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Value Distribution for Eigenfunctions of Desymmetrized Quantum Maps

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1 Introduction

In the past few years, much attention has been devoted to the behavior of eigenfunctions of classically chaotic quantum systems. One aspect of this topic concerns their value distribution and specifically their extreme values (see [Be], [AS], [S1], [IS], [HeR], [ABST]). Our aim is to explore this topic for one of the best-studied models in quantum chaotic dynamics—the *quantized cat map* (see [HB]). This is the quantization of a hyperbolic linear map A of the torus. For a brief background about this model, see Section 2.

For each integer $N \geq 1$ (the inverse Planck constant), let $U_N(A)$ denote the quantization of A as a unitary operator on $\mathcal{H}_N = L^2(\mathbf{Z}/N\mathbf{Z})$. In our previous paper [KR], it was observed that there are *quantum symmetries* of $U_N(A)$—a commutative group of unitary operators that commute with $U_N(A)$. We called these *Hecke operators*, in analogy with the classical theory of modular forms. The eigenspaces of $U_N(A)$ thus admit an orthonormal basis consisting of eigenfunctions of all the Hecke operators, which we called *Hecke eigenfunctions*. These can be thought of as the eigenfunctions for the desymmetrized quantum map.

In [KR] we showed that the Hecke eigenfunctions become uniformly distributed as $N \to \infty$. In this note we investigate suprema and value distributions of the Hecke eigenfunctions. For general N, we obtain a nontrivial bound on the supremum norm of these Hecke eigenfunctions. For prime values of N for which A is diagonalizable modulo N (the "split primes" for A), we obtain much more refined, optimal results via the modern

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theory of exponential sums. We show that for these values of N the Hecke eigenfunctions are uniformly bounded and their absolute values (amplitudes) are either constant or have a semicircle value distribution as $N \to \infty$. Moreover, in the latter case different eigenfunctions become statistically independent.

The following is a detailed description of our results.

1.1 Suprema for general N

The trivial bound

$$\|\psi\|_\infty \le N^{1/2} \tag{1}$$

is a consequence of the L^2-normalization

$$\|\psi\|_2^2 = \frac{1}{N} \sum_{Q \in \mathbf{Z}/N\mathbf{Z}} \left|\psi(Q)\right|^2 = 1.$$

Equidistribution of the eigenfunctions (see [KR]) implies that one can do better: $\|\psi\|_\infty = o(N^{1/2})$. Our first result gives a quantitative improvement on this.

Theorem 1. Let ψ be a Hecke eigenfunction normalized by $\|\psi\|_2 = 1$. Then the supremum of ψ satisfies

$$\|\psi\|_\infty \ll_\epsilon N^{3/8+\epsilon}$$

for all $\epsilon > 0$, the implied constant depending only on ϵ and not on ψ. □

1.2 The split primes

We next consider the case when N is a prime for which A is diagonalizable modulo N. (These constitute half the primes.) In this case the group of Hecke operators is isomorphic to the multiplicative group $(\mathbf{Z}/N\mathbf{Z})^*$ and the Hecke eigenfunctions correspond to eigencharacters of the group of Hecke operators, namely, to Dirichlet characters χ mod N. For nontrivial χ, the corresponding eigenspace is 1-dimensional, while for the trivial character χ_0 the eigenspace is 2-dimensional. For nontrivial χ, denote by $\psi_{\chi,N}$ a Hecke eigenfunction of norm 1. For simplicity, we also assume that A is not triangular modulo N (which holds for all but finitely many N). The suprema of the Hecke eigenfunctions in this case are given by the following theorem.

Theorem 2. If N is a split prime for A such that A is not triangular modulo N, then we have the following.

(1) The Hecke eigenspace corresponding to the trivial character has an orthonormal basis $\psi_{0,N}$, $\psi_{\infty,N}$ for which the amplitude is constant: $|\psi_{\infty,N}| = 1$ and $|\psi_{0,N}| = 1/\sqrt{1-1/N}$.

(2) For nontrivial $\chi \bmod N$, $\|\psi_{\chi,N}\|_\infty \le 2/\sqrt{1-1/N}$. □

We finally turn to the question of value distribution. The issue here is only for nontrivial characters χ. We show that the value distribution of $(1/2)|\psi_{\chi,N}|$ tends to the *semicircle measure*, which is the measure μ_{sc} on $[0,1]$ such that

$$\mu_{sc}(I) = \int_I \frac{4}{\pi}\sqrt{1-u^2}\,du$$

for any interval $I \subset [0,1]$. This measure is the image of Haar measure on $SU(2)$ under the map $g \mapsto |\operatorname{tr}(g)|/2$.

The precise result is the following theorem.

Theorem 3. If N is restricted to vary only over split primes for A, then we have the following.

(1) For any nontrivial character $\chi \bmod N$, the amplitude $|\psi_{\chi,N}(t)|$ has a semicircle limit distribution as $N \to \infty$; that is, for any subinterval $I \subset [0,1]$, we have

$$\frac{1}{N}\#\left\{t \bmod N : \frac{|\psi_{\chi,N}(t)|}{2} \in I\right\} \longrightarrow \mu_{sc}(I)$$

as $N \to \infty$ through split primes for A.

