



Pair arithmetical equivalence for quadratic fields

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Received: 31 July 2020 / Accepted: 24 January 2021

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Abstract

Given two nonisomorphic number fields K and M , and finite order Hecke characters χ of K and η of M respectively, we say that the pairs (χ, K) and (η, M) are arithmetically equivalent if the associated L-functions coincide:

$$L(s, \chi, K) = L(s, \eta, M).$$

When the characters are trivial, this reduces to the question of fields with the same Dedekind zeta function, investigated by Gassmann in 1926, who found such fields of degree 180, and by Perlis (J Number Theory 9(3):342–360, 1977) and others, who showed that there are no nonisomorphic fields of degree less than 7. We construct infinitely many such pairs where the fields are quadratic. This gives dihedral automorphic forms induced from characters of different quadratic fields. We also give a classification of such characters of order 2 for the quadratic fields of our examples, all with odd class number.

Keywords L-functions · Arithmetic equivalence of number fields · Dihedral modular forms · Idele class characters

Mathematics Subject Classification 11R42 · 11F80 · 11F11

1 Introduction

1.1 Arithmetic equivalence of fields

Two number fields K and L are *arithmetically equivalent* if their Dedekind zeta functions coincide: $\zeta_K(s) = \zeta_L(s)$. A field is *arithmetically solitary* if it is isomorphic to any field

The research of Li is partially supported by Simons Foundation Grant # 355798. Z.R received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant agreement no. 786758).

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with the same Dedekind zeta function. Examples are normal extensions of the rationals. The first non-solitary fields were found in 1926 by Gassmann [5], who discovered a pair of non-isomorphic fields of degree 180 which are arithmetically equivalent. Perlis [9] showed that all the fields K with $[K : \mathbb{Q}] \leq 6$ are arithmetically solitary, and constructed a non-solitary field of degree 7.

A variant for Artin L-functions was investigated by Klüners and Nicolae [7]. For $j = 1, 2$ let K_j/\mathbb{Q} be a finite Galois extension, and let χ_j be a faithful character of the Galois group $G_j = \text{Gal}(K_j/\mathbb{Q})$. If the corresponding Artin L-functions coincide: $L(s, \chi_1, K_1/\mathbb{Q}) = L(s, \chi_2, K_2/\mathbb{Q})$, then $K_1 = K_2$ and $\chi_1 = \chi_2$. They also showed that if the base field is not the rationals, this need not be true. For other variations on this theme, see [3,4,8,10,15].

1.2 Arithmetic pair equivalence

In this paper we consider a different variant of field arithmetical equivalence, which we call arithmetic pair equivalence. For a number field K denote by G_K the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/K)$. Given two nonisomorphic number fields K and M , let χ and η be two finite order characters of G_K and G_M respectively. Two pairs (χ, K) and (η, M) are called *arithmetically equivalent* if the associated L-functions coincide:

$$L(s, \chi, K) = L(s, \eta, M).$$

An immediate consequence of pair arithmetical equivalence is that the two fields K and M must have the same degree over \mathbb{Q} . Clearly, pair arithmetical equivalence reduces to field arithmetical equivalence when both characters are trivial.

The problem that we study here is the existence of such pairs, when the base field K is a quadratic extension of the rationals. More precisely, given a quadratic extension K of \mathbb{Q} , we wish to find a nontrivial finite order character χ of G_K so that the pair (χ, K) is arithmetically equivalent to another pair (η, M) .

For instance we take the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-1})$ and the real quadratic field $M = \mathbb{Q}(\sqrt{q})$ where q is any prime satisfying $q \equiv 1 \pmod{8}$, and construct quadratic characters χ of K and η of M such that $L(s, \chi, K) = L(s, \eta, M)$ (see Sect. 5.1).

1.3 Connection with dihedral modular forms

We now connect with the theory of automorphic forms. Recall that by class field theory, finite order characters of the Galois group G_K may be identified with finite order characters of the idele class group of K (Hecke characters), and we shall freely identify the two. For K quadratic over \mathbb{Q} , and a Hecke character χ of K , there is a unique normalized automorphic Hecke-eigenform g_χ of $GL(2)$ over \mathbb{Q} , which is cuspidal if χ is not self-conjugate, with associated L-function $L(s, g_\chi) = L(s, \chi, K)$. It corresponds to the two-dimensional dihedral representation $\rho_\chi := \text{Ind}_{G_K}^{G_\mathbb{Q}} \chi$ of $G_\mathbb{Q}$. We call ρ_χ odd if it has eigenvalues ± 1 at the complex conjugation c in $G_\mathbb{Q}$, otherwise it is called even, in which case $\rho_\chi(c) = \pm Id$. The newform g_χ is holomorphic of weight one if ρ_χ is odd, and is a Maass form with Laplacian eigenvalue $1/4$ if ρ_χ is even.

