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On the Distribution of Lattice Points in Thin Annuli

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1 Introduction

Let $N(t)$ be the number of integer lattice points in a disk of radius t centered at the origin. Thus, $N(t) = \sum_{n \le t^2} r(n)$, where $r(n)$ is the number of ways of writing $n = x^2 + y^2$ as a sum of two squares. As is well known, $N(t)$ is asymptotic to the area πt^2 of the disk. Much effort has gone into understanding the growth of the remainder term. Heath-Brown [9] considered the *distribution* of the normalized remainder term $(N(t)-\pi t^2)/\sqrt{t}$ and proved that it has a limiting value distribution in the sense that there exists a probability distribution function ν such that, for any interval $\mathcal{A}$,

$$\frac{1}{T}\,\mathrm{meas}\left\{t \in [T, 2T] : \frac{N(t)-\pi t^2}{\sqrt{t}} \in \mathcal{A}\right\} \longrightarrow \int_{\mathcal{A}} \nu(x)\,dx, \tag{1.1}$$

where the measure is the ordinary Lebesgue measure. It is known that $\nu(x)$ is not the Gaussian measure; for instance, the tails have been shown to decay roughly like $\exp(-x^4)$ (see [4, 10]).

Bleher, Dyson, and Lebowitz [3, 5, 6] investigated the distribution of a similarly scaled remainder term of the number $N(t,\rho) := N(t+\rho) - N(t)$ of lattice points in an annulus of inner radius t and width $\rho(t)$ depending on t. The "expected" number of points is the area $\pi(2t\rho + \rho^2)$ of the annulus. Define a normalized remainder term by

$$S(t,\rho) := \frac{N(t+\rho) - N(t) - \pi\bigl(2t\rho + \rho^2\bigr)}{\sqrt{t}}. \tag{1.2}$$

The picture that emerges is that there is a number of distinct regimes.

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(1) *The "global", or "macroscopic", regime* $\rho(t) \to \infty$ (*but* $\rho = o(t)$). In such case, Bleher and Lebowitz [6] showed that $S(t,\rho)$ has a limiting distribution with tails which decay roughly as $\exp(-x^4)$. In fact, the distribution is that of the difference of two i.i.d. random variables whose distribution is the limiting distribution of $(N(t) - \pi t^2)/\sqrt{t}$.

(2) *The intermediate, or "mesoscopic", regime* $\rho \to 0$ (*but* $\rho t \to \infty$). The variance of $S(t,\rho)$ is given by [7]

$$\frac{1}{T}\int_T^{2T} \left|S(t,\rho)\right|^2 dt \sim \sigma^2 := 16\rho \log\frac{1}{\rho} \tag{1.3}$$

and Bleher and Lebowitz [6] conjectured that $S(t,\rho)/\sigma$ has a standard Gaussian distribution.

(3) *The "saturation regime":* here $0 < \rho(t) < \infty$ is fixed as $t \to \infty$, where it has been shown [6] that $S(t,\rho)$ has a distribution with rapidly decaying tails. As $\rho \to \infty$, the distribution converges to that found in the macroscopic regime, and as $\rho \to 0$, it converges to the conjectured mesoscopic distribution.

(4) *The local regime* $\rho \approx 1/t$: if the annulus was centered at a generic point rather than at a lattice point, or if we consider "generic" lattices instead of the integer lattice $\mathbb{Z}^2$, then it is consistent with conjectures of Berry and Tabor [1] that the statistics are Poissonian (see [8, 12, 15] for some progress on this, as well as [11, 13, 16]).

In this paper, we prove part of the Gaussian distribution conjecture of Bleher and Lebowitz. We show that $S(t,\rho)$ has a Gaussian distribution when ρ shrinks to zero sufficiently slowly.

Theorem 1.1. If $\rho \to 0$ but $\rho \gg T^{-\delta}$ for all $\delta > 0$, then, for any interval $\mathcal{A}$,

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ t \in [T, 2T] : \frac{S(t,\rho)}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-x^2/2} dx, \tag{1.4}$$

where $\sigma^2 = 16\rho\log(1/\rho)$. □

The structure of the argument is as follows. We replace the sharp counting function $N(t)$ by a smooth counting function $\widetilde{N}_M(t)$ whose smoothness parameter $M = M(T)$ depends on T (note that though t and T are formally independent, we always think of t as being around T). Since we are only interested in $\rho \to 0$, we set $\rho = 1/L$, where $L = L(T)$ tends to infinity with T, and we define the corresponding normalized remainder term to be

$$\widetilde{S}_{M,L}(t) := \frac{\widetilde{N}_M\left(t + \frac{1}{L}\right) - \widetilde{N}_M(t) - \frac{2\pi t}{L} - \frac{\pi}{L^2}}{\sqrt{t}}. \tag{1.5}$$

We compute the moments of $\widetilde{S}_{M,L}(t)$ when t is chosen at random with respect to a smooth measure. We show in Section 3 that the mth moment of $\widetilde{S}_{M,L}/\sigma$ converges to that of a standard normal random variable provided $L \ll T^{\nu(m)}$, with $0 < \nu(m) < 1/(2^{m-1}-1)$. Thus, $\widetilde{S}_{M,L}$ has a normal distribution if $L \to \infty$ but $L \ll T^{\delta}$ for all $\delta > 0$. In Section 4, we show that the variance of the difference $(S(t, 1/L) - \widetilde{S}_{M,L}(t))/\sigma$ goes to zero, and hence, $S(t,\rho)/\sigma$ has a normal distribution with respect to the smooth measure. Finally, we use an approximation argument to pass from smooth measures to the Lebesgue measure used in Theorem 1.1.

2 Smoothing

To obtain Theorem 1.1, we will replace sharp cutoffs by smooth ones. First, we will replace Lebesgue measure with a smooth average of t around T, that is, we pick t at random by taking a smooth function $\omega \geq 0$, of total mass unity, such that both ω and its Fourier transform $\widehat{\omega}$ are rapidly decaying in the sense that for any $A > 2$,

$$\omega(t) \ll \frac{1}{(1+|t|)^A}, \qquad \widehat{\omega}(t) \ll \frac{1}{(1+|t|)^A} \tag{2.1}$$

for all t. (In fact, we also choose ω to be supported on the positive reals as this makes the analysis simpler.)

Define the averaging operator

$$\langle f \rangle = \frac{1}{T}\int_{-\infty}^{\infty} f(t)\omega\left(\frac{t}{T}\right)dt \tag{2.2}$$

(this is the expected value of f with respect to this measure) and let $\mathbb{P}_{\omega,T}$ be the associated probability measure

$$\mathbb{P}_{\omega,T}(f \in \mathcal{A}) = \frac{1}{T}\int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{A}}\big(f(t)\big)\omega\left(\frac{t}{T}\right)dt. \tag{2.3}$$

(Throughout the paper we will extend $N(t)$, $S(t,\rho)$, and similar functions, initially defined for $t > 0$, to the whole real line. Since $\omega(t) = 0$ for $t \leq 0$, we are free to choose whichever extension makes the analysis most simple.)