(2) For $r \ge 2$ and any choice of distinct nontrivial characters $\chi_1, \dots, \chi_r$, the amplitudes $|\psi_{\chi_1,N}|, \dots, |\psi_{\chi_r,N}|$ are statistically independent; that is, for any choice of subintervals $I_1, \dots, I_r \subset [0,1]$, we have

$$\frac{1}{N}\#\left\{t \bmod N : \frac{|\psi_{\chi_i,N}(t)|}{2} \in I_i, \forall i = 1,\dots,r\right\} \longrightarrow \prod_{i=1}^{r}\mu_{sc}(I_I)$$

as $N \to \infty$ through split primes for A. □

Theorem 3 is a consequence of a corresponding theorem by N. Katz [Ka] on the value distribution of certain exponential sums (see Section 5).

1.3 Comparison with Maass forms

To put our results in perspective, we briefly survey the situation in another well-studied example in quantum chaology—the eigenfunctions of the hyperbolic Laplacian Δ on modular surfaces. These are the quotient $\mathbf{H}/\Gamma$ of the hyperbolic plane $\mathbf{H}$ by a congruence subgroup Γ of the units of a quaternion algebra (in the compact case) or of the modular group $SL(2, \mathbf{Z})$ (see P. Sarnak's survey [S1]). In these cases there is a commuting family of self-adjoint operators that commute with the Laplacian, and thus there is an orthonormal basis ψ_j, $j = 0, 1, \ldots$, of $L^2(\mathbf{H}/\Gamma)$ consisting of (real-valued) eigenfunctions of the Laplacian ($\Delta\psi_j + \lambda_j\psi_j = 0$) and of all the Hecke operators. These are called *Maass-Hecke forms*. To compare with our result, note that the Laplace eigenvalue λ scales with Planck's constant h like $1/h^2$. In the case of the cat map, the inverse Planck constant N equals $1/h$.

A general bound for the eigenfunctions of the Laplacian on any compact Riemannian surface gives (see [Hö])

$$||\psi_j||_\infty \ll \lambda_j^{1/4} \sim \left(\frac{1}{h}\right)^{1/2},$$

which is analogous to trivial bound (1). H. Iwaniec and Sarnak [IS] studied the supremum of the Maass-Hecke forms and showed that for L^2-normalized forms one has

$$||\psi_j||_\infty \ll_\epsilon \sqrt{\lambda_j}^{\,5/12+\epsilon} \sim \left(\frac{1}{h}\right)^{5/12+\epsilon}$$

for all $\epsilon > 0$. This is analogous to our result in Theorem 1.

Unlike in the cat map case, for modular surfaces equidistribution of eigenfunctions is still open, though recent work of Sarnak [S2] establishes this for a subsequence of eigenfunctions of "CM-type" for congruence subgroups of $SL(2, \mathbf{Z})$, and a recent formula of T. Watson [W] confirms that it is implied by the generalized Riemann hypothesis for certain automorphic L-functions.

Concerning the question of value distribution for modular surfaces, numerical experiments indicate that the Hecke eigenfunctions have a locally Gaussian value distribution (see [HeR], [AS]). In the compact case, this means that if ψ is an L^2-normalized cusp form with eigenvalue $\lambda > 0$, which is an eigenfunction of all Hecke operators, then the measure of the set of $z \in \mathbf{H}/\Gamma$, where the amplitude $|\psi(z)| < r$, is asymptotic to $\sqrt{2/\pi}\int_0^r e^{-t^2/2}\,dt$ as $\lambda \to \infty$. Here *measure* means the hyperbolic measure $dx\,dy/y^2$ on $\mathbf{H}/\Gamma$ normalized to have total area unity. Moreover, these experiments indicate that eigenfunctions corresponding to different eigenvalues are statistically independent (see [HeR], [AS]).

At present, we are far from being able to prove such statements for modular surfaces (not to mention doing so for a generic system). Recently, some progress has been made toward Gaussian value distribution. Watson [W] showed that the third moment $\int_{\mathbf{H}/\Gamma} \psi^3$ of the eigenfunctions vanishes as $\lambda \to \infty$. (All odd moments vanish for the Gaussian distribution.) Sarnak showed that for "CM-forms" the fourth moment agrees with the Gaussian moment; that is, $\int_{\mathbf{H}/\Gamma} |\psi|^4 \to 3$ as $\lambda \to \infty$. (Recall that we normalized the total area of $\mathbf{H}/\Gamma$ to be unity.)

2 Background

The full details on the cat map and its quantization can be found in [KR]. For the reader's convenience we briefly recall the setup.

2.1 Classical dynamics

The classical dynamics are given by a hyperbolic linear map $A \in SL(2, \mathbf{Z})$ so that $x = \binom{p}{q} \in \mathbf{T}^2 \mapsto Ax$ is a symplectic map of the torus. Given an observable $f \in C^\infty(\mathbf{T}^2)$, the classical evolution defined by A is $f \mapsto f \circ A$, where $(f \circ A)(x) = f(Ax)$.