In terms of Fourier expansions, g_χ is given as follows: Assume χ is not self-conjugate. For ρ_χ odd, the holomorphic weight one cusp form g_χ is

$$g_\chi(z) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) e^{2\pi i N(\mathfrak{a})z},$$

summing over all integral ideals \mathfrak{a} of K coprime to the conductor of χ , where $z = x + iy$ with $x, y \in \mathbb{R}$ and $y > 0$ and $N(\mathfrak{a})$ is the norm of \mathfrak{a} . For ρ_χ even, the Fourier expansion of the Maass cusp form g_χ also involves the K -Bessel function K_0 :

$$g_\chi(z) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \sqrt{y} K_0(2\pi N(\mathfrak{a})y) 2 \begin{cases} \cos(2\pi N(\mathfrak{a})x), & \text{if } \rho_\chi(c) = Id; \\ \sin(2\pi N(\mathfrak{a})x), & \text{if } \rho_\chi(c) = -Id. \end{cases}$$

Therefore, if (χ, K) is arithmetically equivalent to (η, M) for quadratic extensions K and M , then the modular form $g_\chi = g_\eta$ arises from Hecke characters of two different fields. An example was first found by Hecke, see [13, page 243]. In Sect. 5 we exhibit examples of this phenomenon.

From the viewpoint of Galois representations, Rohrlich studied such examples, called ‘‘Hecke-Shintani representations’’ in [11], in the course of deriving an asymptotic formula for the number of isomorphism classes of two-dimensional irreducible monomial representations of $G_\mathbb{Q}$ of bounded conductor.

1.4 The method

We interpret the equality

$$L(s, \chi, K) = L(s, \eta, M)$$

on L-functions of characters as the equality

$$L(s, \text{Ind}_{G_K}^{G_\mathbb{Q}} \chi) = L(s, \text{Ind}_{G_M}^{G_\mathbb{Q}} \eta)$$

on L-functions of induced degree-two representations $\text{Ind}_{G_K}^{G_\mathbb{Q}} \chi$ and $\text{Ind}_{G_M}^{G_\mathbb{Q}} \eta$ of the Galois group $G_\mathbb{Q}$. This then converts the problem on pair arithmetical equivalence to a problem on equivalence of induced representations. Our first main result, Theorem 2.2 in Sect. 2, gives a criterion for pair arithmetical equivalence in terms of the character involved.

Theorem 1.1 *Let K be a quadratic extension of \mathbb{Q} with $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Suppose a finite order character χ of G_K is not equal to its conjugate χ^c . Then the pair (χ, K) is arithmetically equivalent to another pair if and only if $\chi^c = \chi \cdot \delta$ for a quadratic character δ of G_K .*

We note that Rohrlich [12] gave several criteria for non-arithmetically solitary pairs from the perspective of Galois representations.

In view of Theorem 1.1, our problem becomes one of finding finite order characters χ of G_K satisfying

$$\chi^c = \chi \cdot \delta \tag{1}$$

for some quadratic character δ of G_K . From the viewpoint of idele class characters of K , a character μ is self-conjugate, that is, $\mu = \mu^c$, if and only if it comes from an idele class character of \mathbb{Q} by composing with the norm map, in other words, it comes from base change. Therefore it suffices to consider finite order idele class characters χ of K up to base change. As explained in Sect. 3.3, we may further assume that the order of χ is a power of 2.

We construct such χ for imaginary quadratic fields $\mathbb{Q}(\sqrt{-p})$ where p is prime, $p = 3 \pmod 4$ or $p = 2$, and also for $\mathbb{Q}(\sqrt{-1})$, and for the real quadratic fields $\mathbb{Q}(\sqrt{q})$ where $q = 1 \pmod 4$ is prime or $q = 2$. A key feature of these fields is that they have odd class

number. We also classify quadratic idele class characters χ up to base change for these quadratic extensions K and determine their conductors. Note that for χ quadratic, $\chi \neq \chi^c$ if and only if χ and χ^c differ by a quadratic character; therefore we have classified, for such K , all pairs (χ, K) with quadratic χ arithmetically equivalent to another pair. The results for imaginary quadratic K are given in Theorems 4.3 and 4.4. The parallel result for real quadratic fields is given in Theorem 4.7.

