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Small Scale Equidistribution of Eigenfunctions on the Torus

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Abstract: We study the small scale distribution of the L^2 mass of eigenfunctions of the Laplacian on the flat torus $\mathbb{T}^d$. Given an orthonormal basis of eigenfunctions, we show the existence of a density one subsequence whose L^2 mass equidistributes at small scales. In dimension two our result holds all the way down to the Planck scale. For dimensions $d = 3, 4$ we can restrict to individual eigenspaces and show small scale equidistribution in that context. We also study irregularities of quantum equidistribution: We construct eigenfunctions whose L^2 mass does not equidistribute at all scales above the Planck scale. Additionally, in dimension $d = 4$ we show the existence of eigenfunctions for which the proportion of L^2 mass in small balls blows up at certain scales.

1. Introduction

1.1. The semiclassical eigenfunction hypothesis. Let M be a compact Riemannian manifold (smooth, connected and with no boundary), with associated Laplace-Beltrami operator Δ, and $\{\psi_n\}$ an orthonormal basis of $L^2(M, \mathrm{dvol})$ consisting of eigenfunctions: $-\Delta\psi_n = \lambda_n\psi_n$, where dvol is the normalized Riemannian volume form. If the geodesic flow is ergodic, the Quantum Ergodicity Theorem [5,31,37] says that for any choice of orthonormal basis (ONB) $\{\psi_n\}$ consisting of eigenfunctions of the Laplacian, there is a density one subsequence of these eigenfunctions that are uniformly distributed in the unit cotangent bundle S^*M, where a density one subsequence $\{\psi_{n_\ell}\} \subset \{\psi_n\}$ of eigenfunctions is one such that

$$\lim_{\Lambda\to\infty} \frac{\#\{\psi_{n_\ell} : \lambda_{n_\ell} \le \Lambda\}}{\#\{\lambda_n \le \Lambda\}} = 1.$$

(For certain chaotic billiards, exceptional eigenfunctions do exist, see [11].) In particular, there is a density-one subsequence of the eigenfunctions so that the probability densities $|\psi_n(x)|^2$ converge weakly to the uniform distribution in configuration space M along this subsequence, i.e., for any (nice) fixed subset $\Omega \subseteq M$ of positive measure,

$$\frac{1}{\mathrm{vol}(\Omega)}\int_\Omega |\psi_n(x)|^2\,\mathrm{dvol}(x)\to 1.$$

Uniform distribution in configuration space is not only a feature of ergodicity: Marklof and Rudnick [23] show that this is also the case for rational polygons and for flat tori.

Berry [1,2] in his work on the "Semiclassical Eigenfunction Hypothesis" (see also [35]), proposed to go beyond uniform distribution, and study the amplitudes $|\psi_n(x)|^2$ when smoothed over regions in M, whose diameter shrinks as $\lambda_n\to\infty$, but at a rate slower than the Planck scale $\hbar\approx 1/\sqrt{\lambda_n}$, that is to study the local averages

$$\frac{1}{\mathrm{vol}\,B(x_n,r_n)}\int_{B(x_n,r_n)} |\psi_n(x)|^2\,\mathrm{dvol}(x) \tag{1.1}$$

where $B(x_n,r_n)$ is a geodesic ball of radius r_n centered at $x_n\in M$, so that as $\lambda_n\to\infty$, $r_n\to 0$, but $r_n\sqrt{\lambda_n}\to\infty$. We will say that small scale equidistribution of the eigenfunctions $\{\psi_n\}$ holds if (1.1) tends to 1.

There are very few rigorous results on small scale equidistribution in the literature. Luo and Sarnak [21] studied the case of the modular surface, and the orthonormal set of eigenfunctions of the Laplacian which are eigenfunctions of all Hecke operators, showing that for these, small scale equidistribution holds along a density one subsequence for radii $r\gg\lambda^{-\alpha}$, for some small $\alpha>0$. Under the assumption of the Generalized Riemann Hypothesis, Young [36] showed that small scale equidistribution holds for *all* such eigenfunctions for radii $r\gg\lambda^{-1/4+o(1)}$.

The case of compact manifolds with negative sectional curvature was recently investigated independently by Hezari and Rivière [12] and Han [9] who obtained commensurability of the masses along a density one subsequence for logarithmically small radii $r=(\log\lambda)^{-\alpha}$ $(0\le\alpha<\frac{1}{3\dim M})$:

$$a_1\le\frac{1}{\mathrm{vol}(B(x_n,r_n))}\int_{B(x_n,r_n)} |\psi_n(x)|^2\,\mathrm{dvol}(x)\le a_2$$

along the subsequence, where the constants $0<a_1<a_2$ are independent of the centers of the balls x_n and of the subsequence.

1.2. Small scale equidistribution on the flat torus. The case of interest for us is that of the flat d-dimensional torus $\mathbb{T}^d=\mathbb{R}^d/2\pi\mathbb{Z}^d$. The "Semiclassical Eigenfunction Hypothesis" predicts that (1.1) converges to 1 in this setting for radii $r_n\to 0$ with $r_n\sqrt{\lambda_n}\to\infty$, as $\lambda\to\infty$. Hezari and Rivière [13] have recently studied small scale equidistribution in $\mathbb{T}^d$. They show that for a fixed center $x_0\in\mathbb{T}^d$, for any ONB of eigenfunctions $\{\psi_n\}$, there is a density one subsequence so that for all balls $B(x_0,r_n)$ of radius $r_n>\lambda_n^{-\frac{1}{4(d+1)}}$ one has that (1.1) tends to 1 along the subsequence.

Note that below the Planck scale $r=\lambda^{-1/2}$, equidistribution fails badly. For example, consider the ONB of eigenfunctions $\psi_\mu^-(x)=\sqrt{2}\sin(\langle\mu,x\rangle)$, $\psi_\mu^+(x)=\sqrt{2}\cos(\langle\mu,x\rangle)$, $\mu\in\mathbb{Z}^d/\{\pm1\}$, with eigenvalue $\lambda=|\mu|^2$. For $r=o(\lambda^{-1/2})$, and $x\in B(0,r)$ one has $\psi_\mu^\pm(x)\sim\psi_\mu^\pm(0)=1\pm1$, so that every eigenfunction in this ONB is not equidistributed below the Planck scale.

One of our goals is to prove small scale equidistribution on $\mathbb{T}^d$, uniformly for all not too small balls. We succeed for radii $r_n\gg\lambda_n^{-\frac{1}{2(d-1)}+o(1)}$, in particular in dimension $d=2$, our result extends all the way down to the Planck scale $r\gg\lambda^{-1/2+o(1)}$:

Theorem 1.1. *Let* $\{\psi_n\}$ *be an orthonormal basis of eigenfunctions of* $L^2(\mathbb{T}^d, \mathrm{dvol})$*, and*

$$\mathcal{B}_n = \left\{ B(y,r) \subset \mathbb{T}^d : r > \lambda_n^{\frac{-1}{2(d-1)}+o(1)} \right\}.$$

Then along a density one subsequence,

$$\lim_{n\to\infty} \sup_{B(y,r)\in\mathcal{B}_n} \left| \frac{1}{\mathrm{vol}(B(y,r))} \int_{B(y,r)} |\psi_n(x)|^2 \, \mathrm{dvol}(x) - 1 \right| = 0.$$

This result gives that the L^2 mass of "almost all" eigenfunctions in the given orthonormal basis is uniformly distributed in every small ball $B(y,r)$. Even though our result does not reach the Planck scale for dimensions $d > 2$, the scale we achieve is actually optimal (up to the $\lambda^{o(1)}$ factor). This was pointed out to us by Jean Bourgain (see Remark 1.3 after Theorem 1.2).

1.3. Irregularities in quantum equidistribution. Theorem 1.1 leaves open the existence of exceptional sequences of eigenfunctions. In Theorem 3.1 we show that these do exist, so that one cannot improve the "almost all" statement. We show that there is a sequence of eigenvalues $\lambda_n \to \infty$ and corresponding L^2-normalized eigenfunctions ψ_n so that for any choice of radii r_n so that $r_n \to 0$, but $r_n\sqrt{\lambda_n} \to \infty$,

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(B(0,r_n))} \int_{B(0,r_n)} |\psi_n(x)|^2 \, \mathrm{dvol}(x) = 2.$$

For a fixed radius $r \approx 1$, see [18] for information on possible "quantum limits".

In dimension $d = 4$, we can also create "massive irregularities", where we find an infinite sequence of eigenvalues $\lambda_n \to \infty$, so that given any sequence of balls $B(x_n, r_n)$ of radius $r_n \ll \lambda^{-\frac{1}{2(d-1)}-o(1)}$, there are normalized eigenfunctions ψ_n whose L^2-mass on the specific balls $B(x_n, r_n)$ blows up:

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(B(x_n,r_n))} \int_{B(x_n,r_n)} |\psi_n(x)|^2 \, \mathrm{dvol}(x) = \infty.$$

A related feature was found on certain negatively curved surfaces by Iwaniec and Sarnak [17], who found eigenfunctions of the Laplacian whose values blow up at special points, see also [24].