2.2 Kinematics: The space of states

As the Hilbert space of states, we take distributions $\psi(q)$ on the line $\mathbf{R}$ which are periodic in both the position and the momentum representation. This restricts h, Planck's constant, to taking only inverse integer values. With $h = 1/N$, the space of states, denoted $\mathcal{H}_N$, is of dimension N and consists of periodic point masses at the coordinates $q = Q/N$, $Q \in \mathbf{Z}$. We identify $\mathcal{H}_N$ with $L^2(\mathbf{Z}/N\mathbf{Z})$, where the inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle \phi, \psi \rangle = \frac{1}{N} \sum_{Q \bmod N} \phi(Q)\overline{\psi}(Q).$$

2.3 Observables

The basic observables are given by the operators $T_N(n_1, n_2)$ acting on $\psi \in L^2(\mathbf{Z}/N\mathbf{Z})$ via

$$\left(T_N(n_1, n_2)\psi\right)(Q) = e^{(i\pi n_1 n_2)/N}\, e\left(\frac{n_2 Q}{N}\right)\psi(Q + n_1). \tag{2}$$

For any smooth classical observable $f \in C^\infty(\mathbf{T}^2)$ with Fourier expansion

$$f(x) = \sum_{n_1, n_2 \in \mathbf{Z}} \widehat{f}(n_1, n_2) e(n_1 p + n_2 q), \quad x = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{T}^2,$$

its quantization, $\mathrm{Op}_N(f)$, is given by

$$\mathrm{Op}_N(f) := \sum_{n_1, n_2 \in \mathbf{Z}} \widehat{f}(n_1, n_2) T_N(n_1, n_2).$$

2.4 Dynamics

We let $\Gamma(4, 2N) \subset SL(2, \mathbf{Z})$ be the subgroup of matrices that are congruent to the identity matrix modulo 4 (resp., 2) if N is even (resp., odd). For $A \in \Gamma(4, 2N)$, we can assign unitary operators $U_N(A)$, acting on $L^2(\mathbf{Z}/N\mathbf{Z})$, having the following important properties.

• For all observables $f \in C^\infty(\mathbf{T}^2)$,

$$U_N(A)^{-1} \mathrm{Op}_N(f) U_N(A) = \mathrm{Op}_N(f \circ A).$$

• The quantization depends only on $A \bmod 2N$. If $A, B \in \Gamma(4, 2N)$ and $A \equiv B \bmod 2N$, then

$$U_N(A) = U_N(B).$$

• The quantization is multiplicative. If $A, B \in \Gamma(4, 2N)$, then

$$U_N(AB) = U_N(A) U_N(B). \tag{3}$$

2.5 Hecke eigenfunctions

If α is an eigenvalue of A, form $\mathfrak{O} = \mathbf{Z}[\alpha]$, which is an *order* in the real quadratic field $K = \mathbf{Q}(\alpha)$. (Note that $\mathfrak{O}$ is not necessarily equal to $\mathfrak{O}_K$, the full ring of integers in K.) Let $v = (v_1, v_2) \in \mathfrak{O}^2$ be a vector such that $vA = \alpha v$. Let $I := \mathbf{Z}[v_1, v_2] \subset \mathfrak{O}$. Then I is an $\mathfrak{O}$-ideal, and the matrix of α acting on I by multiplication in the basis v_1, v_2 is precisely A. Choosing the basis of I gives an identification

$$\iota : I \longrightarrow \mathbf{Z}^2. \tag{4}$$

The action of $\mathfrak{O}$ on the ideal I by multiplication gives a ring homomorphism $\iota : \mathfrak{O} \to \mathrm{Mat}_2(\mathbf{Z})$ with the property that the determinant of $\iota(\beta)$, $\beta \in \mathfrak{O}$, is given by $\mathcal{N}(\beta)$, where $\mathcal{N} : \mathbf{Q}(\alpha) \to \mathbf{Q}$ is the norm map.

Reducing the norm map modulo 2N gives a well-defined map

$$\mathcal{N}_{2N} : \mathfrak{O}/2N\mathfrak{O} \longrightarrow \mathbf{Z}/2N\mathbf{Z},$$

and we let ${\mathcal{C}_A}^{\theta}(2N)$ be the elements in the kernel of this map which are congruent to $1 \bmod 4\mathfrak{O}$ (resp., $2\mathfrak{O}$) if N is even (resp., odd).

Now, reducing $\iota \bmod 2N$ gives a map

$$\iota_{2N} : {\mathcal{C}_A}^{\theta}(2N) \longrightarrow \mathrm{SL}_2\left(\mathbf{Z}/2N\mathbf{Z}\right).$$

Since ${\mathcal{C}_A}^{\theta}(2N)$ is commutative, the properties in Section 2.4 imply that

$$\left\{U_N(\iota_{2N}(\beta)) : \beta \in {\mathcal{C}_A}^{\theta}\right\}$$

forms a family of commuting operators. Analogously with modular forms, we call these *Hecke operators*, and functions $\psi \in \mathcal{H}_N$ which are simultaneous eigenfunctions of all the Hecke operators are denoted *Hecke eigenfunctions*. Note that a Hecke eigenfunction is an eigenfunction of $U_N(\iota_{2N}(\alpha)) = U_N(A)$.

3 Proof of Theorem 1

3.1 Spectral expansions

We first display the intensity $|\psi(Q)|^2$ for $\psi \in \mathcal{H}_N$ as an expectation value of an N-dependent observable. Choose $f \in C_c^{\infty}(\mathbf{R})$ so that $f(0) = 1$, $\int_{-\infty}^{\infty} f(x)\,dx = 0$, and f is supported in $(-1/2, 1/2)$. The function

$$G_N(x) = N \sum_{k \in \mathbf{Z}} f\big(N(x-k)\big)$$

is periodic, and its Fourier coefficients are given by

$$\widehat{G_N}(m) = \widehat{f}\left(\frac{m}{N}\right),$$

where $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\,dx$ is the Fourier transform of f on the line. For $Q \in \mathbf{Z}/N\mathbf{Z}$, set

$$g_{N,Q}(p,q) = G_N\left(q - \frac{Q}{N}\right), \quad (p,q) \in \mathbf{T}^2.$$

We obtain a function on the torus which is independent of the momentum variable p and is strongly localized in the position variable q around Q/N.