This paper is organized as follows. The main purpose of Srt. 2 is to prove Theorem 1.1. Proposition 2.1 in Sect. 2.1 establishes the necessity using Mackey’s theory. Sections 2.2–2.5 explores the structure of the Galois group of a degree-two dihedral representation induced from a character χ satisfying (1). Among other things, it provides information on the quadratic character δ , and leads to the proof of sufficiency in Sect. 2.6. An idele class character χ of K of finite order gives rise to a primitive multiplicative character $\xi(\chi)$ of the quotient of the ring of integers \mathbb{Z}_K of K by the conductor \mathfrak{f}_χ of χ . In Sect. 3.1 we investigate the lifting problem: Given a nonzero integral ideal \mathfrak{a} of \mathbb{Z}_K and a primitive character ξ of $(\mathbb{Z}_K/\mathfrak{a})^\times$, when does $\xi = \xi(\chi)$ come from an idele class character χ of K ? For quadratic fields K with odd class number as specified above and ξ of order a power of 2, we obtain an easy condition, Corollary 3.3, which is repeatedly used in the paper. Sects. 3.2 and 3.3 concerns base change and reduction to characters of order powers of 2, while Sect. 3.4 describes possible conductors for such characters. With this information, in Sect. 4 we characterize quadratic characters χ up to base change for quadratic fields K specified above and determine their conductors, with imaginary fields in Sect. 4.1 and real fields in Sect. 4.2. Finally, in Sect. 5 explicit examples of infinite families of cusp forms induced from quadratic characters of different quadratic fields are exhibited; holomorphic weight one forms are given in Sect. 5.1, and Maass forms with different infinity type in Sect. 5.2.

2 A criterion for pair arithmetical equivalences for quadratic extensions

2.1 A representation theoretic viewpoint of pair arithmetical equivalence

Given a quadratic field K/\mathbb{Q} and a finite order character χ of $G_K = \text{Gal}(\bar{\mathbb{Q}}/K)$, we are interested in knowing whether there is another quadratic extension M/\mathbb{Q} and a finite order character η of G_M such that the two Artin L -functions agree:

$$L(s, \chi, K) = L(s, \eta, M).$$

Induce the degree-one representation χ of G_K to a degree-two representation $\text{Ind}_K^{\mathbb{Q}} \chi = \text{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ of $G_{\mathbb{Q}}$. Since the Artin L -function is invariant under induction, we have

$$L(s, \chi, K) = L(s, \text{Ind}_K^{\mathbb{Q}} \chi).$$

Similarly, for the pair (η, M) we have the degree-two induced representation $\text{Ind}_M^{\mathbb{Q}} \eta$, and

$$L(s, \eta, M) = L(s, \text{Ind}_M^{\mathbb{Q}} \eta).$$

Hence if the two degree-2 representations are equivalent

$$\text{Ind}_K^{\mathbb{Q}} \chi \simeq \text{Ind}_M^{\mathbb{Q}} \eta, \tag{2}$$

then we have the desired equality

$$L(s, \chi, K) = L(s, \eta, M).$$

This converts the question on arithmetical equivalence of pairs to a question on equivalence of representations. Suppose

$$\text{Gal}(K/\mathbb{Q}) = \langle c \rangle \quad \text{and} \quad \text{Gal}(M/\mathbb{Q}) = \langle \tau \rangle.$$

Then c acts on G_K by conjugation, which in turn defines the conjugate character χ^c by $\chi^c(h) = \chi(chc^{-1})$ for $h \in G_K$. Similarly define the conjugate η^τ of η . In order that χ and η induce irreducible representations of $G_{\mathbb{Q}}$, it is necessary and sufficient that the two characters are not self-conjugate, namely $\chi^c \neq \chi$ and $\eta^\tau \neq \eta$.

Proposition 2.1 *Suppose $\chi^c \neq \chi$ and $\eta^\tau \neq \eta$. Then (2) holds if and only if the restrictions to the subgroup $G_K \cap G_M = G_{KM}$ coincide:*

$$\chi|_{G_K \cap G_M} = \eta|_{G_K \cap G_M}. \tag{3}$$

Moreover, we have

$$\chi^c = \chi \cdot \delta_{KM/K}, \quad \eta^\tau = \eta \cdot \delta_{KM/M}.$$

Here $\delta_{KM/K}$ and $\delta_{KM/M}$ are the quadratic characters of $\text{Gal}(KM/K)$ and $\text{Gal}(KM/M)$, respectively.

Proof As Galois representations, $\chi, \eta, \text{Ind}_K^{\mathbb{Q}} \chi$, and $\text{Ind}_M^{\mathbb{Q}} \eta$ all factor through finite quotients of their respective Galois groups. So they may be viewed as representations of finite groups. Given a pair of representations π_1, π_2 of a finite group G , denote by

$$[\pi_1, \pi_2]_G = \dim \text{Hom}_G(\pi_1, \pi_2).$$

So for π_1, π_2 irreducible, $[\pi_1, \pi_2]_G = 1$ if $\pi_1 \simeq \pi_2$, and is equal to zero otherwise.