We will also smooth the edges of the circle and show that this modified counting function has a Gaussian distribution. Let χ be the indicator function of the unit disc, and ψ a smooth, even function on the real line, of total mass unity, whose Fourier transform $\widehat{\psi}$ is smooth and has compact support. Define a rotationally symmetric function Ψ on $\mathbb{R}^2$ by setting $\widehat{\Psi}(\vec{y}) = \widehat{\psi}(|\vec{y}|)$, where $|\vec{y}|$ denotes the standard Euclidean norm of $\vec{y} \in \mathbb{R}^2$, and where the Fourier transform is

$$\widehat{f}(\vec{y}) = \int_{\mathbb{R}^2} f(\vec{x}) e^{-2\pi i \langle \vec{x}, \vec{y} \rangle} d\vec{x} \tag{2.4}$$

with $\langle \vec{x}, \vec{y} \rangle$ the usual Euclidean inner product. For $\epsilon > 0$, set

$$\Psi_\epsilon(\vec{x}) = \frac{1}{\epsilon^2} \Psi\left(\frac{\vec{x}}{\epsilon}\right). \tag{2.5}$$

Now, set $\chi_\epsilon = \chi * \Psi_\epsilon$ to be the convolution of χ and Ψ_ϵ, which is a smoothed indicator function of the unit disc with "fuzziness" of width ϵ in the sense that $0 \le \chi_\epsilon \le 1$, and if ψ (rather than its Fourier transform $\widehat{\psi}$) had compact support, then $\chi - \chi_\epsilon$ would be concentrated in the shell $1 - \epsilon < |\vec{x}| < 1 + \epsilon$. Due to the rapid decay of tails, this is essentially still the case when ψ is in the Schwarz class, as it is for us.

Now take $\epsilon = 1/t\sqrt{M}$, where $M = M(T)$ depends on T and tends to infinity with T, and define a smooth counting function, or smooth linear statistic, by

$$\widetilde{N}_M(t) = \sum_{\vec{n} \in \mathbb{Z}^2} \chi_\epsilon\left(\frac{\vec{n}}{t}\right). \tag{2.6}$$

This counts lattice points in a "fuzzy circle" of radius about t, with fuzziness about $t\epsilon = 1/\sqrt{M}$.

The number of lattice points in a smooth annulus of inner radius t and width ρ is therefore given by $\widetilde{N}_M(t+\rho) - \widetilde{N}_M(t)$. Since we are interested in radii t in an interval $[T, 2T]$, we will in what follows freeze the width of the annulus to be $\rho(T)$ as t varies in $[T, 2T]$ rather than allowing it to vary with t; this will simplify some of the calculations. Furthermore, since henceforth we are only concerned with $\rho \to 0$, we will set $\rho = 1/L$ and let $L(T) \to \infty$ as $T \to \infty$.

Set

$$\widetilde{S}_{M,L} = \frac{\widetilde{N}_M\left(t + \frac{1}{L}\right) - \widetilde{N}_M(t) - \frac{2\pi t}{L} - \frac{\pi}{L^2}}{\sqrt{t}}. \tag{2.7}$$

The width of the smoothed sides of $\widetilde{N}_M$ is $\mathcal{O}(\epsilon t) = \mathcal{O}(1/\sqrt{M})$. In order for $\widetilde{S}_{M,L}$ to approximate $S(t, 1/L)$, it must be that $1/L$ is much larger than the width of the sides, so we insist that $L/\sqrt{M} \to 0$.

We show the following theorem.

Theorem 2.1. Suppose that $M(T)$ and $L(T)$ are increasing to infinity with T such that $M = \mathcal{O}(T^{\delta})$ for all $\delta > 0$ and $L/\sqrt{M} \to 0$, then for any interval $\mathcal{A}$,

$$\lim_{T\to\infty} \mathbb{P}_{\omega,T}\left\{\frac{\widetilde{S}_{M,L}}{\sigma} \in \mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2}dx, \tag{2.8}$$

where $\widetilde{S}_{M,L}$ is given by (2.7) and

$$\sigma^2 = \frac{16\log L}{L}. \tag{2.9}$$

□

Remark 2.2. The arguments given below for the proof of Theorem 2.1 will also prove a central limit theorem for smooth linear statistics in higher dimensions. Defining $\chi_\epsilon = \chi * \Psi_\epsilon$, where χ is the indicator function of the unit ball and Ψ_ϵ is defined in analogy with (2.5), we have a smooth counting function $\widetilde{N}_M(t) := \sum_{\vec{n}\in\mathbb{Z}^d} \chi_\epsilon(\vec{n}/t)$, where, as before, $\epsilon = 1/t\sqrt{M}$.

The asymptotic behaviour of $\widetilde{N}_M(t)$ is given by $c_d t^d$ with c_d the volume of the unit ball in $\mathbb{R}^d$.

It may then be shown that if $M = \mathcal{O}(T^{\delta})$ for all $\delta > 0$, then the distribution of the normalized remainder term $\widetilde{S}_M(t) = (\widetilde{N}_M(t) - c_d t^d)/t^{(d-1)/2}$, when averaged over t around T, weakly converges to a Gaussian with mean zero and variance

$$\sigma^2 = \begin{cases} \dfrac{2}{\pi^2}K_3 \log M & \text{when } d = 3, \\ \dfrac{d-1}{\pi^2}K_d \displaystyle\int_0^\infty y^{d-4}\widehat{\psi}(y)^2 dy M^{(d-3)/2} & \text{when } d \geq 4, \end{cases} \tag{2.10}$$

where

$$K_d = \frac{4^{d-1}\pi^{d-1/2}}{2^d - 1}\frac{\Gamma\left(\frac{1}{2}d - \frac{1}{2}\right)}{\Gamma(d)\Gamma\left(\frac{1}{2}d\right)}\frac{\zeta(d-1)}{\zeta(d)}. \tag{2.11}$$

3 The distribution of $\widetilde{N}_M$

Lemma 3.1. As $t \to \infty$,

$$\widetilde{N}_M(t) = \pi t^2 - \frac{\sqrt{t}}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n^{3/4}}\cos\left(2\pi t\sqrt{n} + \frac{1}{4}\pi\right)\widehat{\psi}\left(\sqrt{\frac{n}{M}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \tag{3.1}$$

with the error term independent of M. □

Proof. By Poisson summation,

$$\widetilde{N}_M(t) := \sum_{\vec{n}\in\mathbb{Z}^2} (\chi * \Psi_\epsilon)\left(\frac{\vec{n}}{t}\right) = t^2 \sum_{\vec{k}\in\mathbb{Z}^2} \widehat{\chi}(t\vec{k})\widehat{\Psi}_\epsilon(t\vec{k}). \tag{3.2}$$