Lemma 4. Let $Q \in \mathbb{Z}/N\mathbb{Z}$. Then, for all $\psi \in \mathcal{H}_N$,

$$\left|\psi(Q)\right|^2 = \langle \operatorname{Op}_N(g_{N,Q})\psi, \psi\rangle. \qquad \square$$

Proof. Recall that

$$\operatorname{Op}_N(g_{N,Q}) = \sum_{m,n\in\mathbb{Z}} \widehat{g_{N,Q}}(m,n) T_N(m,n).$$

Since $g_{N,Q}$ is independent of p, we have $\widehat{g_{N,Q}}(m,n) = 0$ unless $m = 0$, in which case we have

$$\widehat{g_{N,Q}}(0,n) = e\left(-\frac{nQ}{N}\right)\widehat{G}_N(n) = e\left(-\frac{nQ}{N}\right)\widehat{f}\left(\frac{n}{N}\right). \tag{5}$$

Since $T_N(0,n)\psi(Q') = e(nQ'/N)\psi(Q')$, we get

$$\begin{aligned}
\operatorname{Op}_N(g_{N,Q})\psi(Q') &= \sum_{m,n\in\mathbb{Z}} \widehat{g_{N,Q}}(m,n) T_N(m,n)\psi(Q') \\
&= \sum_{n\in\mathbb{Z}} \widehat{g_{N,Q}}(0,n) T_N(0,n)\psi(Q') \\
&= \sum_{n\in\mathbb{Z}} \widehat{g_{N,Q}}(0,n) e\left(\frac{nQ'}{N}\right)\psi(Q') \\
&= g_{N,Q}(0,Q')\psi(Q') = G_N\left(\frac{Q'-Q}{N}\right)\psi(Q').
\end{aligned}$$

Since the support of $G_N(x)$ is contained in $(-1/2N, 1/2N) \bmod 1$, and $G_N(0) = N$, we get

$$\operatorname{Op}_N(g_{N,Q})\psi(Q') = \begin{cases} N\psi(Q) & \text{if } Q' = Q \bmod N, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\langle \operatorname{Op}_N(g_{N,Q})\psi, \psi\rangle = \frac{1}{N}\sum_{Q' \bmod N} G_N\left(\frac{Q'-Q}{N}\right)\left|\psi(Q')\right|^2 = \left|\psi(Q)\right|^2. \qquad \blacksquare$$

Lemma 5. Let $\psi_1, \ldots, \psi_N \in \mathcal{H}_N$. Then, for all $Q \in \mathbb{Z}/N\mathbb{Z}$,

$$\sum_{j=1}^{N} \left|\psi_j(Q)\right|^8 \le \left(\sum_{n\in\mathbb{Z}} \left|\widehat{f}\left(\frac{n}{N}\right)\right| \left(\sum_{j=1}^{N} \left|\langle T_N(0,n)\psi_j, \psi_j\rangle\right|^4\right)^{1/4}\right)^4. \qquad \square$$

Proof. For ease of notation we put

$$t_j(n) = \left|\left\langle T_N(0,n)\psi_j, \psi_j\right\rangle\right|.$$

By Lemma 4 and identity (5),

$$\begin{aligned}
\left|\psi_j(Q)\right|^2 &= \left\langle \operatorname{Op}_N\left(g_{N,Q}\right)\psi_j, \psi_j\right\rangle \\
&\le \sum_{n\in Z} \left|\widehat{g_{N,Q}}(0,n)\right| t_j(n) \\
&= \sum_{n\in Z} \left|\widehat{f}\left(\frac{n}{N}\right)\right| t_j(n).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{N} \left|\psi_j(Q)\right|^8 &\le \sum_{j=1}^{N} \left(\sum_{n\in Z} \left|\widehat{f}\left(\frac{n}{N}\right)\right| t_j(n)\right)^4 \\
&= \sum_{j=1}^{N} \sum_{n_1,n_2,n_3,n_4\in Z} \prod_{k=1}^{4} \left|\widehat{f}\left(\frac{n_k}{N}\right)\right| t_j(n_k).
\end{aligned}$$

Applying the Cauchy-Schwartz inequality twice, we get that

$$\begin{aligned}
\sum_{j=1}^{N} t_j(n_1)t_j(n_2)t_j(n_3)t_j(n_4) &\le \sqrt{\sum_{j=1}^{N} t_j(n_1)^2 t_j(n_2)^2}\sqrt{\sum_{j=1}^{N} t_j(n_3)t_j(n_4)} \\
&\le \prod_{k=1}^{4}\left(\sum_{j=1}^{N} t_j(n_k)^4\right)^{1/4},
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{j=1}^{N} \left|\psi_j(Q)\right|^8 &\le \sum_{n_1,n_2,n_3,n_4\in Z} \prod_{k=1}^{4} \left(\left|\widehat{f}\left(\frac{n_k}{N}\right)\right| \left(\sum_{j=1}^{N} t_j(n_k)^4\right)^{1/4}\right) \\
&= \left(\sum_{n\in Z} \left|\widehat{f}\left(\frac{n}{N}\right)\right| \left(\sum_{j=1}^{N} t_j(n)^4\right)^{1/4}\right)^4. \qquad \blacksquare
\end{aligned}$$

3.2 A counting problem

Recall that we identified the action of the matrix A on $\mathbf{Z}^2$ with multiplication by its eigenvalue α on the ideal $I \subseteq \mathbf{Z}[\alpha]$. Let $\iota : I \to \mathbf{Z}^2$ be the identification given in (4). We need the following proposition (see [KR, Proposition 11]).

