substack{\rho\\|\gamma|\le T}}R^{\rho}\mathcal{I}_1(\rho)+O\left(\frac{R^2(\log R)^2}{T}\right). \tag{3.4}$$

with the Mellin transform $\mathcal{I}_1(\rho)$ given by (2.6).

Now we handle the second term. By (3.3) again, we have

$$\begin{aligned}\mathrm{II}&=-\sum_{m\le aR}\Lambda(m)\sum_{\substack{\rho\\|\gamma|\le T}}\frac{R^{\rho}f(u/R)^{\rho}}{\rho}=-\sum_{\substack{\rho\\|\gamma|\le T}}\frac{R^{\rho}}{\rho}\sum_{m\le aR}\Lambda(m)f(u/R)^{\rho}\\&=-R\sum_{\substack{\rho\\|\gamma|\le T}}\frac{R^{\rho}}{\rho}\int_0^a f(v)^{\rho}\mathrm{d}v+\sum_{\substack{\rho\\|\gamma|\le T}}\sum_{\substack{\rho'\\|\gamma'|\le T'}}\frac{R^{\rho+\rho'}}{\rho}\int_0^a f(v)^{\rho}v^{\rho'-1}\mathrm{d}v\\&\quad+O\left(\frac{TR^{3/2}(\log R)^3}{T'}\right).\end{aligned}$$

We change variable $u=f(v)$, so $v=g(u)$, to transform

$$\begin{aligned}\frac{1}{\rho}\int_0^a f(v)^{\rho}\mathrm{d}v&=-\int_0^b\frac{u^{\rho}}{\rho}\mathrm{d}g(u)=-\frac{u^{\rho}}{\rho}g(u)\Big|_0^b+\int_0^b u^{\rho-1}g(u)\mathrm{d}u\\&=\int_0^b u^{\rho-1}g(u)\mathrm{d}u=:\mathcal{I}_2(\rho)\end{aligned}$$

and obtain

$$\mathrm{II}=-R\sum_{\substack{\rho\\|\gamma|\le T}}R^{\rho}\mathcal{I}_2(\rho)+\sum_{\substack{\rho\\|\gamma|\le T}}\sum_{\substack{\rho'\\|\gamma'|\le T'}}\frac{R^{\rho+\rho'}}{\rho}\int_0^a f(v)^{\rho}v^{\rho'-1}\mathrm{d}v + O\Big(\frac{TR^{3/2}(\log R)^3}{T'}\Big). \tag{3.5}$$

Combining (3.2), (3.4), and (3.5), and assuming that $2 < T \le T' \ll R$, we have

$$\psi_\Omega(R) = \operatorname{area}(\Omega)R^2 - 4R\sum_{\substack{\rho\\|\gamma|\le T}} R^\rho\Big(\mathcal{I}_1(\rho)+\mathcal{I}_2(\rho)\Big) + 4\mathcal{S} + O\left(\frac{R^2(\log R)^2}{T} + \frac{TR^{3/2}(\log R)^3}{T'}\right), \tag{3.6}$$

where

$$\mathcal{S} := \sum_{\substack{\rho\\|\gamma|\le T}}\sum_{\substack{\rho'\\|\gamma'|\le T'}} \frac{R^{\rho+\rho'}}{\rho}\int_0^a f(v)^\rho v^{\rho'-1}\mathrm{d}v. \tag{3.7}$$

Then we have

$$\begin{aligned}
\mathcal{S} &= \sum_{\substack{\rho\\|\gamma|\le T}}\sum_{\substack{\rho'\\|\gamma'|\le T'}} \frac{R^{\rho+\rho'}}{\rho\rho'}\left(f(v)^\rho v^{\rho'}\Big|_0^a - \int_0^a \rho f(v)^{\rho-1}f'(v)v^{\rho'}\mathrm{d}v\right)\\
&= -\sum_{\substack{\rho\\|\gamma|\le T}}\sum_{\substack{\rho'\\|\gamma'|\le T'}} \frac{R^{\rho+\rho'}}{\rho'}\int_0^a f(v)^{\rho-1}f'(v)v^{\rho'}\mathrm{d}v\\
&= -\int_0^a f'(v)\sum_{\substack{\rho\\|\gamma|\le T}} R^\rho f(v)^{\rho-1}\sum_{\substack{\rho'\\|\gamma'|\le T'}} \frac{R^{\rho'}}{\rho'}v^{\rho'}\mathrm{d}v.
\end{aligned}$$