On the other hand, in dimension $d = 2$ we rule out the existence of such "massive irregularities" at scales above $r > \lambda^{-1/4+o(1)}$ and expect that they do not exist at all for $r > \lambda^{-1/2+o(1)}$, i.e., just above the Planck scale. We will show that for every eigenfunction $\psi(x)$ in dimension $d = 2$ that for radii $r > \lambda^{-1/4+o(1)}$

$$\sup_{y\in\mathbb{T}^2} \frac{1}{\mathrm{vol}(B(y,r))} \int_{B(y,r)} |\psi(x)|^2 \, \mathrm{dvol}(x) \ll 1. \tag{1.2}$$

The problem of obtaining an upper bound for the proportion of L^2 mass of eigenfunctions in small balls was previously studied by Sogge [32], who showed for any compact d-dimensional Riemannian manifold (smooth, connected and with no boundary) M and an L^2-normalized eigenfunction of the Laplace-Beltrami operator ψ that

$$\sup_{y\in M} \frac{1}{\mathrm{vol}(B(y,r))} \int_{B(y,r)} |\psi(x)|^2 \, \mathrm{dvol}(x) \ll r^{1-d},$$

for $r > \lambda^{-1/2}$.

1.4. Localizing on eigenspaces. In higher dimensions ($d \geq 3$), the eigenspaces have fairly large dimension, and we can also localize on each λ-eigenspace in dimensions $d = 3, 4$. That is, prove analogues of Theorem 1.1 when we restrict to an orthonormal basis of an individual eigenspace. For instance, in dimension $d = 3$ for $\lambda \not\equiv 0, 4, 7 \pmod 8$, the dimension of the λ-eigenspace, which we denote N_λ, is quite large of size $\approx \lambda^{\frac{1}{2} \pm o(1)}$; for $d = 4$ and λ odd, $\lambda \leq N_\lambda \ll \lambda^{1+o(1)}$. Using results from the arithmetic theory of quadratic forms, we show

Theorem 1.2. *Suppose that $d = 3$ and $\lambda \not\equiv 0, 4, 7 \pmod 8$, or, $d = 4$ and λ is odd. Let $\{\psi_n\}_{\lambda_n = \lambda}$ be an ONB of eigenfunctions of the λ-eigenspace and let*

$$\mathcal{B}_\lambda = \left\{ B(y, r) \subset \mathbb{T}^d : r > \lambda^{-\frac{1}{2(d-1)} + o(1)} \right\}.$$

Then there exists a subset $S_\lambda \subseteq \{\psi_n\}_{\lambda_n = \lambda}$ of cardinality $N_\lambda(1 + o(1))$, as $\lambda \to \infty$, which consists of eigenfunctions such that

$$\sup_{B(y,r) \in \mathcal{B}_\lambda} \left| \frac{1}{\operatorname{vol}(B(y, r))} \int_{B(y,r)} |\psi_n(x)|^2 \operatorname{dvol}(x) - 1 \right| = o(1), \quad \lambda \to \infty, \ \ \psi_n \in S_\lambda.$$

Theorem 1.2 reaches the same scale $r > \lambda^{\frac{-1}{2(d-1)} + o(1)}$ as Theorem 1.1. Moreover, it gives that the L^2 mass of "almost all" eigenfunctions in the λ-eigenspace equidistributes inside balls of radii $r > \lambda^{\frac{-1}{2(d-1)} + o(1)}$ whereas Theorem 1.1 does not guarantee the existence of even one such eigenfunction. We believe that the analogue of Theorem 1.2 also holds in dimensions $d \geq 5$.

Remark 1.3. Bourgain (private communication) has pointed out that our result is sharp, in that for $d \geq 3$, under the conditions on λ of Theorem 1.2, for radii $\lambda^{-1/2} < r < \lambda^{-\frac{1}{2(d-1)} - o(1)}$ each λ-eigenspace has an ONB for which a positive proportion of the eigenfunctions fail to equidistribute in $B(0, r)$, in fact for which

$$\frac{1}{\operatorname{vol}(B(0, r))} \int_{B(0,r)} |\psi_n(x)|^2 \operatorname{dvol}(x) \sim 0, \quad \lambda \to \infty.$$

The construction is detailed in § 4.

1.5. Discrepancy. Given an ONB $\{\psi_n\}$ consisting of eigenfunctions of the Laplacian, and $a \in C^\infty(\mathbb{T}^d)$, let

$$V_2(a, \Lambda) := \frac{1}{\#\{\lambda_n \leq \Lambda\}} \sum_{\lambda_n \leq \Lambda} \left| \int_{\mathbb{T}^d} a(x) |\psi_n(x)|^2 \operatorname{dvol}(x) - \int_{\mathbb{T}^d} a(x) \operatorname{dvol}(x) \right|^2.$$

Here $\operatorname{dvol}(x) = dx/(2\pi)^d$ where dx is Lebesgue measure. Marklof and Rudnick [23] showed that $V_2(a, \Lambda)$ decays as $\Lambda \to \infty$. This was done via arguing as in Schnirelman's theorem and using Kronecker's theorem that generic geodesics are uniformly distributed when projected to configuration space; the point of [23] was that this argument extends to rational polygons. Hezari and Rivière [13] arrive at their results on small scale equidistribution by giving a quantitative rate of decay of $V_2(a, \Lambda)$.

We will derive Theorem 1.1 from an upper bound on the L^1 discrepancy

$$V_1(a, \Lambda) := \frac{1}{\#\{\lambda_n \leq \Lambda\}} \sum_{\lambda_n \leq \Lambda} \left| \int_{\mathbb{T}^d} a(x) |\psi_n(x)|^2 \operatorname{dvol}(x) - \int_{\mathbb{T}^d} a(x) \operatorname{dvol}(x) \right|.$$

For a fixed a trigonometric polynomial, we will show that

$$V_1(a, \Lambda) \ll_a \Lambda^{-1/2}. \tag{1.3}$$

Note that for chaotic systems, it is expected that the L^1 discrepancy $V_1(a, \Lambda)$ is larger, of size about $\Lambda^{-1/4}$, see [6,7] giving physical arguments for generic chaotic systems, and [19,22] for rigorous results of this quality for the L^2 discrepancy in arithmetic settings, and [38] for logarithmic upper bounds for the general negatively curved case (see also [29]).

1.6. About the proofs. Our arguments rely upon lattice point estimates in place of dynamical properties of the geodesic flow. In particular, the proof of the bound (1.3), given in Sect. 2.1, combines harmonic analysis and a lattice point argument from the geometry of numbers (see Lemma 2.3). The proof of Theorem 1.2 in Sect. 5 replaces this lattice point count with a more refined statistic, which counts lattice points on a sphere that lie within a small spherical cap (see Remark 5.4). To estimate this quantity, we require deeper arithmetic information on the number of representations of a positive definite binary quadratic form by sums of squares of linear forms. This is also used in the construction of "massive irregularities" in high dimensions in § 6. Bourgain's argument, which shows Theorem 1.1 reaches the optimal scale, is detailed in § 4 and also relies upon estimates for the number of lattice points within spherical caps. The construction of quantum irregularities in § 3 relies on more direct arguments.

1.7. Notation. Throughout we use the notation, $f(x) \ll g(x)$, by which we mean $f(x) = O(g(x))$. In addition we write $f(x) \gg g(x)$ provided there exists a $c > 0$ such that $|f(x)| \geq cg(x)$ for all x under consideration, and, if $f(x) \ll g(x)$ and $f(x) \gg g(x)$ we write $f(x) \approx g(x)$.

2. Small Scale Equidistribution

2.1. L^1 discrepancy on the torus. The goal of this section is to prove the upper bound (1.3) on the L^1 discrepancy.

On the torus $\mathbb{T}^d$ each eigenfunction ψ_n of $-\Delta$ with eigenvalue λ_n is of the following form

$$\psi_n = \sum_{\mu \in \mathbb{Z}^d : |\mu|^2 = \lambda_n} c_n(\mu) e_\mu$$

where $e_\mu(x) = e^{\mathrm{i}\langle \mu, x\rangle}$. Throughout, we assume ψ_n is L^2-normalized so that

$$\int_{\mathbb{T}^d} |\psi_n(x)|^2 \,\mathrm{dvol}(x) = \sum_{|\mu|^2 = \lambda_n} |c_n(\mu)|^2 = 1.$$

Lemma 2.1. *For $\mu \in \mathbb{Z}^d$ such that $|\mu|^2 = \lambda$ we have*

$$\sum_{\lambda_n = \lambda} |c_n(\mu)|^2 = 1.$$

Proof. The functions $\{\psi_n : \lambda_n = \lambda\}$ and $\{e_\mu : |\mu|^2 = \lambda\}$ are both orthonormal bases of the λ-eigenspace of $-\Delta$, with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}^d} f(x)\overline{g(x)}\,\mathrm{dvol}(x).$$

Hence in the expansion

$$\psi_n = \sum_{|\mu|^2=\lambda} \langle \psi_n, e_\mu\rangle e_\mu$$

we have

$$\langle \psi_n, e_\mu\rangle = c_n(\mu)$$

and hence the expansion of e_μ is

$$e_\mu = \sum_{\lambda_n=\lambda} \langle e_\mu, \psi_n\rangle \psi_n = \sum_{\lambda_n=\lambda} \overline{c_n(\mu)}\psi_n$$

and therefore

$$\sum_{\lambda_n=\lambda} |c_n(\mu)|^2 = \sum_{\lambda_n=\lambda} |\langle e_\mu, \psi_n\rangle|^2 = \langle e_\mu, e_\mu\rangle = 1.$$

□

Lemma 2.2. *If* $|\zeta| \le 2\sqrt{\lambda}$ *then*

$$\sum_{\lambda_n=\lambda} \left| \int_{\mathbb{T}^d} e_\zeta(x)|\psi_n(x)|^2\,\mathrm{dvol}(x)\right| \le \#\{\mu \in \mathbb{Z}^d : |\mu|^2 = \lambda = |\mu+\zeta|^2\}.$$