We also recall Frobenius reciprocity for an induced representation from a subgroup H of G :

$$[\text{Ind}_H^G \xi, \pi]_G = [\xi, \text{Res}_H \pi]_H.$$

Hence in our case

$$[\text{Ind}_K^{\mathbb{Q}} \chi, \text{Ind}_M^{\mathbb{Q}} \eta]_{G_{\mathbb{Q}}} = [\text{Res}_{G_M} \text{Ind}_K^{\mathbb{Q}} \chi, \eta]_{G_M}.$$

Next we recall ‘‘Mackey theory’’, which says that the restriction of an induced representation has a direct sum decomposition

$$\text{Res}_{G_M} \text{Ind}_K^{\mathbb{Q}} \chi \simeq \bigoplus_{s \in G_K \backslash G_{\mathbb{Q}} / G_M} \text{Ind}_{s^{-1}G_K s \cap G_M}^{G_M} (\chi^s), \tag{4}$$

where $\chi^s(h) = \chi(shs^{-1})$.

In our case, G_K, G_M have index two in $G_{\mathbb{Q}}$, hence are normal, and moreover $G_K G_M = G_{\mathbb{Q}}$ since $G_K \neq G_M$. Hence there is only one double coset, and Mackey’s formula (4) reduces to

$$\text{Res}_{G_M} \text{Ind}_K^{\mathbb{Q}} \chi \simeq \text{Ind}_{G_K \cap G_M}^{G_M} \chi.$$

Hence

$$[\text{Ind}_K^{\mathbb{Q}} \chi, \text{Ind}_M^{\mathbb{Q}} \eta]_{G_{\mathbb{Q}}} = [\text{Res}_{G_M} \text{Ind}_K^{\mathbb{Q}} \chi, \eta]_{G_M} = [\text{Ind}_{G_K \cap G_M}^{G_M} \chi, \eta]_{G_M}.$$

Applying again Frobenius reciprocity gives

$$[\text{Ind}_{G_K \cap G_M}^{G_M} \chi, \eta]_{G_M} = [\chi, \eta]_{G_K \cap G_M} = \begin{cases} 1, & \text{if } \chi|_{G_K \cap G_M} = \eta|_{G_K \cap G_M} \\ 0, & \text{otherwise,} \end{cases}$$

which proves the claim (3).

Moreover, we have $\text{Ind}_K^{\mathbb{Q}} \chi = \text{Ind}_K^{\mathbb{Q}} \chi^c$, so the same conclusion holds with χ replaced by χ^c , and in particular we must have

$$\chi|_{G_K \cap G_M} = \chi^c|_{G_K \cap G_M}.$$

Since $[G_K : G_K \cap G_M] = 2$, this means that $\chi^{-1} \chi^c$ is a quadratic character of G_K , nontrivial because $\chi \neq \chi^c$ for irreducibility, which is trivial on $G_K \cap G_M = G_{KM}$. Hence it must equal $\delta_{KM/K}$.

Likewise $\eta^{-1} \eta^\tau$ is a quadratic character of G_M , which is trivial on $G_K \cap G_M$. Hence it must equal $\delta_{KM/M}$.

Proposition 2.1 provides a necessary condition for $\text{Ind}_K^{\mathbb{Q}} \chi$ to be equivalent to another induced representation of the same type, namely χ^c differs from χ by a quadratic character. The theorem below says that this condition is also sufficient.

Theorem 2.2 *Let K/\mathbb{Q} be a quadratic extension with $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Let χ be a finite order non self-conjugate character of G_K . Then*

- (a) $\text{Ind}_K^{\mathbb{Q}} \chi \simeq \text{Ind}_M^{\mathbb{Q}} \eta$ for some pair (η, M) with $M \neq K$ if and only if
- (b) $\chi^c = \chi \cdot \delta$ for some quadratic character δ of G_K .

As discussed above, (a) implies (χ, K) is arithmetically equivalent to (η, M) . Hence we have the following immediate corollary, which provides a convenient sufficient condition for (χ, K) to be arithmetically equivalent to another pair.

Corollary 2.3 *Let K/\mathbb{Q} be a quadratic extension with $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Let χ be a finite order character of G_K . If $\chi^c = \chi \cdot \delta$ for some quadratic character δ of G_K , then there is a pair arithmetically equivalent to (χ, K) .*

Note that the non self-conjugate requirement for χ automatically follows from the condition $\chi^c = \chi \cdot \delta$.

Theorem 2.2 will be proved in the last subsection of this section, after studying the structure of the Galois group of a dihedral representation induced by a character χ satisfying the condition (b).