We now assume RH, and write the zeros as $\rho = \frac{1}{2} + i\gamma$, $\rho' = \frac{1}{2} + i\gamma'$. By Cauchy–Schwarz,

$$\begin{aligned}
\mathcal{S} &\ll R\left(\int_0^a |f'(v)|\Bigg|\sum_{\substack{\rho\\|\gamma|\le T}} R^{i\gamma} f(v)^{\rho-3/4}\Bigg|^2\mathrm{d}v\right)^{1/2}\\
&\quad\cdot\left(\int_0^a |f'(v)|f(v)^{-1/2}\Bigg|\sum_{\substack{\rho'\\|\gamma'|\le T'}} \frac{R^{i\gamma'}}{\rho'}v^{1/2+i\gamma'}\Bigg|^2\mathrm{d}v\right)^{1/2}\\
&\ll R\left(\int_0^a |f'(v)|\Bigg|\sum_{\substack{\rho\\|\gamma|\le T}} R^{i\gamma} f(v)^{-1/4+i\gamma}\Bigg|^2\mathrm{d}v\right)^{1/2}\\
&\quad\cdot\left((\log R)^4\int_0^a |f'(v)|f(v)^{-1/2}\mathrm{d}v\right)^{1/2}
\end{aligned}$$

on using, for $|v| \le a$,

$$\sum_{\substack{\rho' \\ |\gamma'| \le T'}} \left| \frac{R^{i\gamma'}}{\rho'} v^{1/2+i\gamma'} \right| \ll \sum_{\substack{\rho' \\ |\gamma'| \le T'}} \frac{1}{|\rho'|} \ll (\log T')^2 \ll (\log R)^2.$$

Note that $\int_0^a |f'(v)| f(v)^{-1/2} \mathrm{d}v = -2 f(v)^{1/2} \Big|_0^a = 2\sqrt{b}$. So we have

$$\mathcal{S} \ll R(\log R)^2 \left(\int_0^a |f'(v)| f(v)^{-1/2} \sum_{\substack{\rho \\ |\gamma| \le T}} \sum_{\substack{\rho' \\ |\gamma'| \le T}} (R f(v))^{i(\gamma-\gamma')} \mathrm{d}v \right)^{1/2}$$

$$\ll R(\log R)^2 \left(\sum_{\substack{\rho \\ |\gamma| \le T}} \sum_{\substack{\rho' \\ |\gamma'| \le T}} R^{i(\gamma-\gamma')} \int_0^a f(v)^{-1/2+i(\gamma-\gamma')} (-f'(v)) \mathrm{d}v \right)^{1/2}.$$

Hence we obtain

$$\mathcal{S} \ll R(\log R)^2 \left(\sum_{\substack{\rho \\ |\gamma| \le T}} \sum_{\substack{\rho' \\ |\gamma'| \le T}} \left| \int_0^a f(v)^{-1/2+i(\gamma-\gamma')} \mathrm{d} f(v) \right| \right)^{1/2}$$

$$\ll R(\log R)^2 \left(\sum_{\substack{\rho \\ |\gamma| \le T}} \sum_{\substack{\rho' \\ |\gamma'| \le T}} \frac{1}{1 + |\gamma - \gamma'|} \right)^{1/2} \ll R T^{1/2} (\log R)^{7/2}.$$