Changing into polar coordinates and using the fact that χ is rotationally symmetric, the 2-dimensional Fourier transform of χ is

$$\widehat{\chi}(\vec{y}) = \int_0^1 r \int_0^{2\pi} e^{-2\pi i r|\vec{y}|\cos\theta}\, d\theta dr = \frac{-\cos\left(2\pi|\vec{y}| + \frac{1}{4}\pi\right)}{\pi|\vec{y}|^{3/2}} + \mathcal{O}\left(\frac{1}{|\vec{y}|^{5/2}}\right) \tag{3.3}$$

as $|\vec{y}| \to \infty$. By its definition in (2.5), $\widehat{\Psi}_\epsilon(\vec{y}) = \widehat{\Psi}(\epsilon\vec{y}) = \widehat{\psi}(\epsilon|\vec{y}|)$. Therefore, inserting this into (3.2), treating the mean (when $\vec{k} = \vec{0}$) separately, and setting $\epsilon = 1/t\sqrt{M}$, we get that

$$\begin{aligned} \widetilde{N}_M(t) &= \pi t^2 - \frac{\sqrt{t}}{\pi} \sum_{\vec{k}\neq\vec{0}} \left\{ \frac{\cos\left(2\pi t|\vec{k}| + \frac{1}{4}\pi\right)}{|\vec{k}|^{3/2}} \widehat{\psi}(\epsilon t|\vec{k}|) + \mathcal{O}\left(\frac{1}{t}\frac{\widehat{\psi}(\epsilon t|\vec{k}|)}{|\vec{k}|^{5/2}}\right) \right\} \\ &= \pi t^2 - \frac{\sqrt{t}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \cos\left(2\pi t\sqrt{n} + \frac{1}{4}\pi\right) \widehat{\psi}\left(\sqrt{\frac{n}{M}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \end{aligned} \tag{3.4}$$

with the constant implicit in the error term independent of $M(T)$. ■

Note that the compact support of $\widehat{\psi}$ means that the sum truncates at $n \approx M$. Thus, we need $M \gg 1$ in order to have any terms in the sum.

Now, since

$$\widetilde{S}_{M,L} = \frac{\widetilde{N}_M\left(t + \frac{1}{L}\right) - \widetilde{N}_M(t) - \pi\left(\frac{2t}{L} + \frac{1}{L^2}\right)}{\sqrt{t}}, \tag{3.5}$$

then for $t \geq 1$ and $L \geq 1$,

$$\begin{aligned} \widetilde{S}_{M,L} &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \left[\cos\left(2\pi t\sqrt{n} + \frac{\pi}{4}\right) \right. \\ &\qquad \left. - \cos\left(2\pi\left(t + \frac{1}{L}\right)\sqrt{n} + \frac{\pi}{4}\right) \right] \widehat{\psi}\left(\sqrt{\frac{n}{M}}\right) + \mathcal{O}\left(\frac{1}{t}\right) \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \sin\left(\frac{\pi\sqrt{n}}{L}\right) \sin\left(2\pi\left(t + \frac{1}{2L}\right)\sqrt{n} + \frac{\pi}{4}\right) \widehat{\psi}\left(\sqrt{\frac{n}{M}}\right) + \mathcal{O}\left(\frac{1}{t}\right). \end{aligned} \tag{3.6}$$

Note that we have three independent variables. The variable t, which we always consider to be large, is the radius of the annulus. This is the variable we average over. The width of the annulus is $1/L$. Since we want a thin annulus, we let $L \to \infty$, and a Gaussian behaviour is not seen if this condition does not hold. The annulus does not have sharp sides, but smoothed edges, and the third independent variable is M; the larger M is, the sharper the annulus' sides are (in the sense that it approximates the indicator function better). We must have $L/\sqrt{M} \to 0$ in order for the annulus to have some width and not to be "just sides"; that is, the annulus should not be too smooth.

Proof of Theorem 2.1. First, we show that the mean is $\mathcal{O}(1/T)$. Since $\omega(t)$ is real,

$$\left\langle \sin\left(2\pi\left(t+\frac{1}{2L}\right)\sqrt{n}+\frac{1}{4}\pi\right)\right\rangle = \Im\mathrm{m}\{\widehat{\omega}(-T\sqrt{n})e^{i\pi(\sqrt{n}/L+1/4)}\} \ll \frac{1}{T^A n^{A/2}} \tag{3.7}$$

for any $A > 2$, where we have used the rapid decay of $\widehat{\omega}$. Thus,

$$\langle \widetilde{S}_{M,L}\rangle \ll \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \frac{1}{T^A n^{A/2}} + \mathcal{O}\left(\frac{1}{T}\right) = \mathcal{O}\left(\frac{1}{T}\right). \tag{3.8}$$

Setting

$$\mathcal{M}_m := \left\langle \left(\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n^{3/4}}\sin\left(\frac{\pi\sqrt{n}}{L}\right)\sin\left(2\pi\left(t+\frac{1}{2L}\right)\sqrt{n}+\frac{1}{4}\pi\right)\widehat{\psi}\left(\sqrt{\frac{n}{M}}\right)\right)^m\right\rangle, \tag{3.9}$$

then, from (3.6), the Cauchy-Schwartz inequality implies that the mth moment of $\widetilde{S}_{M,L}$ is

$$\begin{aligned}\langle(\widetilde{S}_{M,L})^m\rangle &= \left\langle\left\{\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n^{3/4}}\sin\left(\frac{\pi\sqrt{n}}{L}\right)\sin\left(2\pi\left(t+\frac{1}{2L}\right)\sqrt{n}+\frac{1}{4}\pi\right)\widehat{\psi}\left(\sqrt{\frac{n}{M}}\right)+\mathcal{O}\left(\frac{1}{T}\right)\right\}^m\right\rangle \\ &= \mathcal{M}_m + \mathcal{O}\left(\sum_{j=1}^{m}\binom{m}{j}\frac{\sqrt{\mathcal{M}_{2m-2j}}}{T^j}\right).\end{aligned} \tag{3.10}$$

The conditions of Theorem 2.1 are that $M = \mathcal{O}(T^\delta)$ for all $\delta > 0$, and that $L \to \infty$ in such a way that $L/\sqrt{M} \to 0$. In such case, Proposition 3.2 allows us to deduce that

$\sigma^2 := \mathcal{M}_2 \sim 16 \log L/L$ and Proposition 3.3 shows that for all $m > 2$,

$$\frac{\mathcal{M}_m}{\sigma^m} = \begin{cases} \dfrac{m!}{2^{m/2}\left(\dfrac{m}{2}\right)!} + \mathcal{O}\left(\dfrac{1}{L^{1-\delta'}}\right) & \text{if } m \text{ is even,} \\ \mathcal{O}\left(\dfrac{1}{L^{1-\delta'}}\right) & \text{if } m \text{ is odd.} \end{cases} \tag{3.11}$$

These are the moments of the standard normal distribution, and inserting these into (3.10), we see that this is sufficient to prove that the distribution of $\widetilde{S}_{M,L}/\sigma$ weakly converges as $T \to \infty$ to a Gaussian with mean zero and variance 1. ■

3.1 The variance

Proposition 3.2. If $M = \mathcal{O}(T^{2(1-\delta)})$ for fixed $\delta > 0$, then the variance of $\widetilde{S}_{M,L}$ is asymptotic to

$$\sigma^2 := \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} \sin^2\left(\frac{\pi\sqrt{n}}{L}\right) \widehat{\psi}^2\left(\sqrt{\frac{n}{M}}\right). \tag{3.12}$$