If $|\zeta| > 2\sqrt{\lambda}$ *then each summand is zero.*

Proof. Expand ψ_n to get

$$\begin{aligned}\left|\int_{\mathbb{T}^d} e_\zeta(x)|\psi_n(x)|^2\,\mathrm{dvol}(x)\right| &= \left|\sum_{|\mu|^2=\lambda_n=|\mu+\zeta|^2} c_n(\mu)\overline{c_n(\mu+\zeta)}\right| \\ &\le \sum_{|\mu|^2=\lambda_n=|\mu+\zeta|^2} \frac{1}{2}|c_n(\mu)|^2 + \frac{1}{2}|c_n(\mu+\zeta)|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\lambda_n=\lambda}\left|\int_{\mathbb{T}^d} e_\zeta(x)|\psi_n(x)|^2\,\mathrm{dvol}(x)\right| &\le \sum_{\lambda_n=\lambda}\sum_{|\mu|^2=\lambda_n=|\mu+\zeta|^2} \frac{1}{2}|c_n(\mu)|^2 + \frac{1}{2}|c_n(\mu+\zeta)|^2 \\ &= \sum_{|\mu|^2=\lambda=|\mu+\zeta|^2} \frac{1}{2}\sum_{\lambda_n=\lambda}|c_n(\mu)|^2 + \frac{1}{2}\sum_{\lambda_n=\lambda}|c_n(\mu+\zeta)|^2 \\ &= \sum_{|\mu|^2=\lambda=|\mu+\zeta|^2} \frac{1}{2} + \frac{1}{2}\end{aligned}$$

since by Lemma 2.1 both inner sums equal one. Hence,

$$\sum_{\lambda_n=\lambda}\left|\int_{\mathbb{T}^d} e_\zeta(x)|\psi_n(x)|^2\,\mathrm{dvol}(x)\right| \leq \#\{\mu\in\mathbb{Z}^d : |\mu|^2=\lambda=|\mu+\zeta|^2\}.$$

Now if $|\zeta|>2\sqrt{\lambda}$ then there is no (real) solution of $|\mu|^2=\lambda=|\mu+\zeta|^2$ and hence all terms above vanish. □

Lemma 2.3. *For a nonzero integer vector $\zeta\in\mathbb{Z}^d$ write $\zeta=m\widehat{\zeta}$, with $m\geq 1$ and $\widehat{\zeta}$ a primitive integer vector. If $0<|\zeta|\leq 2\sqrt{X}$ then*

$$\#\{\mu\in\mathbb{Z}^d : |\mu|^2\leq X, |\mu|^2=|\mu+\zeta|^2\} \ll \frac{(\sqrt{X})^{d-1}}{|\widehat{\zeta}|},$$

while for $|\zeta|>2\sqrt{X}$, the set above is empty.

Proof. Suppose we have a solution $\mu\in\mathbb{Z}^d$ with $|\mu+\zeta|=|\mu|\leq\sqrt{X}$ then $|\zeta|\leq|\mu+\zeta|+|\mu|\leq 2\sqrt{X}$ and hence if $|\zeta|>2\sqrt{X}$ then there are no solutions. So from now on assume $|\zeta|\leq 2\sqrt{X}$.

The equality $|\mu|^2=|\mu+\zeta|^2$ is equivalent to

$$2\langle\mu,\zeta\rangle=-|\zeta|^2 \tag{2.1}$$

which only has solutions if $|\zeta|^2$ is even.

If there are no solutions to (2.1) with $|\mu|\leq\sqrt{X}$, then we are done. Otherwise, there exists a solution μ_0 and any other such solution satisfies

$$\langle\mu-\mu_0,\zeta\rangle=0,\quad |\mu-\mu_0|\leq 2\sqrt{X}.$$

We see that the number of solutions $|\mu|\leq\sqrt{X}$ to (2.1) is bounded by the number of $\nu\in\mathbb{Z}^d$ such that

$$\langle\nu,\zeta\rangle=0,\quad |\nu|\leq 2\sqrt{X}.$$

That is, we are counting lattice points in the $(d-1)$-dimensional sub-lattice which is the ortho-complement of ζ, which lie in a ball. The co-volume (discriminant) of this sub-lattice is $|\widehat{\zeta}|$, where $\zeta=m\widehat{\zeta}$ with $\widehat{\zeta}$ primitive, $m\geq 1$ integer, and by [28, Section 2] the number of such integer solutions is

$$c_d\frac{(2\sqrt{X})^{d-1}}{|\widehat{\zeta}|}+O((\sqrt{X})^{d-2}) \ll \frac{X^{(d-1)/2}}{|\widehat{\zeta}|},$$

since $|\widehat{\zeta}|\leq|\zeta|\ll\sqrt{X}$. Here $c_d=\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the d-dimensional unit ball in $\mathbb{R}^d$. □

Proposition 2.4. *If $|\zeta|\leq 2\sqrt{\Lambda}$ then*

$$V_1(e_\zeta,\Lambda)\ll\frac{\Lambda^{-1/2}}{|\widehat{\zeta}|}.$$

If $|\zeta|>2\sqrt{\Lambda}$ then $V_1(e_\zeta,\Lambda)=0$.

Proof. By Lemma 2.2,

$$V_1(e_\zeta, \Lambda) = \frac{1}{\#\{\lambda_n \le \Lambda\}} \sum_{\lambda \le \Lambda} \sum_{\lambda_n = \lambda} \left| \int_{\mathbb{T}^d} e_\zeta(x) |\psi_n(x)|^2 \, \mathrm{dvol}(x) \right|$$
$$\le \frac{1}{\#\{\lambda_n \le \Lambda\}} \sum_{\lambda \le \Lambda} \#\{\mu \in \mathbb{Z}^d : |\mu|^2 = \lambda = |\mu + \zeta|^2\}$$
$$= \frac{1}{\#\{\lambda_n \le \Lambda\}} \#\{|\mu| \le \sqrt{\Lambda} : |\mu|^2 = |\mu + \zeta|^2\}.$$

The denominator is $\#\{\lambda_n \le \Lambda\} \approx \Lambda^{d/2}$ (Weyl's law, which follows from an elementary argument since $\#\{\lambda_n \le \Lambda\} = \#\{\mu \in \mathbb{Z}^d : |\mu|^2 \le \Lambda\}$), while by Lemma 2.3, the numerator is $\ll (\sqrt{\Lambda})^{d-1}/|\widehat{\zeta}|$, which gives the claim. □

Note that the upper bound (1.3) on the L^1 discrepancy $V_1(a, \Lambda)$ for a general trigonometric polynomial follows from Proposition 2.4.

2.2. Proof of Theorem 1.1. We will need majorants and minorants for the indicator function of a ball. We now cite Lemma 4 of Harman [10] (see also the work of Holt [14] and Holt and Vaaler [15]), which constructs an appropriate version of Beurling-Selberg polynomials:

Lemma 2.5. *Let $B(0, r) \subset \mathbb{T}^d$ be the ball of radius r around the origin. Let $T, r > 0$ with $Tr \ge 1$. There exist trigonometric polynomials $a^\pm$ such that:*

(i) $a^-(x) \le \mathbf{1}_{B(0,r)}(x) \le a^+(x)$;
(ii) $\widehat{a}^\pm(\zeta) = 0$ *if* $|\zeta| \ge T$;
(iii) $\widehat{a}^\pm(0) = \mathrm{vol}(B_d(0, r)) + O_d(r^{d-1}/T)$;
(iv) $|\widehat{a}(\zeta)| \ll_d r^d$.

Proof of Theorem 1.1. Let

$$\mathcal{B}_n = \left\{ B(y, r) \subset \mathbb{T}^d : r > \lambda_n^{-\theta_1} \right\}$$

with θ_1 to be determined later. Also, for $r > \lambda_n^{-\theta_1}$ let $a_n^\pm$ be the Beurling-Selberg polynomials from Lemma 2.5, which majorize and minorize the indicator function of the ball $B(0, r)$ and also satisfy $\widehat{a}_n^\pm(\zeta) = 0$ for $|\zeta| \ge T_n = \lambda_n^{\theta_2}$, with $\theta_2 > \theta_1$. The trigonometric polynomials

$$b_{n,y}^\pm(x) := a_n^\pm(x - y)$$

majorize and minorize the translated ball $B(y, r) = y + B(0, r)$, and their Fourier coefficients are given by $\widehat{b}_{n,y}^\pm(\zeta) = e^{-i\zeta \cdot y} \widehat{a}_n^\pm(\zeta)$, which therefore satisfy the same inequalities as $\widehat{a}_n^\pm(\zeta)$ in Lemma 2.5 (independently of y). In particular, $|\widehat{b}_{n,y}^\pm(\zeta)| \ll r_n^d$ and for $T_n = \lambda_n^{\theta_2}$ with $\theta_2 > \theta_1 \ge 0$ it follows that $\widehat{b}_{n,y}^\pm(0) = \mathrm{vol}(B(0, r))(1 + O(\lambda_n^{\theta_1 - \theta_2}))$.