2.2 The Galois group of a dihedral representation

Consider a two-dimensional faithful¹ irreducible representation

$$\rho : G := \text{Gal}(E/\mathbb{Q}) \rightarrow \text{GL}(2, \mathbb{C})$$

where E/\mathbb{Q} is a finite extension, which is *dihedral*, in the sense that the projectivation

$$\bar{\rho} : G \rightarrow \text{GL}(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C})$$

¹ Otherwise it factors through a subfield of E , that is through $\text{Gal}(E'/\mathbb{Q})$ for $\mathbb{Q} \subset E' \subset E$.

is a dihedral group D_n of order $2n$:

$$\bar{\rho}(G) \simeq D_n.$$

Thus there is an index-two subgroup $H \subset G$ and a character $\chi : H \rightarrow \mathbb{C}^\times$ so that

$$\rho \simeq \text{Ind}_H^G(\chi).$$

The subgroup H corresponds to a quadratic extension K/\mathbb{Q} , a subfield of E , namely the fixed points of H in E , so that

$$H = \text{Gal}(E/K).$$

Let c be an element of $\text{Gal}(E/\mathbb{Q})$ such that when restricted to K it gives the Galois involution of K so that we can write

$$\text{Gal}(K/\mathbb{Q}) = \{1, c\} \simeq \text{Gal}(E/\mathbb{Q})/\text{Gal}(E/K) = G/H.$$

Assume further that there is a quadratic character $\delta : H \rightarrow \{\pm 1\}$, so that

$$\chi^c = \chi \cdot \delta,$$

where χ^c is the character on H given by $\chi^c(h) = \chi(chc^{-1}) = \chi(c^{-1}hc)$ (since $c^2 \in H$) arising from the short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow \text{Gal}(K/\mathbb{Q}) \rightarrow 1.$$

We study the structure of $G = \text{Gal}(E/\mathbb{Q})$.

Theorem 2.4 *The Galois group $G = \text{Gal}(E/\mathbb{Q})$ is of order $4m$, with the center $Z = \ker \delta$ being a cyclic subgroup of order m . The projectivization $G/Z \simeq \bar{\rho}(G) \subset \text{PGL}(2, \mathbb{C})$ is the Klein 4-group $D_2 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof Let \mathbf{v}_1 be a basis of the 1-dimensional representation χ of H . Then $\mathbf{v}_2 = \rho(c)(\mathbf{v}_1)$ is linearly independent of \mathbf{v}_1 since G is generated by H and c , and ρ is 2-dimensional. In the basis $\mathbf{v}_1, \mathbf{v}_2$ of $\text{Ind}_H^G \chi$, we have

$$\rho(c) = \begin{pmatrix} 0 & \chi(c^2) \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(h) = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi^c(h) \end{pmatrix} \quad \text{for } h \in H$$

because $c^2 \in H$.

Since we assume that $\chi^c = \chi \cdot \delta$, we have

$$\rho(h) = \chi(h) \begin{pmatrix} 1 & 0 \\ 0 & \delta(h) \end{pmatrix}$$

and since δ is a quadratic character, we have that on the index two subgroup $\ker \delta \subset H$, $\rho(h)$ is a scalar matrix

$$\rho(h) = \chi(h)I, \quad h \in \ker \delta.$$

Therefore the center of G equals $\ker \delta$, which has index 4 in G :

$$\text{Center}(G) = \ker \delta$$

and the image in $\text{PGL}(2, \mathbb{C})$ of ρ is therefore a group of order 4, isomorphic to $G/\ker \delta$. Since we are given that it is dihedral, we therefore conclude that it equals D_2 .

We note that since $\rho = \text{Ind}_H^G \chi$ is faithful, the same holds for its restriction to H , that is, the subgroup

$$\rho(H) = \left\{ \chi(h) \begin{pmatrix} 1 & 0 \\ 0 & \delta(h) \end{pmatrix} : h \in H \right\}.$$

We now explore the implications of our conditions on the structure of G , knowing that the center is $Z = \ker \delta$, a subgroup of index 2 in H . In particular the order of H is even: $|H| = 2m$.

The center is $Z = \ker \delta$, the image under ρ being

$$\rho(Z) = \{ \chi(h)I : h \in Z \}$$

and the restriction of χ to the center Z must be faithful, so $\chi(Z)$ being a finite subgroup of the multiplicative group of the field of complex numbers must be cyclic, say is cyclic of order m : $Z = \langle z_0 : z_0^m = 1 \rangle$ and $G/Z \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

It remains to restrict the structure of G .

Theorem 2.5 (i) m is even: $m = 2\mu$, so that $|G| = 4m = 8\mu$ has order divisible by 8.