If $L \to \infty$ but $L/\sqrt{M} \to 0$, then

$$\sigma^2 \sim \frac{16 \log L}{L}. \tag{3.13}$$

□

Proof. Expanding out (3.9), we have

$$\mathcal{M}_2 = \frac{4}{\pi^2} \sum_{m,n} \frac{r(m)r(n) \sin\left(\frac{\pi\sqrt{m}}{L}\right) \sin\left(\frac{\pi\sqrt{n}}{L}\right) \widehat{\psi}\left(\sqrt{\frac{m}{M}}\right) \widehat{\psi}\left(\sqrt{\frac{n}{M}}\right)}{(mn)^{3/4}} \\ \times \left\langle \sin\left(2\pi\left(t + \frac{1}{2L}\right)\sqrt{m} + \frac{1}{4}\pi\right) \sin\left(2\pi\left(t + \frac{1}{2L}\right)\sqrt{n} + \frac{1}{4}\pi\right)\right\rangle. \tag{3.14}$$

Now, the average on the second line of the previous equation is

$$\frac{1}{4}\Big[\widehat{\omega}\big(T(\sqrt{m} - \sqrt{n})\big)e^{i\pi(1/L)(\sqrt{n}-\sqrt{m})} \\ + \widehat{\omega}\big(T(\sqrt{n} - \sqrt{m})\big)e^{i\pi(1/L)(\sqrt{m}-\sqrt{n})} \\ - \widehat{\omega}\big(T(\sqrt{m} + \sqrt{n})\big)e^{-i\pi(1/2+(1/L)(\sqrt{m}+\sqrt{n}))} \\ - \widehat{\omega}\big(-T(\sqrt{m} + \sqrt{n})\big)e^{i\pi(1/2+(1/L)(\sqrt{m}+\sqrt{n}))}\Big]. \tag{3.15}$$

The support condition on $\widehat{\psi}$ means that m and n are both constrained to be $\mathcal{O}(M)$, and so, either $m = n$ or $|\sqrt{m} - \sqrt{n}| \gg 1/\sqrt{M}$. Using the bound $\widehat{\omega}(t) \ll (1 + |t|)^{-A}$ for all $A > 0$, the off-diagonal terms contribute at most

$$\sum_{1 \le n \neq m \le M} \left(\frac{\sqrt{M}}{T}\right)^{A} \ll \frac{M^{A/2+2}}{T^{A}} \ll T^{4-\delta A} \tag{3.16}$$

using the assumption that $M = \mathcal{O}(T^{2(1-\delta)})$. Therefore, for any $B > 0$,

$$\mathcal{M}_2 = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} \sin^2\left(\frac{\pi\sqrt{n}}{L}\right) \widehat{\psi}^2\left(\sqrt{\frac{n}{M}}\right) + \mathcal{O}(T^{-B}). \tag{3.17}$$

Define σ^2 to be the above infinite sum. Since $r(n) \ll n^{\eta}$ for all $\eta > 0$, σ^2 is bounded for all L. To find the asymptotics as $L \to \infty$, we use a formula of Ramanujan [14]:

$$\sum_{n \le X} r(n)^2 = 4X \log X + O(X). \tag{3.18}$$

We then have

$$\begin{aligned} \sigma^2 &:= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} \sin^2\left(\frac{\pi\sqrt{n}}{L}\right) \widehat{\psi}^2\left(\sqrt{\frac{n}{M}}\right) \\ &\sim \frac{8}{\pi^2} \int_1^{\infty} \frac{\log x}{x^{3/2}} \sin^2\left(\frac{\pi\sqrt{x}}{L}\right) \widehat{\psi}^2\left(\sqrt{\frac{x}{M}}\right) dx \\ &= \frac{32}{L\pi^2} \int_{1/L}^{\infty} \log(yL) \frac{\sin^2(\pi y)}{y^2} \widehat{\psi}^2\left(\frac{yL}{\sqrt{M}}\right) dy \\ &\sim \frac{\log L}{L} \frac{32}{\pi^2} \int_0^{\infty} \frac{\sin^2(\pi y)}{y^2} \widehat{\psi}^2\left(\frac{yL}{\sqrt{M}}\right) dy \end{aligned} \tag{3.19}$$

on changing variables to $x = y^2 L^2$ and using the fact that we assume that $L \to \infty$. Now, using the additional restriction (caused by the fuzziness of the annulus' sides) that $L/\sqrt{M} \to 0$, we see that since $\widehat{\psi}(yL/\sqrt{M}) \sim 1$ for all $y = o(\sqrt{M}/L)$, the integral can be evaluated asymptotically to equal $\pi^2/2$, and so

$$\sigma^2 \sim \frac{16 \log(L)}{L}. \tag{3.20}$$

Since $L = o(T^{1-\delta})$, the error terms in (3.10) are all smaller than σ^2, and so the variance of $\widetilde{S}_{M,L}$ is asymptotic to σ^2 as $T \to \infty$. ∎

The constraints on M, that $M = \mathcal{O}(T^{2-2\delta})$ but $L/\sqrt{M} \to 0$, illustrate the role of smoothing. The first constraint, that M is not too big, comes from requiring that the annulus is sufficiently smooth to handle the averages easily (to enable us to reduce to the diagonal). The second constraint, that M is not too small, is to ensure that the function is not too smooth so that the width of the edges is greater than the size of the annulus. (That $L \to \infty$ forces M to go to infinity. If it did not, the function would be so smooth as to have no fluctuations!)

3.2 The higher moments

Proposition 3.3. For fixed $\delta > 0$, if $M = \mathcal{O}(T^{2(1-\delta)/(2^{m-1}-1)})$, and if $L \to \infty$ such that $L/\sqrt{M} \to 0$, then for arbitrary $\delta' > 0$,

$$\frac{\mathcal{M}_m}{\sigma^m} = \begin{cases} \dfrac{m!}{2^{m/2}\left(\dfrac{m}{2}\right)!} + \mathcal{O}\left(\dfrac{1}{L^{1-\delta'}}\right) & \text{if } m \text{ is even,} \\ \mathcal{O}\left(\dfrac{1}{L^{1-\delta'}}\right) & \text{if } m \text{ is odd,} \end{cases} \tag{3.21}$$

where $\mathcal{M}_m$ is given in (3.9) and σ^2 is given in (3.12). □

We will need to give lower bounds for alternating sums $\sum \pm\sqrt{n_j}$. To do so, we use the following lemma, a form of Liouville's theorem, (cf. [9]).

Lemma 3.4. For $j = 1, \dots, m$, let $n_j \le M$ be positive integers. Then, either $\sum \epsilon_j \sqrt{n_j} = 0$ for some $\epsilon_j = \pm 1$ or, for all $\epsilon_j = \pm 1$,

$$\left| \sum_{j=1}^{m} \epsilon_j \sqrt{n_j} \right| \ge \frac{1}{(m\sqrt{M})^{2^{m-1}-1}}. \tag{3.22}$$

□

Proof. Assume that $\sum \epsilon_j \sqrt{n_j} \ne 0$ for all choices of $\epsilon_j = \pm 1$. Then

$$P := \prod_{\epsilon_j = \pm 1} \left(\sum_{j=1}^{m} \epsilon_j \sqrt{n_j} \right) \tag{3.23}$$

is nonzero. By Galois theory, since $\sum \epsilon_j \sqrt{n_j}$ is an algebraic number and P is the product over all possible symmetries, P is an integer. Since we assumed that no term in P vanishes, $|P| \ge 1$. Since both $\sum \epsilon_j \sqrt{n_j}$ and $-\sum \epsilon_j \sqrt{n_j}$ are terms in P, if

$$Q := \prod_{\substack{\epsilon_j = \pm 1 \\ j=2,3,\dots,m}} \left(\sqrt{n_1} + \sum_{j=2}^{m} \epsilon_j \sqrt{n_j} \right), \tag{3.24}$$

then $P = (-1)^{2^{m-1}} Q^2$, and so $|Q| = \sqrt{|P|} \ge 1$.