For $\delta > 0$ let

$$\mathcal{S}^\pm = \left\{ \lambda_n : \sup_{B(y,r) \in \mathcal{B}_n} \left| \frac{\int_{\mathbb{T}^d} b_{n,y}^\pm(x) |\psi_n(x)|^2 \, \mathrm{dvol}(x)}{\int_{\mathbb{T}^d} b_{n,y}^\pm(x) \, \mathrm{dvol}(x)} - 1 \right| \ge \lambda_n^{-\delta} \right\}.$$

We will now show that for $\theta_1 < \theta_2 < \frac{1}{2(d-1)} - \delta$ the sets $\mathcal{S}^{\pm}$ have zero density. First note that

$$\sup_{B(y,r)\in\mathcal{B}_n}\left|\frac{\int_{\mathbb{T}^d} b^{\pm}_{n,y}(x)|\psi_n(x)|^2 \ \mathrm{dvol}(x)}{\int_{\mathbb{T}^d} b^{\pm}_{n,y}(x) \ \mathrm{dvol}(x)} - 1\right| \leq \sum_{1\leq|\zeta|\leq\lambda_n^{\theta_2}} \sup_{B(y,r)\in\mathcal{B}_n}\left|\frac{\widehat{b}^{\pm}_{n,y}(\zeta)}{\widehat{b}^{\pm}_{n,y}(0)}\langle e_\zeta\psi_n, \psi_n\rangle\right|$$
$$\ll \sum_{1\leq|\zeta|\leq\lambda_n^{\theta_2}} \left|\langle e_\zeta\psi_n, \psi_n\rangle\right|.$$

Next, apply Chebyshev's inequality, the above estimate, and Lemma 2.2 to get that

$$\frac{\#\{\lambda_n\in\mathcal{S}^{\pm} : \lambda_n \leq \Lambda\}}{\#\{\lambda_n\leq\Lambda\}} \ll \frac{1}{\Lambda^{d/2-\delta}}\sum_{\lambda_n\leq\Lambda}\sup_{B(y,r)\in\mathcal{B}_n}\left|\frac{\int_{\mathbb{T}^d} b^{\pm}_{n,y}(x)|\psi_n(x)|^2 \ \mathrm{dvol}(x)}{\int_{\mathbb{T}^d} b^{\pm}_{n,y}(x) \ \mathrm{dvol}(x)} - 1\right|$$
$$\ll \frac{1}{\Lambda^{d/2-\delta}}\sum_{\lambda\leq\Lambda}\sum_{1\leq|\zeta|\leq\lambda^{\theta_2}}\sum_{\lambda_n=\lambda}\left|\langle e_\zeta\psi_n, \psi_n\rangle\right|$$
$$\ll \frac{1}{\Lambda^{d/2-\delta}}\sum_{\lambda\leq\Lambda}\sum_{1\leq|\zeta|\leq\lambda^{\theta_2}}\#\{\mu\in\mathbb{Z}^d : |\mu|^2 = \lambda = |\mu+\zeta|^2\}.$$

By Lemma 2.3

$$\sum_{\lambda\leq\Lambda}\sum_{1\leq|\zeta|\leq\lambda^{\theta_2}}\#\{\mu\in\mathbb{Z}^d : |\mu|^2 = \lambda = |\mu+\zeta|^2\} \ll \Lambda^{(d-1)/2}\sum_{1\leq|\zeta|\leq\Lambda^{\theta_2}}\frac{1}{|\widehat{\zeta}|},$$

where $\zeta = m\widehat{\zeta}$ and $\widehat{\zeta}$ is primitive. The last sum is bounded by

$$\sum_{1\leq m\leq\Lambda^{\theta_2}}\sum_{1\leq|\widehat{\zeta}|\leq\Lambda^{\theta_2}/m}\frac{1}{|\widehat{\zeta}|} \ll \Lambda^{\theta_2(d-1)}\sum_{1\leq m\leq\Lambda^{\theta_2}}\frac{1}{m^{d-1}} \ll \begin{cases}\Lambda^{\theta_2}\log\Lambda \text{ if } d=2,\\ \Lambda^{\theta_2(d-1)} \text{ if } d\geq 3.\end{cases}$$

Collecting estimates, we have shown that

$$\frac{\#\{\lambda_n\in\mathcal{S}^{\pm} : \lambda_n\leq\Lambda\}}{\#\{\lambda_n\leq\Lambda\}} \ll \Lambda^{\theta_2(d-1)-\frac{1}{2}+\delta}\log\Lambda$$

which tends to zero for $\theta_2 < \frac{1}{2(d-1)} - \delta$.

To conclude the proof first observe that if $\lambda_n \notin \mathcal{S}^{+}$ with $\theta_1 < \theta_2 < \frac{1}{2(d-1)} - \delta$ (so λ_n lies in a set of density one) it follows by parts (i) and (iii) of Lemma 2.5 that

$$\int_{B(y,r)}|\psi_n(x)|^2\,\mathrm{dvol}(x) - \mathrm{vol}(B(y,r))$$
$$\leq \int_{\mathbb{T}^d} b^{+}_{n,y}(x)|\psi_n(x)|^2 \ \mathrm{dvol}(x) - \widehat{b}^{+}_{n,y}(0) + O(r^d\lambda_n^{\theta_1-\theta_2}). \tag{2.2}$$

A similar analysis holds for $\lambda_n \notin \mathcal{S}^-$ with the inequality reversed. Therefore, for $\lambda_n \notin (\mathcal{S}^+ \cup \mathcal{S}^-)$ and $\theta_1 < \theta_2 < \frac{1}{2(d-1)} - \delta$ (so λ_n lies in a density one set)

$$\begin{aligned} &\sup_{B(y,r)\in\mathcal{B}_n} \left| \int_{B(y,r)} |\psi_n(x)|^2 \,\mathrm{dvol}(x) - \mathrm{vol}(B(y,r)) \right| \\ &\le \max_{\pm} \sup_{B(y,r)\in\mathcal{B}_n} \left| \int_{\mathbb{T}^d} b^{\pm}_{n,y}(x)|\psi_n(x)|^2 \,\mathrm{dvol}(x) - \widehat{b^{\pm}_{n,y}}(0) \right| + O(r^d \lambda_n^{\theta_1-\theta_2}) \\ &\ll r^d \lambda^{-\delta} + r^d \lambda_n^{\theta_1-\theta_2}, \end{aligned} \tag{2.3}$$

so the claim follows. □

3. Irregularities of quantum equidistribution

In the previous section we saw that given an ONB of eigenfunctions $\{\psi_n\}$ the L^2 mass of almost all eigenfunctions ψ_n equidistributes within balls with radii $r_n \ge \lambda_n^{-\frac{1}{2(d-1)}+o(1)}$. We will show the existence of a sequence of eigenvalues $\{\lambda_m\}$ which tends to infinity with corresponding eigenfunctions whose L^2 mass is not equidistributed within balls with radii $r_m \ge \lambda_m^{-1/2+o(1)}$, which is just above the Planck scale.

Theorem 3.1. *There exists a sequence $\{\lambda_m\}_m$ of eigenvalues of $-\Delta$ on $\mathbb{T}^d$ with $\lambda_m \to \infty$ and corresponding L^2-normalized eigenfunctions ψ_m so that for any choice of radii r_m so that $r_m \to 0$, but $r_m\sqrt{\lambda_m} \to \infty$,*

$$\frac{1}{\mathrm{vol}(B(0,r_m))} \int_{B(0,r_m)} |\psi_n(x)|^2 \,\mathrm{dvol}(x) = 2 + o(1) \qquad (m \to \infty).$$

Proof. Let $\lambda_m = m^2 + (m+1)^2$ and take

$$\psi_m(x) = \cos(mx_1 + (m+1)x_2) + \cos((m+1)x_1 + mx_2),$$

where $x = (x_1, x_2, \dots, x_d)$, which are L^2-normalized eigenfunctions on $(\mathbb{T}^d, \mathrm{dvol})$ with eigenvalue λ_m. See Fig. 1 for a plot of the intensities $|\psi_m(x)|^2$.

Squaring out we get

$$\begin{aligned} |\psi_m(x)|^2 = 1 &+ \cos(x_1 - x_2) + \cos((2m+1)(x_1+x_2)) \\ &+ \frac{1}{2}\cos((2m+2)x_1 + 2mx_2) + \frac{1}{2}\cos(2mx_1 + (2m+2)x_2) \end{aligned}$$

and we wish to average this over the ball $B(0, r_m)$.

For the term $\cos(x_1 - x_2)$, observe that its average over $B(0, r_m)$ tends to 1, because on this shrinking ball, we have $|x_1 - x_2| \le 2r_m$ and hence $\cos(x_1 - x_2) = 1 + O(r_m^2)$, so that

$$\frac{1}{\mathrm{vol}(B(0,r_m))} \int_{B(0,r_m)} \cos(x_1 - x_2) \,\mathrm{dvol}(x) = 1 + O(r_m^2) \to 1, \quad \text{as } r_m \to 0.$$

To handle the other three terms, note that if $\mu \in \mathbb{Z}^d$ is any frequency vector, then changing variables we find

$$\frac{1}{\mathrm{vol}(B(0,r_m))} \int_{B(0,r_m)} \cos(\langle \mu, x\rangle) \,\mathrm{dvol}(x) = \frac{1}{\mathrm{vol}(B(0,1))} \int_{B(0,1)} \cos(\langle r_m\mu, y\rangle) \,\mathrm{dvol}(y)$$

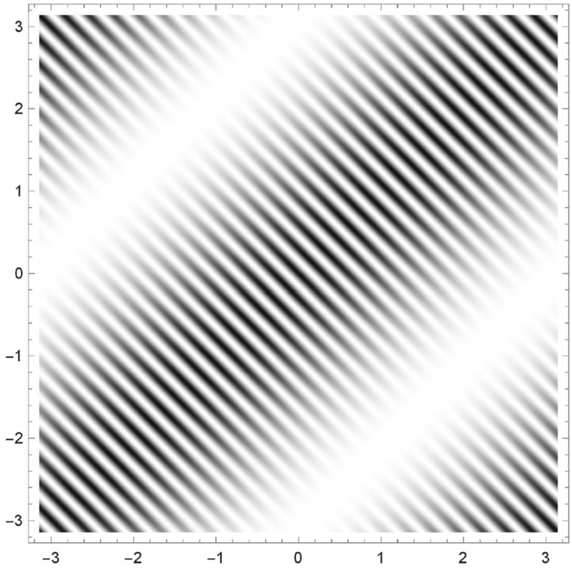

Fig. 1. Plot of the intensities of $|\psi_m(x)|^2$ for $m = 10$ in dimension $d = 2$, where $\psi_m(x) = \cos(mx_1 + (m + 1)x_2) + \cos((m + 1)x_1 + mx_2)$

that is we get the Fourier transform of the unit ball at the frequency $r_m\mu$. As is well known, the Fourier transform of the unit ball decays in *all directions*:

$$\frac{1}{\mathrm{vol}(B(0,1))}\int_{B(0,1)} \cos(\langle \xi, x\rangle)\,\mathrm{dvol}(x) \to 0, \quad \text{as } |\xi| \to \infty.$$