(ii) The action of c on H is given by

$$chc^{-1} = \begin{cases} h, & h \in Z \\ z_1 h, & h \notin Z \end{cases}$$

where $z_1 \in Z$ is the unique involution in Z (corresponding to the element $m/2 \in \mathbb{Z}/m\mathbb{Z}$).

(iii) There are two possibilities for H

- (a) $H \simeq \mathbb{Z}/2m\mathbb{Z}$ is cyclic;
- (b) $H = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

To prove this theorem, there are two possibilities to consider: χ is a faithful character of H , or not, which are carried out in the next two subsections.

2.3 The case χ is faithful

In this case $H \simeq \chi(H)$ is embedded as a finite subgroup of the multiplicative group of the complex numbers, hence is cyclic, of order $2m$, say $H = \langle h_0 : h_0^{2m} = 1 \rangle$, then $Z = \langle h_0^2 \rangle$ consists of the squares in H .

Now conjugation by c acts by an automorphism, hence takes the generator h_0 to h_0^k :

$$ch_0c^{-1} = h_0^k$$

with k coprime to $|H| = 2m$. In particular k is odd.

We can obtain further information by using the matrix representation: Recall $\delta(h_0) = -1$, and $\chi(h_0) = \zeta_{2m}$ is a primitive $2m$ -th root of unity, and

$$\rho(h_0) = \chi(h_0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \zeta_{2m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$\rho(c)\rho(h_0)\rho(c)^{-1} = \begin{pmatrix} 0 & \chi(c^2) \\ 1 & 0 \end{pmatrix} \zeta_{2m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \chi(c^2)^{-1} & 0 \end{pmatrix} = \zeta_{2m} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

while on the other hand

$$\rho(c)\rho(h_0)\rho(c)^{-1} = \rho(ch_0c^{-1}) = \rho(h_0^k) = \rho(h_0)^k = \zeta_{2m}^k \begin{pmatrix} 1 & 0 \\ 0 & (-1)^k \end{pmatrix}$$

giving

$$\zeta_{2m}^{k-1} = -1.$$

Since ζ_{2m} is a primitive $2m$ -th root of unity, we find

$$k - 1 = m \pmod{2m}$$

so that the only possibility is (given $1 \leq k < 2m$)

$$k = m + 1.$$

Combined with the fact that k is odd, as observed above, this implies that $m = 2\mu$ is even.

We can rewrite the action of c on H as follows: As c commutes with the center Z , which has index 2 in H , it suffices to compute

$$ch_0c^{-1} = h_0^{m+1} = h_0 \cdot (h_0^2)^{m/2} = h_0z_1,$$

where $z_1 = z_0^{m/2}$ is the unique element of order 2 in Z . This proves

$$chc^{-1} = \begin{cases} h, & h \in Z \\ hz_1, & h \notin Z, \end{cases}$$

as described in (ii).

2.4 The case χ is not faithful

Recall that $Z = \langle z_0 : z_0^m = 1 \rangle$ is cyclic of order m , so that $\rho(z_0) = \zeta_m I$ for ζ_m a primitive m th root of unity. Since $\chi(H) = \chi(Z)$ by assumption and χ is faithful on Z , $\ker \chi$ is generated by an element $h_1 \in H$ of order 2. As Z commutes with h_1 and Z intersects $\langle h_1 \rangle$ trivially, we have

$$H = Z \amalg h_1Z = Z \times \langle h_1 \rangle = \ker \delta \times \ker \chi \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The action of ρ on H is given by

$$\rho(z_0^j) = \zeta_m^j I, \quad \rho(h_1z_0^j) = \zeta_m^j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 0 \leq j \leq m - 1. \tag{5}$$

We prove that m is even: Conjugating by c , we must have $ch_1c^{-1} \in H$, so that

$$\rho(c)\rho(h_1)\rho(c)^{-1} = \begin{pmatrix} 0 & \chi(c^2) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \chi(c^2)^{-1} & 0 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \rho(H),$$

that is,

$$- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \zeta_m^j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for some j , forcing $\zeta_m^j = -1$. But if m is odd then -1 is not an m -th root of unity, giving a contradiction. Hence m must be even. Furthermore, the above computation shows that

$$\rho(c)\rho(h_1)\rho(c)^{-1} = -\rho(h_1) = \zeta_m^{m/2} \rho(h_1) = \rho(z_0^{m/2}h_1).$$

Together with c commuting with Z , we obtain the action of c on H :

$$chc^{-1} = \begin{cases} h, & h \in Z \\ hz_1, & h \notin Z, \end{cases}$$

where $z_1 = z_0^{m/2}$ is the unique element of order 2 in Z , as described in (ii).