Proposition 6. Fix $\nu \in I$. Then, for any orthonormal basis of Hecke eigenfunctions ψ_j,

$$\sum_{j=1}^{N} \left|\left\langle T_N(\iota(\nu))\psi_j, \psi_j \right\rangle\right|^4 \leq \frac{N}{\left|\mathcal{C}_A{}^{\theta}(2N)\right|^4} \operatorname{Sol}(N, \nu),$$

where $\operatorname{Sol}(N, \nu)$ is the number of solutions of

$$\nu\left(\beta_1 - \beta_2 + \beta_3 - \beta_4\right) \equiv 0 \mod N \tag{6}$$

with $\beta_1, \ldots, \beta_4$ in $\mathcal{C}_A{}^{\theta}(N)$. $\square$

It was also shown that (6) has fewer than $N^{2+\epsilon}$ solutions for ν fixed as N tends to infinity. However, in the current setup, we need to make the dependence on ν more explicit.[1] Recall that α is an eigenvalue of A and that $\mathcal{N} : \mathbf{Q}(\alpha) \to \mathbf{Q}$ is the norm map.

Lemma 7. We have

$$\sum_{j=1}^{N} \left|\left\langle T_N(0, n)\psi_j, \psi_j \right\rangle\right|^4 \leq \gcd(n, N)^2 N^{-1+\epsilon}. \qquad \square$$

Proof. It is sufficient to show that (6) has fewer than

$$\gcd(n, N)^2 N^{2+\epsilon}$$

solutions. We argue as in [KR, Section 7] except for a small twist. If $(0, 1) \in \mathbf{Z}^2$ corresponds to $\omega \in I$, then $(0, n)$ corresponds to $\nu = n\omega$ since the action is $\mathbf{Z}$-linear. We may thus make ν Galois stable by multiplying by $\overline{\omega}$. Since ω is in $\mathbf{Z}[\alpha]$, the number of solutions to

$$\nu\left(\beta_1 - \beta_2 + \beta_3 - \beta_4\right) \equiv 0 \mod N, \ \beta_i \in \mathcal{C}_A{}^{\theta}(N)$$

is bounded by the number of solutions of

$$n\mathcal{N}(\omega)\left(\beta_1 - \beta_2 + \beta_3 - \beta_4\right) \equiv 0 \mod N, \ \beta_i \in \mathcal{C}_A{}^{\theta}(N),$$

which in turn is given by the number of solutions to

$$\beta_1 - \beta_2 + \beta_3 - \beta_4 \equiv 0 \mod \left(\frac{N}{\gcd\left(N, n\mathcal{N}(\omega)\right)}\right), \ \beta_i \in \mathcal{C}_A{}^{\theta}(N). \tag{7}$$

Let $N' = N/\gcd(N, n\mathcal{N}(\omega))$. The number of solutions of (7) then equals the product of

[1] If $n \equiv 0 \mod N$, then there are about N^4 solutions; it is essential to control the contribution from such terms.

- the number of solutions to

$$\beta_1' - \beta_2' + \beta_3' - \beta_4' \equiv 0 \mod N', \ \beta_i' \in \mathcal{C}_A{}^{\theta}(N'),$$

where $\beta_1', \ldots, \beta_4'$ ranges over all elements in $\mathcal{C}_A{}^{\theta}(N')$, and

- the number of elements $(\beta_1, \beta_2, \beta_3, \beta_4) \in \mathcal{C}_A{}^{\theta}(N)^4$ which reduce to the same element $(\beta_1', \beta_2', \beta_3', \beta_4') \in \mathcal{C}_A{}^{\theta}(N')^4$.

From [KR, Proposition 14], applied with $\nu = 1$, it follows that the first term is less than $(N')^{2+\epsilon}$. For the second term, [KR, Lemma 20] gives that the cardinality of the cokernel of the reduction map $\mathcal{C}_A{}^{\theta}(N) \to \mathcal{C}_A{}^{\theta}(N')$ is uniformly bounded in N. Hence there are less than $(\mathcal{C}_A{}^{\theta}(N)/\mathcal{C}_A{}^{\theta}(N'))^4$ elements $(\beta_1, \beta_2, \beta_3, \beta_4)$ that reduce to $(\beta_1', \beta_2', \beta_3', \beta_4') \mod N'$.

Finally, from [KR, Lemma 8] we have

$$N^{1-\epsilon} \ll \left|\mathcal{C}_A{}^{\theta}(N)\right| \ll N^{1+\epsilon},$$

and thus

$$\frac{\mathcal{C}_A{}^{\theta}(N)}{\mathcal{C}_A{}^{\theta}(N')} \ll \left(\frac{N}{N'}\right)^{1+\epsilon}.$$

The number of solutions is therefore bounded by

$$(N')^{2+\epsilon}\left(\frac{N}{N'}\right)^{4+\epsilon} \ll N^{2+\epsilon} \gcd\left(n\mathcal{N}(\omega), N\right)^{2+\epsilon} \ll N^{2+2\epsilon} \gcd(n, N)^2. \qquad \blacksquare$$

Lemma 8. For all $\epsilon > 0$,

$$\sum_{n \in Z} \left|\widehat{f}\left(\frac{n}{N}\right)\right| \gcd(n, N)^{1/2} \ll_\epsilon N^{1+\epsilon}. \qquad \square$$

Proof. Let $R = 1/\epsilon$. Since f is smooth and compactly supported, $|\widehat{f}(n/N)| \ll 1$ for $|n| \leq N^{1+1/R}$, and $|\widehat{f}(n/N)| \ll 1/(n/N)^R$ for $|n| \geq N^{1+1/R}$.