By assumption, $n_j \le M$ for all j, and so, independently of the ϵ_j,

$$\left|\sqrt{n_1} + \sum_{j=2}^{m} \epsilon_j \sqrt{n_j}\right| \le m\sqrt{M}; \tag{3.25}$$

and so, for any $\eta_j = \pm 1$,

$$\left|\sqrt{n_1} + \sum_{j=2}^{m} \eta_j \sqrt{n_j}\right| = \frac{|Q|}{\prod^* \left|\sqrt{n_1} + \sum_{j=2}^{m} \epsilon_j \sqrt{n_j}\right|} \ge \frac{1}{(m\sqrt{M})^{2^{m-1}-1}}, \tag{3.26}$$

where $\prod^*$ denotes the product over all ϵ_j distinct from η_j, there are $2^{m-1} - 1$ terms in such a product. ■

From this, it is simple to derive the following lemma.

Lemma 3.5. For $j = 1, \dots, m$, let $n_j \le M$ be positive integers and let $\epsilon_j = \pm 1$ be such that

$$\sum_{j=1}^{m} \epsilon_j \sqrt{n_j} \ne 0. \tag{3.27}$$

Then,

$$\left|\sum_{j=1}^{m} \epsilon_j \sqrt{n_j}\right| \ge \frac{1}{(m\sqrt{M})^{2^{m-1}-1}}. \tag{3.28}$$

□

Proof. Either $\sum \eta_j \sqrt{n_j} \ne 0$ for any choice of $\eta_j = \pm 1$, and by Lemma 3.4 we are done, or else there exists a (strict) subset $S \subsetneq \{1, \dots, m\}$ such that

$$\sum_{j \in S} \epsilon_j \sqrt{n_j} - \sum_{j \notin S} \epsilon_j \sqrt{n_j} = 0 \tag{3.29}$$

so that

$$\left|\sum_{j=1}^{m} \epsilon_j \sqrt{n_j}\right| = 2\left|\sum_{j \in S} \epsilon_j \sqrt{n_j}\right|. \tag{3.30}$$

Note that, by assumption, $\sum_{j \in S} \epsilon_j \sqrt{n_j} \ne 0$ and, if m' denotes the number of terms in the sum, then $1 \le m' < m$. Now, repeat the argument: either $\sum_{j \in S} \eta_j \sqrt{n_j} \ne 0$ for any choice of $\eta_j = \pm 1$, in which case Lemma 3.4 gives that

$$\left|\sum_{j \in S} \epsilon_j \sqrt{n_j}\right| \ge \frac{1}{(m'\sqrt{M})^{2^{m'-1}-1}} > \frac{1}{(m\sqrt{M})^{2^{m-1}-1}}, \tag{3.31}$$

or else one can further subdivide the set S as before. Since the number of terms in the sum is a positive integer and reduces upon each subdivision, this process terminates. ■

Proof of Proposition 3.3. Expanding (3.9) out,

$$\mathcal{M}_m = \frac{2^m}{\pi^m} \sum_{n_1,\ldots,n_m \geq 1} \prod_{j=1}^{m} \frac{r(n_j)}{n_j^{3/4}} \sin\left(\frac{\pi\sqrt{n_j}}{L}\right) \widehat{\psi}\left(\sqrt{\frac{n_j}{M}}\right) \times \left\langle \prod_{j=1}^{m} \sin\left(2\pi\left(t+\frac{1}{2L}\right)\sqrt{n_j}+\frac{1}{4}\pi\right)\right\rangle. \tag{3.32}$$

Now,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{m} \sin\left(2\pi\left(t+\frac{1}{2L}\right)\sqrt{n_j}+\frac{1}{4}\pi\right)\right\rangle \\
&= \left\langle \prod_{j=1}^{m} \frac{1}{2i}\left[e^{2\pi i((t+1/2L)\sqrt{n_j}+1/8)} - e^{-2\pi i((t+1/2L)\sqrt{n_j}+1/8)}\right]\right\rangle \\
&= \sum_{\epsilon_j=\pm 1} \frac{1}{2^m i^m} \int_{-\infty}^{\infty} \prod_{j=1}^{m} \epsilon_j \exp\left(\epsilon_j 2\pi i\left(\left(t+\frac{1}{2L}\right)\sqrt{n_j}+\frac{1}{8}\right)\right)\frac{1}{T}\omega\left(\frac{t}{T}\right)dt \\
&= \sum_{\epsilon_j=\pm 1} \frac{\prod \epsilon_j}{2^m i^m} \widehat{\omega}\left(-T\sum_{j=1}^{m}\epsilon_j\sqrt{n_j}\right) e^{\sum_{j=1}^{m}\epsilon_j \pi i((1/L)\sqrt{n_j}+1/4)}.
\end{aligned} \tag{3.33}$$

By the compact support condition of $\widehat{\psi}$, we may always assume that $n_j = \mathcal{O}(M)$. By Lemma 3.5 and the fact that $\widehat{\omega}$ decays faster than any polynomial power, the off-diagonal terms (those terms with $\sum_{j=1}^{m} \epsilon_j \sqrt{n_j} \neq 0$) contribute at most

$$\sum_{1\leq n_1,\ldots,n_m \leq M} \left(\frac{(\sqrt{M})^{2^{m-1}-1}}{T}\right)^A \ll \frac{M^{(2^{m-1}-1)A/2+m}}{T^A} \ll T^{-\delta A + 2m/(2^{m-1}-1)} \tag{3.34}$$

which is vanishingly small since A can be arbitrarily large. Thus, the only contributing terms are those with $\sum_{j=1}^{m} \epsilon_j \sqrt{n_j} = 0$, and using the fact that $\widehat{\omega}(0) = 1$, we therefore have, for any $B > 0$,

$$\mathcal{M}_m = \sum_{n_1,\ldots,n_m} \sum_{\substack{\epsilon_j=\pm 1 \\ \sum \epsilon_j \sqrt{n_j}=0}} \prod_{j=1}^{m} \frac{-i\epsilon_j r(n_j)}{\pi n_j^{3/4}} \sin\left(\frac{\pi\sqrt{n_j}}{L}\right) \widehat{\psi}\left(\sqrt{\frac{n_j}{M}}\right) e^{i\pi\epsilon_j/4} + \mathcal{O}(T^{-B}). \tag{3.35}$$

In order to estimate the size of $\mathcal{M}_m/\sigma^m$ when $L \to \infty$, we need to use Besicovich's theorem [2].