2.5 Completing the proof of Theorem 2.5

In conclusion, G is a semi-direct product

$$G = H \rtimes \mathbb{Z}/2\mathbb{Z}$$

with $H = \mathbb{Z}/2m\mathbb{Z}$ if χ is faithful on H and $H = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ otherwise. This proves (iii). The evenness of m is also established in both cases. This is (i). The proof of Theorem 2.5 is now completed.

Several remarks are in order.

Remarks 1. Since the action of c fixes the elements in Z and interchanges h and z_1h for h outside Z , this shows that Frob_v for the primes v above p inert in K are in Z and those above p splitting in K , if not identity, are outside Z .

2. For $h = c^2 \in H$, since $chc^{-1} = c^2 = h$, we have $c^2 \in Z$ by Theorem 2.5(ii). Therefore $\delta(c^2) = 1$ because $\ker \delta = Z$.

3. The fixed field F of $\ker \delta = Z$ is a biquadratic extension of \mathbb{Q} since G/Z is a Klein-4 group by Theorem 2.4. Thus F contains three quadratic extensions of \mathbb{Q} , one of which is K , and $F = KM$ for any quadratic subfield $M \neq K$. Moreover, δ , being the unique quadratic character on $\text{Gal}(KM/K)$, is equal to the quadratic character $\delta_{KM/K}$ attached to the extension KM/K , which lifts the quadratic character $\delta_{M/\mathbb{Q}}$. This shows that

$$\chi^c = \chi \cdot \delta_{KM/K}$$

for any quadratic subfield M of F other than K . In particular, since c restricted to F has order 2, we may choose M to be the fixed subfield of c on F .

4. Suppose χ has order r . Raising both sides of $\chi^c = \chi \cdot \delta$ to the r th power implies r even since δ has order 2. This is in concert with statement (i) in Theorem 2.5.

The following proposition gives different criteria for the faithfulness of χ on H .

Proposition 2.6 *Suppose χ has order $r \equiv 0 \pmod{4}$. The following statements are equivalent:*

- (a) χ is faithful on H .
- (b) $\ker \delta$ contains $\ker \chi$, equivalently, the fixed field of $\ker \chi$ contains that of $\ker \delta$.
- (c) $\chi^{r/2}$ is δ .
- (d) $\chi^c = \chi^{1+r/2}$.

Note that the assertion (d) is a condition for faithfulness of χ on H in terms of χ alone.

Proof Since $\chi^c = \chi \cdot \delta$, we have (c) \iff (d).

(a) \implies (c). Since χ is faithful on H , so $H \cong \chi(H)$ is cyclic and $\chi^{r/2}$, being the unique character on H of order 2, is equal to δ .

(c) \implies (b). This is because $\ker \chi^{r/2}$ contains $\ker \chi$.

(b) \implies (a). It is shown in Sect. 2.4 that, if χ is not faithful on H , then $\ker \chi = \langle h_1 \rangle$ has order 2 and $\ker \delta = Z$ intersects $\ker \chi$ trivially. So $\ker \delta$ does not contain $\ker \chi$.

As an immediate consequence, we have the following description of the fixed field E of H .

Corollary 2.7 E is the composition of the fixed field of $\ker \chi$ and that of $\ker \delta$.

2.6 A proof of Theorem 2.2

By Proposition 2.1, (a) implies (b) with $\delta = \delta_{KM/K}$, a quadratic character of $\text{Gal}(KM/K)$ and hence of G_K .

Now we consider the converse. Suppose $\chi^c = \chi \cdot \delta$ for some quadratic character δ of G_K . We shall find a quadratic extension M of \mathbb{Q} different from K and a finite order non self-conjugate character η of G_M such that $\chi|_{G_K \cap G_M} = \eta|_{G_K \cap G_M}$, which in turn implies (a) by Proposition 2.1.

Write ρ for the induced representation $\text{Ind}_K^{\mathbb{Q}} \chi$, which is an irreducible degree-2 dihedral representation of $G_{\mathbb{Q}}$ since χ is not self-conjugate. Then by Remark 3 in Sect. 2.5, there is a quadratic subfield $M \neq K$ of the fixed field of $\ker \delta$, pointwisely fixed by c , such that we may choose $\delta = \delta_{KM/K}$. We proceed to find a character η of G_M with the desired properties.

Let \mathbf{v}_1 be a basis of the space of χ and $\mathbf{v}_2 = \rho(c)(\mathbf{v}_1)$ so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of the space V of ρ . In this basis, we have

$$\rho(c) = \begin{pmatrix} 0 & \chi(c^2) \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(h) = \begin{pmatrix} \chi(h) & 0 \\ 0 & \chi^c(h) \end{pmatrix} \quad \text{for } h \in G_K$$

as in the proof of Theorem 2.4. Since $\ker \delta_{KM/K} = G_{KM}$, we have $\chi = \chi^c$ on G_{KM} and $\chi = -\chi^c$ on $G_K \setminus G_{KM}$ by assumption.