Trivially, $\gcd(n, N)^{1/2} \leq N$, so

$$\begin{aligned}
\sum_{|n| \geq N^{1+1/R}} \left|\widehat{f}\left(\frac{n}{N}\right)\right| \gcd(n, N)^{1/2} &\ll \sum_{|n| \geq N^{1+1/R}} \frac{N}{\left(\frac{n}{N}\right)^R} \\
&= N^{R+1} \sum_{|n| \geq N^{1+1/R}} n^{-R} \\
&\ll \frac{N^{R+1}}{(N^{1+1/R})^{R-1}} \\
&= N^{1+1/R}.
\end{aligned}$$

For the sum over small n, we have

$$\sum_{|n|\le N^{1+1/R}} \left|\hat{f}\left(\frac{n}{N}\right)\right| \gcd(n,N)^{1/2} \ll \sum_{|n|\le N^{1+1/R}} \gcd(n,N)^{1/2} \ll N^{1/R} \sum_{n=0}^{N-1} \gcd(n,N)^{1/2}. \tag{8}$$

We note that

$$\sum_{n=0}^{N-1} \gcd(n,N)^{1/2} \ll N^{1+\epsilon}. \tag{9}$$

Indeed,

$$\sum_{n=0}^{N-1} \gcd(n,N)^{1/2} = \sum_{d|N} d^{1/2} \sum_{n\le N,\gcd(n,N)=d} 1 \le \sum_{d|N} d^{1/2}\frac{N}{d} = N\sum_{d|N} d^{-1/2}.$$

Now, $\sum_{d|N} d^{-1/2}$ is bounded by the number of divisors of N and hence is less than N^{ϵ} for all $\epsilon > 0$.

Therefore from (8) and (9) we get

$$\sum_{|n|\le N^{1+1/R}} \left|\hat{f}\left(\frac{n}{N}\right)\right| \gcd(n,N)^{1/2} \ll \sum_{|n|\le N^{1+1/R}} \gcd(n,N)^{1/2} \ll N^{1/R+1+\epsilon}. \qquad \blacksquare$$

3.3 Conclusion of the proof

Let $\{\psi_j\}_{i=1}^N$ be an orthonormal basis of $\mathcal{H}_N$ such that $\psi_1 = \psi$ and each ψ_j is a Hecke eigenfunction. We then bound $|\psi(Q)|^8$ trivially by the sum $\sum_{j=1}^N |\psi_j(Q)|^8$.

By Lemma 5,

$$\sum_{j=1}^N |\psi_j(Q)|^8 \le \left(\sum_{n\in Z}\left|\hat{f}\left(\frac{n}{N}\right)\right|\left(\sum_{j=1}^N \left|\left\langle T_N(0,n)\psi_j,\psi_j\right\rangle\right|^4\right)^{1/4}\right)^4,$$

and from Lemma 7 we have

$$\sum_{j=1}^N \left|\left\langle T_N(0,n)\psi_j,\psi_j\right\rangle\right|^4 \le \gcd(n,N)^2 N^{-1+\epsilon}.$$

Lemma 8 then gives that

$$\sum_{j=1}^{N} \left|\psi_j(Q)\right|^8 \ll N^{-1+\epsilon} \left(\sum_{n\in Z} \left|\widehat{f}\left(\frac{n}{N}\right)\right| \gcd(n,N)^{1/2} \right)^4$$
$$\ll N^{-1+\epsilon} \left(N^{1+\epsilon}\right)^4 \ll N^{3+5\epsilon},$$

and thus

$$\left|\psi(Q)\right| \ll N^{3/8+\epsilon},$$

as required. ∎

4 Uniform boundedness for split primes

4.1 Explicit construction of Hecke eigenfunctions

Let $N = p$ be an odd "split" prime, that is, a prime p that does not divide $(\operatorname{tr} A)^2 - 4$ such that A is diagonalizable modulo p. We also assume that A is not triangular modulo p, that is, p does not divide any of the off-diagonal entries of A. For such p, we give an explicit construction of the Hecke eigenfunctions. (See [DEGI] for an alternative approach to constructing these.) This construction enables us to prove Theorems 2 and 3.

The condition above implies that $N = p$ is an odd prime that splits in K and does not divide the conductor[2] of $\mathfrak{O}$. Since p does not divide the conductor of $\mathfrak{O}$, we have $\mathfrak{O}/p\mathfrak{O} \simeq \mathfrak{O}_K/p\mathfrak{O}_K$. Moreover, since p splits in K, $(\mathfrak{O}_K/p\mathfrak{O}_K)^\times \simeq (\mathbf{Z}/p\mathbf{Z})^\times \times (\mathbf{Z}/p\mathbf{Z})^\times$, and from [KR, Lemma 19] we get that $\mathcal{C}_A{}^\theta(p) \simeq (\mathbf{Z}/p\mathbf{Z})^\times$. On the other hand, since p is odd, we have $\mathcal{C}_A{}^\theta(2p) \simeq \mathcal{C}_A{}^\theta(p)$; hence $\mathcal{C}_A{}^\theta(2p)$ is a cyclic group of order $p-1$. Let β be the generator of this group, and let $B \in \Gamma(4,2p)$ be congruent to $\iota(\beta) \bmod 2p$. The Hecke operators are then given by $U_N(B^k)$. Since the order of $B \bmod p$ is $p-1$, B can be diagonalized modulo p. (All elements in $(\mathbf{Z}/p\mathbf{Z})^\times$ are $(p-1)$-roots of unity.) Let (B_2, B_p) denote the mod 2 and mod p reductions of B. Since $B \in \Gamma(4,2p)$, we find that B_2 is the identity matrix, and, as B_p is diagonalizable, there exist $D, M \in \Gamma(4,2p)$ such that $D \equiv \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \bmod 2p$ is diagonal and