Therefore, applying this to the frequency vectors $\mu = (2m, 2m+2, \vec{0})$, $(2m+2, 2m, \vec{0})$ and $(2m+1, 2m+1, \vec{0})$, which have length $|\mu| \approx m$, we get

$$\frac{1}{\mathrm{vol}(B(0,r_m))}\int_{B(0,r_m)} \cos(\langle \mu, x\rangle)\,\mathrm{dvol}(x) \to 0, \quad r_m|\mu| \to \infty.$$

Thus whenever $r_m \to 0$, with $r_m \cdot m \approx r_m\sqrt{\lambda_m} \to \infty$,

$$\frac{1}{\mathrm{vol}(B(0,r_m))}\int_{B(0,r_m)} |\psi_m(x)|^2\,\mathrm{dvol}(x) = 2 + o(1)$$

giving our claim. □

4. Below the Critical Radius: $r < \lambda^{-\frac{1}{2(d-1)}}$

In this section, we detail Bourgain's argument which gives for balls with radii $r < \lambda^{-\frac{1}{2(d-1)}-o(1)}$ that in each eigenspace there is an orthonormal set of eigenfunctions with size exceeding a positive multiple of the dimension of the eigenspace, which consists of eigenfunctions whose L^2 mass is scarce in the ball $B(0,r)$.

Denote

$$\mathcal{E}_\lambda = \{\mu \in \mathbb{Z}^d : |\mu|^2 = \lambda\}, \qquad N_\lambda = \#\mathcal{E}_\lambda.$$

Theorem 4.1 (Bourgain). *Suppose $d \geq 3$. Also, if $d = 3$ suppose $\lambda \not\equiv 0, 4, 7 \mod 8$, and, if $d = 4$ suppose λ is odd. Then for each such λ-eigenspace there exists an orthonormal set of eigenfunctions $\mathcal{A} \subset \{\psi_{\lambda_n}\}_{\lambda_n=\lambda}$ with size $\#\mathcal{A} \gg N_\lambda$ such that for each $\psi \in \mathcal{A}$*

$$\frac{1}{\operatorname{vol}(B(0,r))} \int_{B(0,r)} |\psi(x)|^2 \, \mathrm{dvol}(x) \to 0 \qquad (\lambda \to \infty)$$

provided that $r < \lambda^{-\frac{1}{2(d-1)}-o(1)}$.

Completing the orthonormal set $\mathcal{A}$ in Theorem 4.1 (in any way) gives an ONB of eigenfunctions $\mathcal{B}$ with the property that a positive proportion of $\psi \in \mathcal{B}$ do not equidistribute within the small balls $B(0,r)$, $r < \lambda^{\frac{1}{2(d-1)}-o(1)}$. Hence, the scale achieved in Theorems 1.1 and 1.2 is sharp.

Before detailing Bourgain's argument we require the following lemma on the distribution of points on spheres. For each point $\mu \in \sqrt{\lambda}S^{d-1}$, we associate the cap $\operatorname{cap}(\mu; Y) = \operatorname{Ball}(\mu, Y) \cap \sqrt{\lambda}S^{d-1}$ of size Y about μ, where $\operatorname{Ball}(x, Y) = \{y \in \mathbb{R}^d : |x - y| \leq Y\}$.

Lemma 4.2. *Suppose for a sequence of λ's, we are given a finite set $\mathcal{A}_\lambda \subset \sqrt{\lambda}S^{d-1}$ of points on the sphere, with cardinality $\#\mathcal{A}_\lambda \to \infty$ as $\lambda \to \infty$. Let $Y = Y_\lambda$ satisfy $Y_\lambda \gg \lambda^{1/2+o(1)}/(\#\mathcal{A}_\lambda)^{\frac{1}{d-1}}$. Then the set $\mathcal{V} \subset \mathcal{A}_\lambda$ consisting of $\nu \in \mathcal{A}_\lambda$ such that*

$$\#\left(\mathcal{A}_\lambda \cap \operatorname{cap}(\nu, Y)\right) \geq 2$$

has density one: $\#\mathcal{V} \sim \#\mathcal{A}_\lambda$ as $\lambda \to \infty$.

Proof. Let

$$\mathcal{U} = \{\mu \in \mathcal{A}_\lambda : \#(\mathcal{A}_\lambda \cap \operatorname{cap}(\mu; Y)) < 2\}.$$

We wish to show that $\#\mathcal{U} = o(\#\mathcal{A}_\lambda)$.

Each point on the sphere $\sqrt{\lambda}S^{d-1}$ is contained in at most one of the caps of size $Y/2$ around $\mu \in \mathcal{U}$, because if $\operatorname{cap}(\mu_1; Y/2) \cap \operatorname{cap}(\mu_2; Y/2)$ is non-empty for distinct $\mu_1 \neq \mu_2 \in \mathcal{U}$ then $\mu_2 \in \operatorname{cap}(\mu_1; Y)$ contradicting the assumption $\mu_2 \in \mathcal{U}$. Consequently the caps $\operatorname{cap}(\mu_1; Y/2)$ and $\operatorname{cap}(\mu_1; Y/2)$ are disjoint, so that we have

$$\operatorname{vol}\Big(\bigcup_{\mu\in\mathcal{U}} \operatorname{cap}(\mu, Y/2)\Big) = \sum_{\mu\in\mathcal{U}} \operatorname{vol}\Big(\operatorname{cap}(\mu, Y/2)\Big) \approx Y^{d-1}\#\mathcal{U}$$

and also we have the trivial bound

$$\operatorname{vol}\Big(\bigcup_{\mu\in\mathcal{U}} \operatorname{cap}(\mu, Y/2)\Big) \leq \operatorname{vol}(\sqrt{\lambda}S^{d-1}) \ll_d \lambda^{(d-1)/2}.$$

Combining these formulas we obtain

$$\frac{\#\mathcal{U}}{\#\mathcal{A}_\lambda} \ll \frac{1}{\#\mathcal{A}_\lambda} \cdot \frac{\lambda^{(d-1)/2}}{Y^{d-1}} \ll \lambda^{-o(1)} \to 0$$

under our assumption on Y, which gives the claim. $\square$

Proof of Theorem 4.1. First observe that if we have two distinct lattice points $\mu \neq \mu' \in \mathcal{E}_\lambda$, which are close: $0 < |\mu - \mu'| < M_\lambda$ (we will take $M_\lambda = \lambda^{\frac{1}{2(d-1)}+o(1)}$), then the eigenfunction

$$\psi_\mu(x) := \frac{1}{\sqrt{2}}\Big(e^{i\langle\mu,x\rangle} - e^{i\langle\mu',x\rangle}\Big)$$

fails to equidistribute in the ball $B(0,r)$ centered at the origin for any $r = o(M_\lambda^{-1})$. Indeed, for $x \in B(0,r)$

$$|\psi_\mu(x)|^2 = 1 - \cos(\langle\mu-\mu',x\rangle) = O\Big((r|\mu-\mu'|)^2\Big)$$

and since $r|\mu-\mu'| \le rM_\lambda = o(1)$, we have

$$|\psi_\mu(x)|^2 = o(1), \quad x \in B(0,r).$$

Therefore

$$\frac{1}{\mathrm{vol}(B(0,r))}\int_{B(0,r)} |\psi_\mu(x)|^2\, \mathrm{dvol}(x) \to 0.$$

Next we claim that there is a set $\mathcal{S} \subset \mathcal{E}_\lambda$ containing a positive proportion of μ's ($\#\mathcal{S}/N_\lambda \gg 1$) such that :

- for each $\mu \in \mathcal{S}$ there is another lattice point μ' which is close to μ: $|\mu-\mu'| < \lambda^{\frac{1}{2(d-1)}+o(1)}$;
- if $\mu \neq \nu \in \mathcal{S}$ are distinct, then the pairs $\{\mu,\mu'\}$ and $\{\nu,\nu'\}$ are disjoint, that is $\nu \neq \mu'$ and $\nu' \neq \mu,\mu'$.