Lemma 3.6. If q_j, for $j = 1, \dots, m$, are distinct square-free positive integers, then $\sqrt{q_1}, \dots, \sqrt{q_m}$ are linearly independent over the rationals. □

Therefore, if $\sum_{j=1}^{m} \epsilon_j \sqrt{n_j} = 0$ with $n_j \geq 1$, then there must exist a division of $\{1, \dots, m\}$ into $\{S_i\}$ such that

$$\{1, \dots, m\} = \coprod_{i=1}^{\ell} S_i, \tag{3.36}$$

where $\sum_{i=1}^{\ell} |S_i| = m$ such that for $i = 1, 2, \dots, \ell$, for all $j \in S_i$, $n_j = q_i f_j^2$, with the q_i being distinct square-free integers, and with the f_j satisfying

$$\sum_{j \in S_i} \epsilon_j f_j = 0. \tag{3.37}$$

Summing over all possible divisions, we see that

$$\begin{aligned} \frac{\mathcal{M}_m}{\sigma^m} = \sum_{\ell=1}^{m} \sum_{\{1,\dots,m\}=\coprod_{i=1}^{\ell} S_i} \left(\frac{1}{\sigma^{|S_1|}} \sum_{q_1} D_{q_1}(S_1) \right) \\ \times \left(\frac{1}{\sigma^{|S_2|}} \sum_{q_2,\, q_2 \neq q_1} D_{q_2}(S_2) \right) \cdots \left(\frac{1}{\sigma^{|S_\ell|}} \sum_{q_\ell,\, q_\ell \neq q_1, \dots, q_{\ell-1}} D_{q_\ell}(S_\ell) \right), \end{aligned} \tag{3.38}$$

where

$$D_q(S) := \frac{1}{q^{3|S|/4}} \sum_{\substack{f_j \geq 1 \\ \epsilon_j = \pm 1 \\ \sum_{j \in S} \epsilon_j f_j = 0}} \prod_{j \in S} \frac{-i\epsilon_j e^{i\pi\epsilon_j/4} r(q f_j^2)}{\pi f_j^{3/2}} \sin\left(\pi \frac{1}{L} f_j \sqrt{q}\right) \widehat{\psi}\left(f_j \sqrt{\frac{q}{M}}\right). \tag{3.39}$$

We will show in Lemma 3.7 that if $L \to \infty$ such that $L/\sqrt{M} \to 0$, then for all $\delta' > 0$,

$$\frac{1}{\sigma^{|S|}} \sum_q D_q(S) = \begin{cases} 0 & \text{if } |S| = 1, \\ 1 & \text{if } |S| = 2, \\ \mathcal{O}\left(\frac{1}{L^{1-\delta'}}\right) & \text{otherwise.} \end{cases} \tag{3.40}$$

Therefore, the only terms in $\mathcal{M}_m/\sigma^m$ which do not vanish as $L \to \infty$ are those where $|S_i| = 2$ for all i. If m is odd, there are no such terms, and if $m = 2k$ is even, then the number of terms is equal to the number of ways of partitioning $\{1, \dots, 2k\}$ into $\coprod_{i=1}^{k} S_i$

with $|S_i| = 2$, which equals

$$\frac{1}{k!}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} = \frac{(2k)!}{k!2^k}. \tag{3.41}$$

This completes the proof of Proposition 3.3. ■

Lemma 3.7. If $L \to \infty$ is such that $L/\sqrt{M} \to 0$, then

$$\frac{\sum_q D_q(S)}{\sigma^{|S|}} = \begin{cases} 1 & \text{if } |S| = 2, \\ \mathcal{O}\left(\frac{1}{L^{1-\delta}}\right) & \text{otherwise} \end{cases} \tag{3.42}$$

for all $\delta > 0$, where $D_q(S)$ is defined in (3.39) and σ^2 is defined in (3.12). □

Proof. For convenience, we assume, without loss of generality, that $S = \{1, 2, \ldots, |S|\}$. Using $r(n) \ll n^\delta$ for all $\delta > 0$, and $\widehat{\psi}(x) \ll 1$, we can upper bound by

$$\sum_q D_q(S) \ll \sum_{q=1}^{\infty} \frac{q^{|S|\delta}}{q^{3|S|/4}} Q(q), \tag{3.43}$$

where

$$Q(q) = \sum_{\epsilon_j = \pm 1} \sum_{\substack{f_j \geq 1 \\ \sum_{j=1}^{|S|} \epsilon_j f_j = 0}} \prod_{j=1}^{|S|} \frac{f_j^\delta}{f_j^{3/2}} \left| \sin\left(\frac{\pi f_j \sqrt{q}}{L}\right) \right|. \tag{3.44}$$

Note that $Q(q) \ll 1$ for all q. When $q \ll L^2$, a sharper result can be deduced by a more careful treatment of $Q(q)$. In order to have $\sum_{j=1}^{|S|} \epsilon_j f_j = 0$, at least two of the ϵ must have different signs, and so, with no loss of generality, we put $\epsilon_{|S|} = -1$ and $\epsilon_{|S|-1} = +1$. Hence,

$$f_{|S|} = f_{|S|-1} + \sum_{j=1}^{|S|-2} \epsilon_j f_j. \tag{3.45}$$

In order to have both $f_{|S|} \geq 1$ and $f_{|S|-1} \geq 1$, it must be that

$$f_{|S|-1} \geq 1 + \max\left\{0, -\sum_{j=1}^{|S|-2} \epsilon_j f_j\right\}. \tag{3.46}$$

Therefore,

$$Q(q) = 2 \sum_{\epsilon_1,\ldots,\epsilon_{|S|-2}=\pm 1} \sum_{f_1,\ldots,f_{|S|-2}\geq 1} \sum_{f_{|S|-1}\geq 1+\max\{0,-\sum_{j=1}^{|S|-2}\epsilon_j f_j\}} \times \left(\prod_{j=1}^{|S|-1} \frac{\left|\sin\left(\frac{\pi f_j\sqrt{q}}{L}\right)\right|}{f_j^{3/2-\delta}}\right) \frac{\left|\sin\left(\pi\frac{\sqrt{q}}{L}\left(f_{|S|-1}+\sum_{j=1}^{|S|-2}\epsilon_j f_j\right)\right)\right|}{\left(f_{|S|-1}+\sum_{j=1}^{|S|-2}\epsilon_j f_j\right)^{3/2-\delta}}. \tag{3.47}$$

Changing sums into integrals gives

$$Q(q) \ll \int\cdots\int_1^\infty dx_1\cdots dx_{|S|-2} \sum_{\epsilon_j=\pm 1} \int_{1+\max\{0,-\sum_{j=1}^{|S|-2}\epsilon_j x_j\}}^\infty dx_{|S|-1} \times \left(\prod_{j=1}^{|S|-1} \frac{\left|\sin\left(\pi\frac{1}{L}x_j\sqrt{q}\right)\right|}{x_j^{3/2-\delta}}\right) \frac{\left|\sin\left(\pi\frac{1}{L}\sqrt{q}\left(x_{|S|-1}+\sum_{j=1}^{|S|-2}\epsilon_j x_j\right)\right)\right|}{\left(x_{|S|-1}+\sum_{j=1}^{|S|-2}\epsilon_j x_j\right)^{3/2-\delta}}, \tag{3.48}$$

and changing variables to $x_j\sqrt{q}/L \to y_j$,

$$Q(q) \ll \frac{q^{|S|/4+1/2-|S|\delta/2}}{L^{|S|/2+1-|S|\delta}} \times \int\cdots\int_{\sqrt{q}/L}^\infty \sum_{\epsilon_j=\pm 1} \int_{\max\{0,-\sum\epsilon_j y_j\}+\sqrt{q}/L}^\infty \times \left(\prod_{j=1}^{|S|-1} \frac{|\sin(\pi y_j)|}{y_j^{3/2-\delta}}\right) \frac{\left|\sin\left(\pi\left(y_{|S|-1}+\sum_{j=1}^{|S|-2}\epsilon_j y_j\right)\right)\right|}{\left(y_{|S|-1}+\sum_{j=1}^{|S|-2}\epsilon_j y_j\right)^{3/2-\delta}} dy_{|S|-1}\, dy_{|S|-2}\cdots dy_1. \tag{3.49}$$