Now by taking $T = R^{2/3}$ and $T' = R^{5/6}$, we have

$$\psi_\Omega(R) = \operatorname{area}(\Omega) R^2 - 4R \sum_{\substack{\rho \\ |\gamma| \le R^{2/3}}} R^\rho \Big(\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho) \Big) + O\left(R^{4/3} (\log R)^{7/2} \right).$$

Using Lemma 2.3, we may extend the sum over all zeros, introducing an error of $O(R^{7/6+o(1)})$ which is negligible relative to the other remainders. Thus define $\tilde{H}_\Omega(R)$ as in (3.1). Then we have

$$\psi_\Omega(R) = \operatorname{area}(\Omega) R^2 + R^{3/2} \tilde{H}_\Omega(R) + O\left(R^{4/3} (\log R)^{7/2} \right).$$

This completes the proof. □

Corollary 3.3 *Assuming RH, we have*

$$\psi_\Omega(R) = \text{area}(\Omega)R^2 + O(R^{3/2}).$$

Proof Indeed, from RH, $|R^{\rho-1/2}| = 1$ and inserting that into the definition (3.1) of $\tilde{H}_\Omega(R)$ and using Lemma 2.3 shows that $\tilde{H}_\Omega(R) = O(1)$. The statement then follows from Proposition 3.2. □

4 The value distribution function p_Ω

We now compare the empirical remainder term

$$H_\Omega(R) = \frac{\psi_\Omega(R) - \text{area}(\Omega)R^2}{R^{3/2}}$$

with the sum (3.1)

$$\tilde{H}_\Omega(R) = -4\sum_\rho R^{\rho-\frac{1}{2}}\Big(\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)\Big)$$

the sum over the nontrivial zeros of the Riemann zeta function. Assuming the Riemann Hypothesis, we write them as $\rho = \frac{1}{2} + i\gamma$, $\gamma \in \mathbb{R}$. As an immediate consequence of Proposition 3.2 we obtain

Lemma 4.1 *The (logarithmic) value distributions of H_Ω and of $\tilde{H}_\Omega$ coincide.*

Therefore, the logarithmic value distribution of H_Ω is the (ordinary) value distribution of the sum

$$h_\Omega(t) = \tilde{H}_\Omega(e^t) = -4\sum_\rho \Big(\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)\Big)e^{it\gamma}.$$

Noting that

$$\overline{\mathcal{I}_j(\rho)} = \mathcal{I}_j(\bar{\rho})$$

we find that

$$h_\Omega(t) = -4\sum_\rho \Big(\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)\Big)e^{it\gamma} = \sum_{\gamma>0} \mathcal{B}_\Omega(\gamma)\cos(t\gamma + \varphi_\gamma) \tag{4.1}$$

where the sum is over zeros with positive imaginary part $\gamma > 0$, and

$$\mathcal{B}_\Omega(\gamma) := 8\Big|\mathcal{I}_1(\rho) + \mathcal{I}_2(\rho)\Big|, \quad \varphi_\gamma := \arg\Big(-\mathcal{I}_1(\rho) - \mathcal{I}_2(\rho)\Big). \tag{4.2}$$

According to Lemma 2.3,

$$\mathcal{B}_\Omega(\gamma) \ll \gamma^{-3/2}, \quad \gamma \to +\infty.$$

Since the n th zero $\gamma_n \approx n/\log n$ by the Riemann–von Mangoldt formula, we see that the sum (4.1) is absolutely convergent, and defines a uniformly almost periodic function, hence has a limiting value distribution measure, (see e.g. [3, Theorem 4.1]), as follows from applying the Kronecker–Weyl ergodic theorem:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T G\Big(h_\Omega(t)\Big)\mathrm{d}t=\int_{-A}^{A}G(u)\mathrm{d}\mu_\Omega(u)$$

for all bounded continuous functions on $[-A,A]$, where

$$A=\max_t |h_\Omega(t)|.$$

Note that by Lemma 2.4, we know that $\mathcal{B}_\Omega(\gamma)$ is nonzero infinitely often.