Given this, we form for each $\mu \in \mathcal{S}$ the eigenfunction ψ_μ, and then for $\mu \neq \nu \in \mathcal{S}$ the pairs $\{\mu,\mu'\}$ and $\{\nu,\nu'\}$ are disjoint, and so the eigenfunctions ψ_μ and ψ_ν are orthogonal. This establishes Bourgain's result Theorem 4.1

It remains to prove the claim. Let $Y_\lambda = \lambda^{\frac{1}{2(d-1)}+o(1)}$. We construct $\mathcal{S}$ as follows: In Lemma 4.2 first take $\mathcal{A}_\lambda^0 = \mathcal{E}_\lambda$, and note that under the assumption of the theorem on λ, we have $\#\mathcal{A}_\lambda^0 = N_\lambda \gg \lambda^{\frac{d}{2}-1-o(1)}$ (see Sect. 5.1) so that $Y_\lambda \gg \lambda^{1/2+o(1)}/N_\lambda^{\frac{1}{d-1}}$. Hence by Lemma 4.2 we get a set $\mathcal{V}$ of density one. Take some $\mu \in \mathcal{V}$; then there exists $\mu' \in \mathcal{A}_\lambda^0$ such that $0 < |\mu-\mu'| < \lambda^{\frac{1}{2(d-1)}+o(1)}$. Now remove the pair $\{\mu,\mu'\}$ from $\mathcal{E}_\lambda$, to obtain a smaller set $\mathcal{A}_\lambda^1 = \mathcal{A}_\lambda^0\backslash\{\mu,\mu'\}$, and repeat this process $(\frac{1}{2}-o(1))N_\lambda$ times, at each time getting a non-empty remainder set $\mathcal{A}_\lambda^j$, of size $\#\mathcal{A}_\lambda^j \gg N_\lambda$, so that still $Y_\lambda \gg \lambda^{1/2+o(1)}/\#\mathcal{A}_\lambda^j$ and we can continue to invoke Lemma 4.2.

We obtain $(\frac{1}{2}-o(1))N_\lambda$ resulting pairs, which by construction are close and disjoint. In this way we obtain a set $\mathcal{S}$ of density $\frac{1}{2}-o(1)$ with the desired properties. □

5. Results for Individual Eigenspaces

5.1. Arithmetic background. We denote by $R_d(n)$ the number of representations of n as a sum of d squares. This is the dimension of the n-eigenspace of the Laplacian on $\mathbb{T}^d$.

For $d = 4$, Jacobi's four square theorem says that $R_4(n) = 8\sum_{d|n,4\nmid d} d$ so that $R_4(n) \ll n^{1+o(1)}$ and for n odd we have a lower bound $R_4(n) \geq 8n$.

For $d = 3$, we have $R_3(n) \ll n^{1/2+o(1)}$ and Siegel's theorem says that for $n \neq 0, 4, 7 \mod 8$, we have a lower bound $R_3(n) \gg n^{1/2-o(1)}$. When $d \geq 5$, a classical result of Hardy and Ramanujan gives $R_d(n) \approx n^{d/2-1}$. For more details on these bounds including more precise formulas see e.g. [16, Chapter 11], [8].

For $n, t \geq 1$ let $A_d(n, t)$ denote the number of representations of the positive definite binary quadratic form

$$Q(x, y) = nx^2 + 2txy + ny^2$$

as a sum of squares of d linear forms. That is,

$$A_d(n, t) = \#\left\{(\mu, \nu) \in \mathbb{Z}^d \times \mathbb{Z}^d : \sum_{j=1}^{d}(\mu_j x + \nu_j y)^2 = Q(x, y)\right\}.$$

where x, y are indeterminates. Equivalently,

$$A_d(n, t) = \#\left\{(\mu, \nu) \in \mathbb{Z}^d \times \mathbb{Z}^d : |\mu|^2 = |\nu|^2 = n \text{ and } \langle \mu, \nu\rangle = t\right\}.$$

The number of representations of quadratic forms by quadratic forms has been widely studied. This generalizes the classical problem of representing integers by quadratic forms, and for a survey of results on these problems see [30]. The study of the more specific case of representing a quadratic form by a sum of squares of linear forms dates back to at least Mordell, who studied the criteria for which such a representation exists in a small number of variables (such a representation always exists if the number of variables is sufficiently large). In the case $d = 3$ Venkov [33] [34, Chapter 4.16] and Pall [25,26] studied $A_3(n, t)$, obtaining an exact, but complicated formula for it. From this one can deduce the following useful bound:

Lemma 5.1. *If $|t| < n$ then*

$$A_3(n, t) \ll \gcd(n, t)^{1/2} n^{o(1)}.$$

This kind of bound was stated and used by Linnik [20], who omitted the factor of $\gcd(n, t)^{1/2}$. A correct version was given by Pall [25, §7], [26, Theorem 4], see also [3, Proposition 2.2].

In the case $d = 4$, Pall and Taussky [27] established an exact formula for $A_4(n, t)$. The relevant case for us will be when n is odd, in this case their formulas states the following.

Lemma 5.2. *If n is odd and $|t| < n$ then setting $e := \gcd(n, t)$, we have*

$$A_4(n, t) = \sum_{h|e} R_4(h) \cdot \#\{\nu \in \mathbb{Z}^3 : |\nu|^2 = n^2 - t^2, \gcd(\nu_1, \nu_2, \nu_3, e) = h\}.$$

In particular, for n odd Lemma 5.2 gives

$$A_4(n, t) \geq 8 \cdot R_3(n^2 - t^2). \tag{5.1}$$

with equality holding if $\gcd(n,t)=1$. This is seen by using $R_4(h)\ge 8$ for odd h and noting that every ν with $|\nu|^2=n^2-t^2$, will satisfy $\gcd(\nu_1,\nu_2,\nu_3,e)=h$ for some $h\mid e$.

To get an upper bound for $A_4(n,t)$, first note that for $|t|<n$ and $h|e$

$$\#\{\nu\in\mathbb{Z}^3 : |\nu|^2=n^2-t^2, \gcd(\nu_1,\nu_2,\nu_3,e)=h\} \\ \le R_3\left(\frac{n^2-t^2}{h^2}\right) \ll \left(\frac{n^2-t^2}{h^2}\right)^{1/2+o(1)},$$

where in the last step we used the bound $R_3(m)\ll m^{1/2+o(1)}$. Now use this estimate in Lemma 5.2 along with the bounds $R_4(h)\ll h^{1+o(1)}$ and $\sum_{h|e}1\ll e^{o(1)}$ to get for n odd and $|t|<n$ that

$$A_4(n,t)\ll n^{1/2+o(1)}(n-t)^{1/2}, \tag{5.2}$$

uniformly for $|t|<n$.

5.2. L^1 discrepancy for each λ-eigenspace. For $a\in C(\mathbb{T}^d)$ define the localized L^1 discrepancy

$$V_1^{\mathrm{loc}}(a,\lambda)=\sum_{\lambda_n=\lambda}|\langle a\psi_n,\psi_n\rangle-\langle a,1\rangle|.$$

Lemma 5.3. *Suppose $T\le\sqrt{2\lambda}$. Then*

$$\sum_{1\le|\zeta|\le T}V_1^{\mathrm{loc}}(e_\zeta,\lambda)\le\sum_{\lambda-T^2/2\le t\le\lambda-1}A_d(\lambda,t).$$

Proof. Applying Lemma 2.2 gives

$$\sum_{1\le|\zeta|\le T}V_1^{\mathrm{loc}}(e_\zeta,\lambda)\le\sum_{2\le|\zeta|^2\le T^2}\#\left\{\mu : |\mu|^2=\lambda=|\mu+\zeta|^2\right\}.$$

(Note we can ignore ζ with $|\zeta|^2$ odd, since for these $\langle e_\zeta\psi_n,\psi_n\rangle=0$.) Next, observe that

$$\sum_{2\le|\zeta|^2\le T^2}\#\left\{\mu : |\mu|^2=\lambda=|\mu+\zeta|^2\right\}=\sum_{2\le\ell\le T^2}\sum_{\substack{\mu,\nu\in\mathbb{Z}^d\\ |\mu|^2=\lambda=|\nu|^2\\ |\mu-\nu|^2=\ell}}1.$$

For $|\mu|^2=|\nu|^2=\lambda$ we have $|\mu-\nu|^2=\ell$ iff $\langle\mu,\nu\rangle=(\lambda-\ell/2)$. Hence,

$$\sum_{2\le\ell\le T^2}\sum_{\substack{\mu,\nu\in\mathbb{Z}^d\\ |\mu|^2=\lambda=|\nu|^2\\ |\mu-\nu|^2=\ell}}1=\sum_{\lambda-T^2/2\le t\le\lambda-1}A_d(\lambda,t).$$

□

5.3. Proof of Theorem 1.2. Suppose that $d = 3$ or $d = 4$. For $d = 3$ suppose $\lambda \not\equiv 0, 4, 7 \pmod 8$, so the dimension of the λ-eigenspace, N_λ, is $\approx \lambda^{1/2\pm o(1)}$; if $d = 4$ suppose λ is odd so that $\lambda \ll N_\lambda \ll \lambda^{1+o(1)}$. Let

$$\mathcal{B}_\lambda = \left\{ B(y, r) \subset \mathbb{T}^d : r \geq \lambda^{-\theta_1} \right\}$$

for θ_1 to be determined later. As in the proof of Theorem 1.1 we take $b^{\pm}_{n,y}$ to be Beurling-Selberg polynomials which majorize and minorize the indicator function of the ball $B(y, r)$ with $r \geq \lambda_n^{-\theta_1} = \lambda^{-\theta_1}$. We take the lengths of the polynomials $b^{\pm}_{n,y}$ to be $T_n = \lambda_n^{\theta_2} = \lambda^{\theta_2}$ with $\theta_2 > \theta_1$. Given an orthonormal basis $\{\psi_n\}_{\lambda_n=\lambda}$ of the λ-eigenspace define

$$\mathcal{S}^{\pm}_\lambda = \left\{ \lambda_n = \lambda : \sup_{B(y,r)\in\mathcal{B}_\lambda} \left| \frac{\int_{\mathbb{T}^d} b^{\pm}_{n,y}(x)|\psi_n(x)|^2 \, \mathrm{dvol}(x)}{\int_{\mathbb{T}^d} b^{\pm}_{n,y}(x) \, \mathrm{dvol}(x)} - 1 \right| \geq \lambda^{-\delta} \right\}.$$