Since the multiple integral is bounded, we may conclude that

$$Q(q) \ll \begin{cases} \frac{q^{|S|/4+1/2-|S|\delta/2}}{L^{|S|/2+1-|S|\delta}} & \text{if } q < L^2, \\ 1 & \text{if } q \geq L^2. \end{cases} \tag{3.50}$$

Substituting this into (3.43), we see that

$$\sum_q D_q(S) \ll \begin{cases} \dfrac{L^{\delta'}}{L} & \text{if } |S| = 2, \\ \dfrac{L^{\delta'}}{L^{|S|/2+1}} & \text{if } |S| \geq 3. \end{cases} \tag{3.51}$$

Hence,

$$\frac{1}{\sigma^{|S|}} \sum_q D_q(S) \ll \begin{cases} L^{\delta'} & \text{if } |S| = 2, \\ \dfrac{1}{L^{1-\delta'}} & \text{if } |S| \geq 3 \end{cases} \tag{3.52}$$

since equation (3.13) gives $\sigma \sim 4\sqrt{\log L}/\sqrt{L}$ when $L \to \infty$ but $L/\sqrt{M} \to 0$. However, in the case $|S| = 2$, by the definition of $D_q(S)$ and σ^2, we see that

$$\sum_q D_q(S) = \sigma^2. \tag{3.53}$$

This completes the proof of the lemma. ■

4 Unsmoothing

Recall that $S(t, 1/L)$ is the normalized remainder term for the number of lattice points in an annulus of inner radius t and width $1/L$. In this section, we prove Theorem 1.1 by showing that the variance of the difference $(S(t, 1/L) - \widetilde{S}_{M,L}(t))/\sigma$ vanishes and then combining this with Chebyshev's inequality to deduce a distribution theorem for $S(t, 1/L)$.

We begin with an approximation result for $N(t)$.

Lemma 4.1. For any $a > 0$, $c > 1$,

$$\begin{aligned} N(t) = \pi t^2 - \frac{\sqrt{t}}{\pi} \sum_{n \leq X} \frac{r(n)}{n^{3/4}} \cos\left(2\pi t\sqrt{n} + \frac{1}{4}\pi\right) \\ + \mathcal{O}\left(|t|^{-1/2}\right) + \mathcal{O}\left(X^a\right) + \mathcal{O}\left(\frac{|t|^{2c-1}}{\sqrt{X}}\right). \end{aligned} \tag{4.1}$$

□

This lemma was already invoked by Heath-Brown [9], with the proof being an argument similar to that which derives [17, equation (12.4.4)].

Lemma 4.2. Suppose that $L \to \infty$ as $T \to \infty$ and choose M so that $L/\sqrt{M} \to 0$ as $T \to \infty$ but $M = \mathcal{O}(T^{2(1-\delta)})$ (for a fixed $\delta > 0$). Then, as $T \to \infty$,

$$\left\langle \left| S\left(t, \frac{1}{L}\right) - \widetilde{S}_{M,L}(t) \right|^2 \right\rangle \ll \frac{\log M}{\sqrt{M}}. \tag{4.2}$$

□

Proof. Putting $a = \delta'$ and $c = 1 + \delta'/2$ for $\delta' > 0$ arbitrarily small in Lemma 4.1, we have

$$S\left(t, \frac{1}{L}\right) := \frac{N\left(t + \frac{1}{L}\right) - N(t) - \pi\left(\frac{2t}{L} + \frac{1}{L^2}\right)}{\sqrt{t}} = \frac{2}{\pi} \sum_{n \le X} \frac{r(n)}{n^{3/4}} \sin\left(\frac{\pi\sqrt{n}}{L}\right) \sin\left(2\pi\left(t + \frac{1}{2L}\right)\sqrt{n} + \frac{1}{4}\pi\right) + R(X, t), \tag{4.3}$$

where

$$R(X, t) \ll \frac{1}{|t|} + \frac{X^{\delta'}}{\sqrt{|t|}} + \frac{|t|^{1/2+\delta'}}{\sqrt{X}}. \tag{4.4}$$

Set $X = T^{2-\delta}$. Since $M = \mathcal{O}(T^{2(1-\delta)})$ and $\widehat{\psi}$ has compact support, the infinite sum in $\widetilde{S}_{M,L}(t)$, given in (3.6), is truncated before $n = T^{2-\delta}$, and so

$$S\left(t, \frac{1}{L}\right) - \widetilde{S}_{M,L}(t) = \frac{2}{\pi} \sum_{n \le T^{2-\delta}} \frac{r(n)}{n^{3/4}} \sin\left(\frac{\pi\sqrt{n}}{L}\right) \sin\left(2\pi\left(t + \frac{1}{2L}\right)\sqrt{n} + \frac{\pi}{4}\right)\left(1 - \widehat{\psi}\left(\sqrt{\frac{n}{M}}\right)\right) + R\left(T^{2-\delta}, t\right). \tag{4.5}$$

Let P denote the sum, then the Cauchy-Schwartz inequality gives

$$\left\langle \left(S\left(t, \frac{1}{L}\right) - \widetilde{S}_{M,L}(t)\right)^2 \right\rangle = \langle P^2 \rangle + \left\langle R\left(T^{2-\delta}, t\right)^2 \right\rangle + \mathcal{O}\left(\sqrt{\langle P^2 \rangle}\sqrt{\left\langle R\left(T^{2-\delta}, t\right)^2 \right\rangle}\right). \tag{4.6}$$

Observe that

$$\left\langle R\left(T^{2-\delta}, t\right)^2 \right\rangle \ll T^{-1+\delta''} \tag{4.7}$$

for arbitrarily small $\delta'' > 0$, and

$$\langle P^2 \rangle = \frac{2}{\pi^2} \sum_{n \le T^{2-\delta}} \frac{r(n)^2}{n^{3/2}} \sin^2\left(\frac{\pi\sqrt{n}}{L}\right)\left(1 - \widehat{\psi}\left(\sqrt{\frac{n}{M}}\right)\right)^2 + \mathcal{O}\left(\sum_{1 \le m \ne n \le T^{2-\delta}} \widehat{\omega}\left(T\left(\sqrt{n} - \sqrt{m}\right)\right)\right). \tag{4.8}$$

The same argument used in Section 3.1 shows that the error term here vanishes like $\mathcal{O}(T^{-B})$ for any $B > 0$.