We now assume the Linear Independence Hypothesis. Wintner [24] studied the value distribution of a sum of infinitely many cosine waves with incommensurate frequences

$$h(t):=\sum_{n=1}^{\infty}a_n\cos(\gamma_n t-\varphi_n)$$

where $a_n>0$, with $A:=\sum_n a_n<\infty$, and $\{\gamma_n\}$ are linearly independent over the rationals, showing that there is a smooth[1] value distribution function $p(u)$, whose characteristic function is given by

$$\int_{-A}^{A}p(u)e^{isu}\mathrm{d}u=\prod_{n=1}^{\infty}J_0(a_n s) \tag{4.3}$$

and that the value distribution is even: $p(u)=p(-u)$.

A form of Nyquist's Sampling Theorem gives a formula for the probability distribution function $p(u)$ that is useful for computational purposes, compare [1, equation (25)].[2]

Lemma 4.2 *Let*

$$h(t)=\sum_{n=1}^{\infty}a_n\cos(\gamma_n t-\varphi_n)$$

where $a_n>0$, with $A:=\sum_n a_n<\infty$, and $\{\gamma_n\}$ are linearly independent over the rationals. Then the value distribution function $p(u)$ of h is smooth and even, and given for $|u|\le A$ by the convergent Fourier series

$$p(u)=\frac{1}{2A}+\frac{1}{A}\sum_{k=1}^{\infty}\left(\prod_{n=1}^{\infty}J_0\left(\frac{\pi k a_n}{A}\right)\right)\cos\left(\frac{\pi k u}{A}\right),\quad |u|\le A \tag{4.4}$$

[1] Smoothness breaks down if we only take a finite sum.

[2] There is an unfortunate typo in [1, equation (25)].

and $p(u) = 0$ *outside the interval* $[-A, A]$.

Proof We define a new, $2A$-periodic function on whole real line by

$$p_{\text{per}}(u) := \sum_{\ell \in \mathbb{Z}} p(u + 2A\ell)$$

which is still smooth, and coincides with $p(u)$ on $[-A, A]$. The Fourier coefficients of $p_{\text{per}}(u)$ are

$$\widehat{p}_{\text{per}}(k) = \frac{1}{2A} \int_{-A}^{A} p_{\text{per}}(u) e^{-2\pi i k \frac{u}{2A}} \mathrm{d}u = \frac{1}{2A} \int_{-\infty}^{\infty} p(u) e^{-2\pi i k \frac{u}{2A}} \mathrm{d}u = \frac{1}{2A} \widehat{p}\left(\frac{k}{2A}\right)$$

where $\widehat{p}(\xi) = \int_{-\infty}^{\infty} p(u) e^{-2\pi i u \xi} \mathrm{d}u$ is the Fourier transform of p. In particular $\widehat{p}_{\text{per}}(0) = 1/(2A)$. Thus we have

$$p_{\text{per}}(u) = \frac{1}{2A} + \sum_{k \neq 0} \frac{1}{2A} \widehat{p}\left(\frac{k}{2A}\right) e^{i\pi k \frac{u}{A}}.$$

The expansion converges pointwise because $p_{\text{per}}(u)$ is smooth. Since $p(u)$ is even, we rewrite

$$p_{\text{per}}(u) = \frac{1}{2A} + \frac{1}{A} \sum_{k=1}^{\infty} \widehat{p}\left(\frac{k}{2A}\right) \cos\left(\pi k \frac{u}{A}\right).$$

Due to (4.3), we have $\widehat{p}(s) = \prod_{n=1}^{\infty} J_0(2\pi a_n s)$. Therefore

$$p_{\text{per}}(u) = \frac{1}{2A} + \frac{1}{A} \sum_{k=1}^{\infty} \left(\prod_{n=1}^{\infty} J_0\left(\frac{\pi k a_n}{A}\right) \right) \cos\left(\frac{\pi k u}{A}\right).$$

Now for $|u| < A$, $p_{\text{per}}(u) = p(u)$ and so we obtain the result. $\square$

Corollary 4.3 *Assume RH and Hypothesis LI. Then the logarithmic value distribution function* $p_\Omega(u)$*of* $H_\Omega(R)$ *is given by* (4.4) *with* $\gamma_n > 0$ *being the imaginary parts of the Riemann zeros, and* $a_n = |\mathcal{B}_\Omega(\gamma_n)|$.