$$B \equiv MDM^{-1} \mod 2p.$$

Because all matrices lie in $\Gamma(4,2p)$, multiplicativity property (3) implies that $U_N(B) = U_N(M)U_N(D)U_N(M)^{-1}$.

[2]Recall that $\mathfrak{O}$ can be written as $\mathbf{Z} + f\mathfrak{O}_K$; the integer f is called the conductor of $\mathfrak{O}$.

Now, U_N is constructed as a tensor product $U_N = \otimes_{p^k || N} U_{p^k}$, and since B_2 is trivial, we find the action is determined by B_p. Recall from [KR, Section 4] that the action of diagonal matrices on $\psi \in L^2(\mathbb{Z}/p\mathbb{Z})$ is given by

$$\big(U_p(D)\psi\big)(x) = \Lambda_p(t)\psi(tx),$$

where $\Lambda_p(t)$ is the quadratic character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Thus if χ is a character on $(\mathbb{Z}/p\mathbb{Z})^\times$, extended to $\mathbb{Z}/p\mathbb{Z}$ by letting $\chi(0) = 0$, we find that χ is an eigenfunction of $U_p(D)$. We also find that $\delta(x)$, where $\delta(0) = 1$ and $\delta(x) = 0$ for $x \neq 0$, is an eigenfunction.

If f is an eigenfunction of $U_p(D)$, then $U_p(M)f$ is an eigenfunction of $U_p(B)$ since

$$U_p(B)U_p(M)f = U_p(M)U_p(D)U_p(M^{-1})U_p(M)f = U_p(M)U_p(D)f.$$

But $U_p(B)$ generates the group of Hecke operators; hence any Hecke eigenfunction is either the form $U_p(M)\chi$ for χ nontrivial, or a linear combination of $U_p(M)\chi_0$ and $U_p(M)\delta$ for χ_0 trivial.

4.2 A reduction to exponential sums

We first note that M is not upper triangular modulo p; otherwise, the same would hold for B. Since A is a power of $B \pmod p$, this would imply that A is upper triangular modulo p, which is contrary to our assumption on p. Thus we may use the Bruhat decomposition of $SL(2, \mathbb{Z}/p\mathbb{Z})$ to write

$$M = \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \tag{10}$$

for some b_1, b_2, t (depending on p). From [KR, Section 4] we obtain that, for $\psi \in L^2(\mathbb{Z}/p\mathbb{Z})$,

$$(U_p(M)\psi)(x) = \Lambda_p(t)\frac{e_p(r_p b_1 x^2)}{\sqrt{p}} \sum_{y=1}^{p} e_p\big(r_p(b_2 y^2 + 2xy)\big)\psi(ty), \tag{11}$$

where r_p is the inverse of $2 \bmod p$ and $e_p(x) := e^{2\pi i x/p}$.

4.2.1 The case $\psi = U_p(M)\chi$. We begin with the following lemma on exponential sums.

Lemma 9. If $r_p \not\equiv 0 \bmod p$, then we have the following.

(1) If χ is nontrivial and $b_2 \neq 0 \bmod p$, then

$$\left|\sum_{y=1}^{p} e_p\left(r_p\left(b_2 y^2 + 2xy\right)\right)\chi(y)\right| \leq 2\sqrt{p}.$$

(2) If $\chi = \chi_0$ is trivial or $b_2 = 0 \bmod p$, then

$$\left|\sum_{y=1}^{p-1} e_p\left(r_p\left(b_2 y^2 + 2xy\right)\right)\chi(y)\right| = \sqrt{p}. \qquad \square$$

Proof. If χ is trivial or $b_2 \equiv 0 \bmod p$, then we can express the sum as a classical Gauss sum, and in this case the result is well known. For χ nontrivial, we note that the degree of $r_p(b_2 y^2 + 2xy)$ is coprime to p; hence we may apply Weil's bound (see [We]) on exponential sums (see [Sch , p. 45, Theorem 2G]) to obtain

$$\left|\sum_{y=1}^{p} e_p\left(r_p\left(b_2 y^2 + 2xy\right)\right)\chi(y)\right| \leq 2\sqrt{p}.$$

Note that the bound is independent of the order of χ. ■

Corollary 10. Let $\psi = \sqrt{p/(p-1)}U_p(M)\chi$. Then $\|\psi\|_2 = 1$, $|\psi| \equiv \sqrt{p/(p-1)}$ if χ is the trivial character, and, for nontrivial characters χ,

$$\|\psi\| \leq 2\sqrt{\frac{p}{(p-1)}}. \qquad \square$$

4.2.2 The case $f = U_p(M)\delta$. We begin with the following lemma.

Lemma 11. Let $\psi_{\infty,p} = \sqrt{p}U_p(M)\delta$. Then $\|\psi_{\infty,p}\|_2 = 1 = \|\psi_{\infty,p}\|_\infty$. □