Using Lemma 5.3 along with the bound $\widehat{b^{\pm}_{n,y}} \ll r^d$ given by Lemma 2.5 (iv) (which holds uniformly in y), we get from Chebyshev's inequality as in the proof of Theorem 1.1 that

$$\begin{aligned} \frac{\#\mathcal{S}^{\pm}_\lambda}{N_\lambda} &\ll \frac{1}{\lambda^{\frac{d}{2}-1-2\delta}} \sum_{1\leq|\zeta|\leq\lambda^{\theta_2}} \sum_{\lambda_n=\lambda} |\langle e_\zeta \psi_n, \psi_n\rangle| \sup_{B(y,r)\in\mathcal{B}_\lambda} \left| \frac{\widehat{b^{\pm}_{n,y}}(\zeta)}{\widehat{b^{\pm}_{n,y}}(0)} \right| \\ &\ll \frac{1}{\lambda^{\frac{d}{2}-1-2\delta}} \sum_{1\leq|\zeta|\leq\lambda^{\theta_2}} V_1^{\mathrm{loc}}(e_\zeta, \lambda) \\ &\ll \frac{1}{\lambda^{\frac{d}{2}-1-2\delta}} \sum_{\lambda-\lambda^{2\theta_2}/2\leq t<\lambda} A_d(\lambda, t). \end{aligned}$$

Since we assume $d = 3$ and $\lambda \not\equiv 0, 4, 7 \pmod 8$ or $d = 4$ and λ odd, combining Lemma 5.1 and (5.2) gives

$$A_d(\lambda, t) \ll \lambda^{(d-3)/2+o(1)} \gcd(\lambda, t)(\lambda - t)^{(d-3)/2}.$$

Thus,

$$\begin{aligned} \sum_{\lambda-\lambda^{2\theta_2}/2\leq t<\lambda} A_d(\lambda, t) &\ll \lambda^{(d-3)/2+\theta_2(d-3)+o(1)} \sum_{\lambda-\lambda^{2\theta_2}/2\leq t<\lambda} \gcd(\lambda, t) \\ &\ll \lambda^{(d-3)/2+\theta_2(d-3)+o(1)} \sum_{e|\lambda} e \sum_{\frac{\lambda-\lambda^{2\theta_2}/2}{e}\leq t_0<\lambda/e} 1 \\ &\ll \lambda^{(d-3)/2+\theta_2(d-1)+o(1)}, \end{aligned}$$

where in the last step we bounded the inner sum as $O(\lambda^{2\theta_2}/e)$ since if $\lambda^{2\theta_2}/(2e) < 1$ then the sum is empty. Collecting estimates gives

$$\frac{\#\mathcal{S}^{\pm}_\lambda}{N_\lambda} \ll \lambda^{\theta_2(d-1)-\frac{1}{2}+3\delta},$$

which tends to zero if $\theta_1 < \theta_2 < \frac{1}{2(d-1)} - 3\delta$.

Thus, the subset of the ONB $\{\psi_n\}_{\lambda_n=\lambda}$, which consists of eigenfunctions ψ_n with $\lambda_n \notin (S_\lambda^+ \cup S_\lambda^-)$ has cardinality $N_\lambda(1+o(1))$ provided $\theta_1 < \theta_2 < \frac{1}{2(d-1)} - 3\delta$. Repeating the same argument given at the end of the proof of Theorem 1.1 (see equations (2.2), (2.3)) we see that each eigenfunction in this subset satisfies

$$\sup_{B(y,r)\in\mathcal{B}_\lambda} \left| \int_{B(y,r)} |\psi_n(x)|^2 \, \mathrm{dvol}(x) - \mathrm{vol}(B(y,r)) \right| \ll r^d \lambda^{-\delta} + r^d \lambda^{\theta_1-\theta_2}.$$

□

Remark 5.4. Our argument reduces the problem of small scale quantum ergodicity to a lattice point estimate, which can be rephrased in terms of statistics of lattice points in caps: For each lattice point $\nu \in \mathcal{E}_\lambda = \{\mu \in \mathbb{Z}^d : |\mu|^2 = \lambda\}$, let

$$\mathfrak{n}(\nu, Y) = \#(\mathcal{E}_\lambda \cap \mathrm{cap}(\nu, Y)) - 1 = \#\{\mu \in \mathcal{E}_\lambda : 0 < |\mu - \nu| \leq Y\} \tag{5.3}$$

be the number of *other* lattice points in a cap of size Y about ν. In fact we actually show that in any dimension $d \geq 3$ if

$$\frac{1}{N_\lambda} \sum_{|\nu|^2=\lambda} \mathfrak{n}(\nu, Y) \to 0, \quad \text{as } \lambda \to \infty$$

then the assertion of Theorem 1.2 holds in dimension d at scales $r > Y^{-1+o(1)}$ (we also assume here that $\lambda \not\equiv 0, 4, 7 \pmod 8$) if $d = 3$ and λ is odd if $d = 4$, for $d \geq 5$ no such restrictions are needed). That is, given the above, small scale quantum ergodicity holds in dimension d at scales above $r > Y^{-1+o(1)}$ on every such λ-eigenspace.

6. Massive Irregularities

In this section we are concerned with the existence of a sequence of eigenfunctions ψ_λ for which the proportion of the L^2 mass of ψ_λ within small balls becomes arbitrarily large as $\lambda \to \infty$. For $d = 4$ we show the existence of such a sequence of eigenfunctions ψ_λ for balls with radii $r_\lambda \leq \lambda^{-1/6-o(1)}$. On the other hand, for $d = 2$ we are able to rule out this behavior for balls with radii that shrink sufficiently slowly.

6.1. Blowup for $d = 4$. Let

$$\psi_\lambda(x) = \frac{1}{\sqrt{N_\lambda}} \sum_{|\mu|^2=\lambda} e_\mu(x). \tag{6.1}$$

We show that at small scales the L^2 mass of ψ_λ blows up in dimension $d = 4$.

Theorem 6.1. *Let $\psi_m = \psi_{\lambda_m}$ be as given in (6.1) in dimension $d = 4$. Then along the sequence of odd eigenvalues λ_m we have for any sequence of radii $r_m < \lambda_m^{-1/6-o(1)}$,*

$$\lim_{m\to\infty} \frac{1}{\mathrm{vol}(B(0, r_m))} \int_{B(0,r_m)} |\psi_m(x)|^2 \, \mathrm{dvol}(x) = \infty.$$

Note that the result is trivial for $r = o(\lambda^{-1/2})$, because then for $x \in B(0,r)$ we can replace $\psi_\lambda(x) \sim \psi(0) = \sqrt{N_\lambda}$ and then the average of $|\psi_\lambda(x)|^2$ over the ball $B(0,r)$ will be large. This also implies that for $r \geq \varepsilon\lambda^{-1/2}$ with $\varepsilon > 0$ sufficiently small

$$\frac{1}{\mathrm{vol}(B(0,r))}\int_{B(0,r)} |\psi_\lambda(x)|^2\,\mathrm{dvol}(x) \geq \frac{1}{\mathrm{vol}(B(0,r))}\int_{B(0,\varepsilon\lambda^{-1/2})} |\psi_\lambda(x)|^2\,\mathrm{dvol}(x) \gg \frac{N_\lambda}{r^d}\cdot\varepsilon^d\lambda^{-d/2}$$

in every dimension $d \geq 2$. Recall for $d \geq 3$, $N_\lambda \gg \lambda^{\frac{d}{2}-1-o(1)}$ provided that λ is odd if $d = 4$ and if $d = 3$, $\lambda \not\equiv 0, 4, 7 \pmod 8$. For such λ the RHS tends to infinity for $r_\lambda \leq \lambda^{-\frac{1}{d}-o(1)}$. Theorem 6.1 shows that massive irregularities extend beyond this trivial regime.

For $T \leq \sqrt{2\lambda}$ let

$$S_d(\lambda, T) = \sum_{\lambda - T^2/2 \leq t < \lambda} A_d(\lambda, t)$$

and note that in the proof of Lemma 5.3 we saw that

$$S_d(\lambda, T) = \sum_{2 \leq |\zeta|^2 \leq T^2} \#\{\mu : |\mu|^2 = \lambda = |\mu + \zeta|^2\}. \tag{6.2}$$

Lemma 6.2. *Let ψ_λ be as in (6.1). Suppose that $r_\lambda \to 0$ as $\lambda \to \infty$. Then for any dimension $d \geq 2$*

$$\frac{1}{\mathrm{vol}\,B(0,r_\lambda)}\int_{B(0,r_\lambda)} |\psi_\lambda(x)|^2\,\mathrm{dvol}(x) \gg \frac{1 + S_d(\lambda, r_\lambda^{-1+o(1)})}{N_\lambda}.$$

Remark 6.3. The RHS is bounded below by the mean value

$$\frac{1}{N_\lambda}\sum_{|\nu|^2=\lambda} \mathfrak{n}(\nu, r_\lambda^{-1+o(1)})$$

of the other lattice points in caps of size $r_\lambda^{-1+o(1)}$, where $\mathfrak{n}(\nu, Y)$ is as defined in (5.3). So if this tends to infinity then the conclusion of Theorem 6.1 holds in dimension d at scales r_λ.