Since $\sum_{n \le X} r(n)^2 \sim 4X \log X$, partial summation gives

$$\begin{aligned}\langle P^2 \rangle &\sim \frac{8}{\pi^2} \int_1^{T^{2-\delta}} \frac{\log x}{x^{3/2}} \sin^2\left(\frac{\pi\sqrt{x}}{L}\right)\left(1 - \widehat{\psi}\left(\sqrt{\frac{x}{M}}\right)\right)^2 dx \\ &= \frac{32}{L\pi^2} \int_{1/L}^{T/L} \frac{\log(Ly)\sin^2(\pi y)}{y^2} \left(1 - \widehat{\psi}\left(\frac{yL}{\sqrt{M}}\right)\right)^2 dy\end{aligned} \tag{4.9}$$

by the change of variables $x = y^2 L^2$. If $yL/\sqrt{M} \ll 1$, then

$$\widehat{\psi}\left(\frac{yL}{\sqrt{M}}\right) = 1 + \mathcal{O}\left(\frac{yL}{\sqrt{M}}\right), \tag{4.10}$$

leading to

$$\begin{aligned}\langle P^2 \rangle &\ll \frac{L}{M} \int_0^{\sqrt{M}/L} \log(Ly)\sin^2(\pi y)\,dy + \frac{1}{L}\int_{\sqrt{M}/L}^{T/L} \frac{\log(Ly)\sin^2(\pi y)}{y^2} dy \\ &\ll \frac{\log M}{\sqrt{M}}.\end{aligned} \tag{4.11}$$

Inserting this into (4.6), using $M = \mathcal{O}(T^{2(1-\delta)})$, and choosing $0 < \delta'' < \delta$ in the estimate of $\langle R(X,t)^2 \rangle$, we have that

$$\begin{aligned}\left\langle \left(S\left(t, \frac{1}{L}\right) - \widetilde{S}_{M,L}(t)\right)^2 \right\rangle &\ll \frac{\log M}{\sqrt{M}} + \frac{1}{T^{1-\delta''}} + \frac{\sqrt{\log M}}{M^{1/4} T^{1/2-\delta''/2}} \\ &= \mathcal{O}\left(\frac{\log M}{\sqrt{M}}\right).\end{aligned} \tag{4.12}$$

■

Lemma 4.3. Under the conditions of Lemma 4.2, for all fixed $\eta > 0$,

$$\mathbb{P}_{\omega,T}\left\{ \left| \frac{S\left(t, \frac{1}{L}\right)}{\sigma} - \frac{\widetilde{S}_{M,L}(t)}{\sigma} \right| > \eta \right\} \longrightarrow 0 \tag{4.13}$$

as $T \to \infty$, where $\sigma^2 = 16 \log L / L$. □

Proof. For fixed $\eta > 0$, Chebychev's inequality gives

$$\mathbb{P}_{\omega,T}\left\{\left|\frac{S\left(t,\frac{1}{L}\right)}{\sigma}-\frac{\widetilde{S}_{M,L}(t)}{\sigma}\right|>\eta\right\} \le \frac{\left\langle\left(S\left(t,\frac{1}{L}\right)-\widetilde{S}_{M,L}(t)\right)^2\right\rangle}{\eta^2\sigma^2} \ll \frac{L}{\log L}\frac{\log M}{\sqrt{M}} \tag{4.14}$$

which tends to zero as $T \to \infty$ by the assumptions placed on M and L. ■

Corollary 4.4. If $L \to \infty$ but $L = \mathcal{O}(T^{\delta})$ for all $\delta > 0$ as $T \to \infty$, then for any interval $\mathcal{A}$,

$$\mathbb{P}_{\omega,T}\left\{\frac{S\left(t,\frac{1}{L}\right)}{\sigma}\in\mathcal{A}\right\} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2}dx, \tag{4.15}$$

where $\sigma^2 = 16\log L/L$. □

Proof. Set $M = L^3$, then $M = \mathcal{O}(T^{\delta})$ for all $\delta > 0$ and $L/\sqrt{M} \to 0$. Thus, $\widetilde{S}_{M,L}/\sigma$ weakly converges to a standard normal distribution as $T \to \infty$ when t is smoothly averaged around T by Theorem 2.1. But Lemma 4.3 implies that $S(t, 1/L)/\sigma$ must also weakly converge to a standard normal distribution. ■

We are now able to prove our main result, Theorem 1.1, which says that if $L \to \infty$ but $L = \mathcal{O}(T^{\delta})$ for all $\delta > 0$, then for any interval $\mathcal{A}$,

$$\lim_{T\to\infty}\frac{1}{T}\,\text{meas}\left\{t\in[T,2T]:\frac{S\left(t,\frac{1}{L}\right)}{\sigma}\in\mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2}dx. \tag{4.16}$$

Proof of Theorem 1.1. Fix $\epsilon > 0$ and approximate the indicator function $\mathbf{1}_{[1,2]}$ above and below by smooth functions $\chi_{\pm} \ge 0$ so that $\chi_{-} \le \mathbf{1}_{[1,2]} \le \chi_{+}$, where both $\chi_{\pm}$ and their Fourier transforms are smooth and of rapid decay, and so that their total masses are within ϵ of unity $|\int\chi_{\pm}(x)dx - 1| < \epsilon$. Now, set $\omega_{\pm} := \chi_{\pm}/\int\chi_{\pm}$. Then $\omega_{\pm}$ are "admissible," and for all t,

$$(1-\epsilon)\omega_{-}(t) \le \mathbf{1}_{[1,2]}(t) \le (1+\epsilon)\omega_{+}(t). \tag{4.17}$$

Now,

$$\operatorname{meas}\left\{t \in [T, 2T] : \frac{S\left(t, \frac{1}{L}\right)}{\sigma} \in \mathcal{A}\right\} = \int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{A}}\left(\frac{S\left(t, \frac{1}{L}\right)}{\sigma}\right) \mathbf{1}_{[1,2]}\left(\frac{t}{T}\right) dt, \tag{4.18}$$

and since (4.17) holds, we find that

$$\begin{aligned} (1-\epsilon)\mathbb{P}_{\omega_-,T}\left\{\frac{S\left(t, \frac{1}{L}\right)}{\sigma} \in \mathcal{A}\right\} &\le \frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] : \frac{S\left(t, \frac{1}{L}\right)}{\sigma} \in \mathcal{A}\right\} \\ &\le (1+\epsilon)\mathbb{P}_{\omega_+,T}\left\{\frac{S\left(t, \frac{1}{L}\right)}{\sigma} \in \mathcal{A}\right\}. \end{aligned} \tag{4.19}$$

By Corollary 4.4, the two extreme sides of this inequality have a limit, as $T \to \infty$, of

$$(1 \pm \epsilon)\frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2} dx, \tag{4.20}$$

and so, we get that

$$(1-\epsilon)\frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2} dx \le \liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] : \frac{S\left(t, \frac{1}{L}\right)}{\sigma} \in \mathcal{A}\right\} \tag{4.21}$$

with a similar statement for limsup; since $\epsilon > 0$ is arbitrary, this shows that the limit exists and equals

$$\lim_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{t \in [T, 2T] : \frac{S\left(t, \frac{1}{L}\right)}{\sigma} \in \mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2} dx, \tag{4.22}$$

which is the Gaussian law. ■

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