Proof. We have $\|\psi_{\infty,p}\|_2 = 1$ since $\|\delta\|_2^2 = 1/p$ and $U_p(M)$ is unitary. From (11) we get

$$\begin{aligned}\left|\psi_{\infty,p}(x)\right| &= \left|\sqrt{p}(U_p(M)\delta)(x)\right| \\ &= \left|\sqrt{p}\Lambda_p(t)\frac{e_p(r_p b_1 x^2)}{\sqrt{p}}\sum_{y=1}^{p} e_p\left(r_p\left(b_2 y^2 + 2xy\right)\right)\delta(ty)\right| = 1\end{aligned}$$

since $\delta(ty) = 0$ unless $y = 0$. Hence $\|\psi_{\infty,p}\|_\infty = 1$. ■

Theorem 2 follows immediately from Corollary 10 and Lemma 11.

5 Value distribution for split primes

Let p be a split prime for our map A. We assume that A is not triangular modulo p. To prove Theorem 3, we again use (11), which says that we can write the normalized Hecke eigenfunctions $\psi_{\chi,p}$ for nontrivial χ as

$$\psi_{\chi,p}(x) = \frac{\Lambda_p(t_p)e_p(r_p b_{1,p} x^2)}{\sqrt{p-1}} \sum_{y \bmod p} e_p\big(r_p\big(b_{2,p} y^2 + 2xy\big)\big)\chi(y),$$

where r_p is the inverse of $2 \bmod p$, Λ_p is the unique quadratic character $\bmod p$, and t_p, $b_{1,p}$, and $b_{2,p}$ come from Bruhat decomposition (10) of the diagonalizing matrix M_p for A.

Note that $b_{2,p} \neq 0 \bmod p$ for p as in our assumptions, since otherwise from (10) we find that

$$M_p = \begin{bmatrix} -b_{1,p} t_p & \dfrac{1}{t_p} \\ -t_p & 0 \end{bmatrix},$$

and consequently the matrix B is upper triangular:

$$B = MDM^{-1} = \begin{bmatrix} \dfrac{1}{t_p} & b_{1,p}\left(t_p - \dfrac{1}{t_p}\right) \\ 0 & t_p \end{bmatrix}.$$

Since A is a power of $B \bmod p$, this implies that A is also upper triangular, contradicting our assumption on p.

Thus we may express the absolute value of $\psi_{\chi,p}$ in terms of the exponential sums

$$H_p(\chi, R)(t) = \sum_{y \bmod p} e_p\big(R(y^2 + ty)\big)\chi(y).$$

First, define (following [Ka]) a normalization $F_p(\chi, R)(t)$ of these sums as ($R \neq 0 \bmod p$)

$$F_p(\chi, R)(t) = -\frac{e_p\left(\dfrac{Rt^2}{8}\right)}{\sqrt{\chi\left(\dfrac{-1}{2}\right)G(R,\chi)G(R,\Lambda_p)}} H_p(\chi, R)(t),$$

where $G(R,\chi) = \sum_{x \bmod p} \chi(x)e_p(Rx)$ are Gauss sums, and $\sqrt{*}$ denotes any choice of the square root. Then we have

$$\left|\psi_{\chi,p}(x)\right| = \frac{1}{\sqrt{1-\dfrac{1}{p}}}\left|F_p(\chi, r_p b_{2,p})\left(\frac{2x}{b_{2,p}}\right)\right|. \tag{12}$$

Concerning the normalized sums $F_p(\chi, R)(t)$, Katz proved the following value distribution and statistical independence theorem.

Theorem 12 [Ka]**.** Let p be an odd prime, let χ be a nontrivial character mod p, and let $R \neq 0 \bmod p$. Then we have the following.

(1) The normalized sums $F_p(\chi, R)(t)$ are real and take values in the interval $[-2, 2]$.

(2) As $p \to \infty$, the p numbers $\{F_p(\chi, R)(t)/2 : t \bmod p\}$ become equidistributed in $[-1, 1]$ with respect to the semicircle measure $(2/\pi)\sqrt{1-u^2}\, du$.

(3) For any $r \geq 2$ and a choice of r distinct nontrivial characters $\chi_1, \ldots, \chi_r$, the p vectors

$$\left\{ \left(\frac{F_p(\chi_1, R)(t)}{2}, \ldots, \frac{F_p(\chi_r, R)(t)}{2} \right) : t \bmod p \right\}$$

become equidistributed in $[-1, 1]^r$ with respect to the product of the semicircle measures. $\square$

By virtue of relation (12) between the normalized eigenfunctions f_χ and the normalized sums $F_p(\chi, R)$, Theorem 3 is an immediate consequence of Katz's theorem and the following general lemma.

Lemma 13. Let $\{f_p(t), t = 1, \ldots, p\}$ be a sequence of p points in the interval $[0, 1]$ which become equidistributed as $p \to \infty$ with respect to a probability measure $\rho(x)\, dx$ having a continuous density ρ. Suppose $\{g_p(t) : t = 1, \ldots, p\}$ is another sequence of points so that $g_p(t) = \theta_p f_p(t)$ with $\theta_p = 1 + o(1)$. Then $\{g_p(t) : t = 1, \ldots, p\}$ is also equidistributed in $[0, 1]$ with respect to $\rho(x)\, dx$. $\square$

We leave the proof of this as a simple exercise for the reader.

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