Proof. We first construct an auxiliary smooth minorant of $\mathbf{1}_{B(0,r_\lambda)}(x)$ on the torus. Let $f \in C_0^\infty(\mathbb{R})$ be a nonzero function such that $0 \leq f(x) \leq 1$ and supp $f = [-\frac{1}{2}, \frac{1}{2}]$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be given by $g(x) = f(|x|)$ and define $F_{r_\lambda} : \mathbb{T}^d \to \mathbb{R}$ by

$$F_{r_\lambda}(x) = \sum_{n \in \mathbb{Z}^d} (g * g)\left(\frac{x + 2\pi n}{r_\lambda}\right).$$

Observe that

$$(g * g)(y) = \int_{\mathbb{R}^d} f(|x|)f(|y - x|)dx < 1.$$

Also, for $|y| \geq 1$

$$0 \leq (g * g)(y) \leq \int_{|y-x|<\frac{1}{2}, |x|<\frac{1}{2}} 1 \, dx = 0.$$

It follows that $\mathbf{1}_{B(0,1)}(x) \geq (g * g)(x) \geq 0$. Write

$$\mathcal{F}(g * g)(\xi) = \int_{\mathbb{R}^d} (g * g)(x) e_{-\xi}(x) \frac{dx}{(2\pi)^d} = \left| \int_{\mathbb{R}^d} g(x) e_{-\xi}(x) \frac{dx}{(2\pi)^d} \right|^2$$

and note by Poisson summation

$$F_{r_\lambda}(x) = \sum_{n \in \mathbb{Z}^d} (g * g) \left(\frac{x + 2\pi n}{r_\lambda} \right) = r_\lambda^d \sum_{\zeta \in \mathbb{Z}^d} \mathcal{F}(g * g)(r_\lambda \zeta) e_\zeta(x).$$

Hence, $F_{r_\lambda} : \mathbb{T}^d \to \mathbb{R}$ is a smooth minorant of $\mathbf{1}_{B(0,r_\lambda)}(x)$ and has *non-negative* Fourier coefficients. Also, observe that $\mathcal{F}(g*g)(\xi) = \mathcal{F}(g*g)(0) + O(|\xi|)$. From these estimates we get that

$$\begin{aligned}
&\int_{B(0,r_\lambda)} |\psi_\lambda(x)|^2 \, \mathrm{dvol}(x) \geq \int_{\mathbb{T}^d} F_{r_\lambda}(x) |\psi_\lambda(x)|^2 \, \mathrm{dvol}(x) \\
&= \frac{r_\lambda^d}{N_\lambda} \sum_{\zeta \in \mathbb{Z}^d} \mathcal{F}(g * g)(r_\lambda \zeta) \#\{\mu : |\mu|^2 = \lambda = |\mu + \zeta|^2\} \\
&\geq \frac{r_\lambda^d}{2N_\lambda} \mathcal{F}(g * g)(0) \left(1 + \sum_{0 \neq |\zeta| \leq r_\lambda^{-1+o(1)}} \#\{\mu : |\mu|^2 = \lambda = |\mu + \zeta|^2\} \right)
\end{aligned}$$

by dropping the large frequencies using the non-negativity of $\mathcal{F}(g * g)$ and also noting note that $\mathcal{F}(g*g)(0) = \left| \int_{\mathbb{R}^d} g(x) \frac{dx}{(2\pi)^d} \right|^2 > 0$. Applying (6.2) to the inner sum completes the proof. □

6.2. Proof of Theorem 6.1. By Lemma 6.2 it suffices to show that for odd values of $\lambda \to \infty$ such that for any sequence $r_\lambda \ll \lambda^{-\frac{1}{6}-o(1)}$, we have

$$S_4(\lambda, r_\lambda^{-1+o(1)})/R_4(\lambda) \to \infty.$$

By definition, if $T = r^{-1+o(1)}$,

$$\begin{aligned}
S_4(\lambda, T) &= \sum_{0<\lambda-t\leq T^2/2} A_4(\lambda, t) \\
&\geq \sum_{\substack{0<\lambda-t\leq T^2/2 \\ t \text{ even}}} A_4(\lambda, t).
\end{aligned}$$

We now assume $r > \lambda^{-1/2}$ so that $|t| < \lambda$. Applying (5.1) for odd λ we have $A_4(\lambda, t) \geq 8R_3(\lambda^2 - t^2)$ so that

$$S_4(\lambda, T) \geq \sum_{\substack{0<\lambda-t\leq T^2/2 \\ t \text{ even}}} R_3(\lambda^2 - t^2).$$

Recall that if $n \not\equiv 0, 4, 7 \pmod 8$ then Siegel's theorem gives $R_3(n) \gg n^{\frac{1}{2}-o(1)}$. Now if λ is odd and t is even then $\lambda^2 - t^2 = 1, 5 \pmod 8$ and in particular Siegel's theorem implies

$$R_3(\lambda^2 - t^2) \gg (\lambda^2 - t^2)^{\frac{1}{2}-o(1)} \gg \lambda^{\frac{1}{2}-o(1)}(\lambda - t)^{\frac{1}{2}}.$$

Hence we find

$$\begin{aligned} S_4(\lambda, T) &\gg \lambda^{\frac{1}{2}-o(1)} \sum_{\substack{0<\lambda-t\leq T^2/2 \\ t \text{ even}}} (\lambda - t)^{\frac{1}{2}} \\ &= \lambda^{\frac{1}{2}-o(1)} \sum_{\substack{1\leq m\leq T^2/2 \\ m \text{ odd}}} m^{1/2} \gg \lambda^{\frac{1}{2}-o(1)} T^3. \end{aligned}$$

Hence for $T \approx r^{-1+o(1)}$ with $\lambda^{-1/2} < r \ll \lambda^{-1/6+o(1)}$

$$S_4(\lambda, r^{-1+o(1)}) \gg \lambda^{\frac{1}{2}-o(1)} r^{-3}.$$

Since $R_4(\lambda) \ll \lambda^{1+o(1)}$, we find that along the sequence of odd integers

$$\frac{S_4(\lambda, r_\lambda^{-1+o(1)})}{R_4(\lambda)} \gg \lambda^{-1/2-o(1)} r^{-3} \to \infty$$

for $r_\lambda \ll \lambda^{-\frac{1}{6}-o(1)}$. □

6.3. Ruling out blowup for $d = 2$ at certain scales. The construction of massive irregularities in the previous section used some features particular to high dimensions. In fact for $d = 2$, we can rule out the existence of this behavior at scales that are not too small, and expect that massive irregularities do not exist at all scales that are at least slightly above the Planck scale. More precisely, if $d = 2$ then for every eigenfunction ψ_λ we will prove that the proportion of L^2 mass inside balls with radii $r_\lambda > \lambda^{-1/4+o(1)}$ is bounded and we expect this should be true as long as $r_\lambda > \lambda^{-1/2+o(1)}$.

Proposition 6.4. *Let $\psi_\lambda(x)$ be an $L^2(\mathbb{T}^2, \mathrm{dvol})$ normalized eigenfuction in dimension $d = 2$ with eigenvalue λ. Then for any ball with radius $r_\lambda > \lambda^{-1/4+o(1)}$*

$$\sup_{y\in\mathbb{T}^2} \frac{1}{\mathrm{vol}(B(y, r))} \int_{B(y,r)} |\psi_\lambda(x)|^2 \,\mathrm{dvol}(x) \ll 1. \tag{6.3}$$

Proof. Let b_y^+ be the translated Beurling-Selberg polynomial described in the proof of Theorem 1.1 which majorizes the indicator function of $B(y, r)$ on $\mathbb{T}^2$ with length $T = 2/r$, so in particular $|\widehat{b_y^+}(\zeta)/\operatorname{vol}(B(y, r))| \ll 1$, uniformly for $y \in \mathbb{T}^2$. Write

$$\psi_\lambda(x) = \sum_{|\mu|^2=\lambda} c(\mu)e_\mu(x)$$

and argue as in the proof of Lemma 5.3 to get

$$\frac{1}{\operatorname{vol}(B(y,r))}\int_{B(y,r)} |\psi_\lambda(x)|^2 \operatorname{dvol}(x) \ll 1 + \sum_{2\le\ell\le T^2} \sum_{\substack{\mu,\nu\in\mathbb{Z}^2 \\ |\mu|^2=\lambda=|\nu|^2 \\ |\mu-\nu|^2=\ell}} |c(\mu)c(\nu)|$$

$$\le 1 + \sum_{|\mu|^2=\lambda} |c(\mu)|^2 \sum_{2\le\ell\le T^2} \sum_{\substack{|\nu|^2=\lambda \\ |\mu-\nu|^2=\ell}} 1,$$

uniformly for $y \in \mathbb{T}^2$. To bound the inner sum, let

$$M(R, \rho) = \max_{|\mu|=R} \#\{\nu \in \mathbb{Z}^2 : |\mu| = R = |\nu|,\ |\mu - \nu| \le \rho\}$$

be the maximal number of lattice points in an arc of size ρ on the circle of radius R. Note that

$$\sum_{2\le\ell\le T^2} \sum_{\substack{|\nu|^2=\lambda \\ |\mu-\nu|^2=\ell}} 1 \le M(\sqrt{\lambda}, T) - 1.$$

Since $T = 2/r$ we conclude

$$\sup_{y\in\mathbb{T}^2} \frac{1}{\operatorname{vol}(B(y,r))}\int_{B(y,r)} |\psi_\lambda(x)|^2 \operatorname{dvol}(x) \ll M\left(\sqrt{\lambda}, \frac{2}{r}\right).$$

A result of Cilleruelo and Córdoba [4] states that for any $0 < \delta < 1/2$,

$$M(R, R^{1/2-\delta}) \ll_\delta 1,$$

thus (6.3) holds for $r > \lambda^{-1/4+o(1)}$ as claimed. Moreover, we expect that $M(R, R^{1-\delta}) \ll_\delta 1$; this would imply that (6.3) holds for $r_\lambda > \lambda^{-1/2+o(1)}$. □

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