Note that $p_\Omega(u)$ is the probability distribution function (PDF) of the random cosine sum

$$g_\Omega(t) = \sum_{n=1}^{\infty} \mathcal{B}_\Omega(\gamma_n) \cos x_n$$

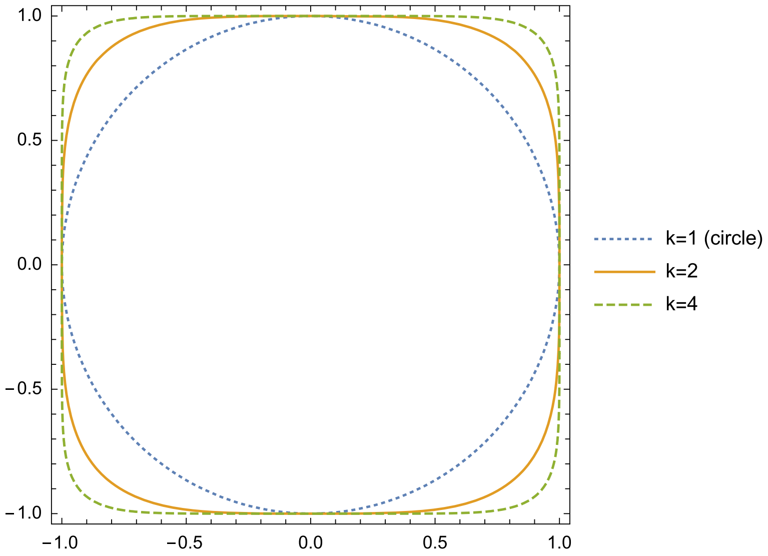

Fig. 4 The superellipse $x^{2k} + y^{2k} \le 1$, for $k = 1$ (a circle), $k = 2$ and $k = 4$

with $x_n \in [0, \pi]$ uniform independent identically distributed (IID) random variables. Random combinations of cosine waves of the form

$$Z = \sum_{n=1}^{N} a_n \cos x_n$$

with $a_n > 0$ and x_n uniform IID random variables, have been studied, starting with Lord Rayleigh [18] in the context of random flights (Pearson's problem of the random walk), where one wants to find the distribution of the sum of N vectors with specified lengths a_n and randomly distributed phases, Z being the real part of the sum of the random vectors $a_n e^{ix_n}$. They were used for in the theory of multi-channel carrier telephony (see [1,20]) or for modeling sea waves (see [8]).

5 Prime points in a superellipse

The superellipse is the planar domain Ω_k bounded by the Lamé curve $x^{2k} + y^{2k} = 1$, where $k \ge 2$ is an integer. The boundary curve $\partial\Omega_k$ is smooth, but the curvature vanishes at the points $(\pm 1, 0)$, $(0, \pm 1)$, which are the vertices of the curve, see Fig. 4.

As noted in the Introduction, the ordinary lattice point count $N_{\Omega_k}(R) = \#\{\mathbb{Z}^2 \cap R\Omega_k\}$ is anomalous in that the remainder term is larger than in ovals, due to the existence of points (namely the vertices) on the boundary $\partial\Omega_k$ where the curvature vanishes to order $2k - 2$ (recall $k \ge 2$), and at which the normal to the curve has rational slope. Indeed, already van der Corput in his thesis (see also [6,17]) showed that the remainder term $N_{\Omega_k}(R) - \mathrm{area}(\Omega_k)R^2$ is as large as $R^{1-1/(2k)}$ for arbitrarily

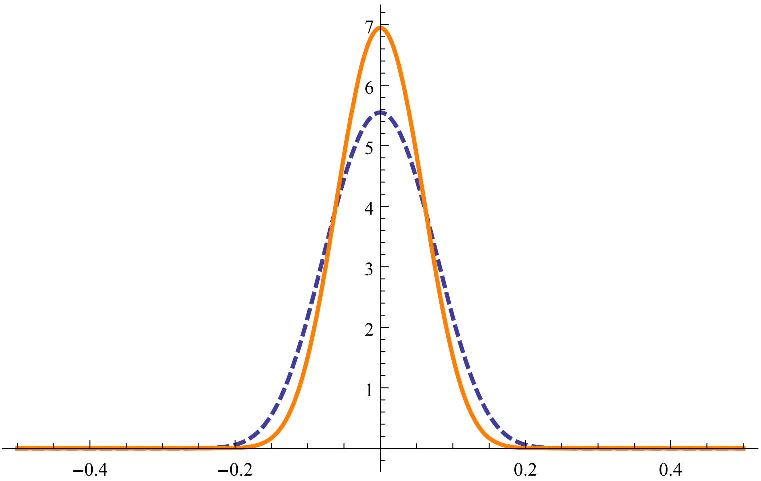

Fig. 5 The value distribution function p_Ω (rescaled) for the superellipse $x^{2k} + y^{2k} \le 1$, for $k = 2$ (dashed) and $k = 4$ (solid)

large R, unlike the upper bound of $O(R^{2/3})$ (and conjecturally $O(R^{1/2+o(1)})$) for ovals.

We examine our prime lattice point count $\psi_{\Omega_k}(R)$ for the superellipse, and find that unlike the ordinary lattice point count, the prime lattice point count behaves in the same way as it does for ovals, namely that

$$\psi_{\Omega_k}(R) = \text{area}(\Omega_k)R^2 + O(R^{3/2})$$

and that the remainder term

$$H_{\Omega_k}(R) = \frac{\psi_{\Omega_k}(R) - \text{area}(\Omega_k)R^2}{R^{3/2}}$$

has a limiting (logarithmic) distribution function, given by a similar formula as for the case of ovals.

Most arguments in Sect. 3 carry over to this case. The only change is in the asymptotic evaluation of the Mellin transforms (2.6) in Lemma 2.3, where nonvanishing curvature at the vertices is used. Here, we can evaluate them directly: The boundary in the positive quadrant is defined as the graph of the function $f(x) = (1 - x^{2k})^{1/(2k)}$ which coincides with its inverse: $g(y) = f(y)$. The Mellin transforms are given by

$$\mathcal{I}_1(\rho) = \mathcal{I}_2(\rho) = \int_0^1 (1 - x^{2k})^{1/(2k)} x^{\rho-1} dx = \frac{1}{2k} B\left(1 + \frac{1}{2k}, \frac{\rho}{2k}\right)$$

and hence

$$\mathcal{I}_1(\rho) \sim \frac{\Gamma(1 + \frac{1}{2k})}{2k} \rho^{-(1+\frac{1}{2k})}$$

by Stirling's formula (note the exponent $1 + \frac{1}{2k}$ is smaller than the exponent $3/2$ obtained in Lemma 2.3 for ovals). Hence we obtain

$$\psi_{\Omega_k}(R) = \text{area}(\Omega_k)R^2 + R^{3/2}\tilde{H}_{\Omega_k}(R) + O\left(R^{4/3}(\log R)^{7/2}\right)$$

with

$$\tilde{H}_{\Omega_k}(R) = -\frac{4}{k}\sum_{\rho} B\left(1 + \frac{1}{2k}, \frac{\rho}{2k}\right)R^{\rho-1/2}.$$

See Fig. 5 for plots of the value distribution p_{Ω_k}, by using (1.1) with 1000 zeros.

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