As c fixes M elementwisely, $c \in G_M$ and $G_M = G_{KM} \cup G_{KM}c$. We study the restriction of ρ to G_M . By Remark 2 of the previous subsection, c^2 lies in G_{KM} so that $\rho(c^2)$ is scalar multiplication by a nonzero constant $a = \chi(c^2)$. Write $a = b^2$. Let $\mathbf{w}_1 = \mathbf{v}_1 + b^{-1}\mathbf{v}_2$ and $\mathbf{w}_2 = \mathbf{v}_1 - b^{-1}\mathbf{v}_2$. In terms of the new basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of V , we find $\rho(c)\mathbf{w}_1 = b\mathbf{w}_1$ and $\rho(c)\mathbf{w}_2 = -b\mathbf{w}_2$. Let $h \in G_{KM}$. With respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of V , the action of $\rho(h)$ is given by scalar multiplication $\begin{pmatrix} \chi(h) & 0 \\ 0 & \chi(h) \end{pmatrix}$ as discussed before, while the action of $\rho(hc)$ is represented by $\begin{pmatrix} b\chi(h) & 0 \\ 0 & -b\chi(h) \end{pmatrix}$, analogous to ρ restricted to G_K . This shows that the 1-dimensional space spanned by \mathbf{w}_1 is invariant under $\rho(G_M)$ and hence the action is given by a character η of G_M , and ρ restricted to G_M is a direct sum of two characters η and $\eta \cdot \delta_{KM/M}$ with $\eta|_{G_K \cap G_M} = \chi|_{G_K \cap G_M}$. Moreover, η has finite order and η is not self-conjugate, as desired.

3 Characters of Galois groups and idele class characters

3.1 Idele class characters

We first set up notation and recall some facts. For a number field K , denote by \mathbb{Z}_K its ring of integers and U_K the group of units in \mathbb{Z}_K . The maximal ideals of \mathbb{Z}_K give rise to finite places of K , while the infinite places of K come from the r_1 distinct real imbeddings and r_2 nonconjugate complex imbeddings of K . Together, they constitute the set $\Sigma(K)$ of all places of K . The completion of K at a place $v \in \Sigma(K)$ is denoted K_v . When v is finite, let \mathcal{O}_v

denote the ring of integers in K_v , \mathcal{U}_v the group of units, and \mathcal{M}_v the unique maximal ideal of \mathcal{O}_v . Any generator π_v of \mathcal{M}_v is a uniformizer of K_v . Then $K_v^\times = \mathcal{U}_v \times \langle \pi_v \rangle$. Clearly \mathcal{U}_v contains the group of global units U_K .

The (topological) group of ideles of K , defined by

$$I_K = \left\{ x = (x_v) \in \prod_{v \in \Sigma(K)} K_v^\times \mid x_v \in \mathcal{U}_v \text{ for almost all } v \right\},$$

is the restricted product of $\{K_v^\times : v \in \Sigma(K)\}$ with respect to $\{\mathcal{U}_v : \text{finite } v \in \Sigma(K)\}$. The field K^\times is diagonally imbedded in I_K as a discrete subgroup, and the quotient I_K/K^\times is called the idele class group of K . We may write $I_K = (I_K)_\infty (I_K)^\infty$, where

$$(I_K)_\infty = \prod_{v \in \Sigma(K) \text{ infinite}} K_v^\times$$

is the subgroup of ideles supported at the infinite places and $(I_K)^\infty$ is the subgroup of ideles supported at the finite places.

A character χ of I_K is a continuous homomorphism from I_K to the unit circle in \mathbb{C}^\times . It can be expressed as a product of χ_v , its restriction to K_v^\times , over all $v \in \Sigma(K)$. Note that χ_v at a finite place v is determined by its values on the group of units \mathcal{U}_v and one uniformizer π_v . Sometimes we write $\chi = \chi_\infty \chi^\infty$, where χ_∞ and χ^∞ are the restrictions of χ to $(I_K)_\infty$ and $(I_K)^\infty$, respectively. We discuss χ^∞ and χ_∞ in more detail below.

As a result of continuity, for almost all finite v , $\chi_v(\mathcal{U}_v) = 1$, in which case we say that χ_v is unramified, or χ is unramified at v . The set S of finite places where χ is ramified is finite. At each $v \in S$, there is a smallest positive integer $n(\chi_v)$ such that χ_v is trivial on $1 + \mathcal{M}_v^{n(\chi_v)}$; call $\mathcal{M}_v^{n(\chi_v)}$ the conductor of χ_v . The conductor of χ is the product $\mathfrak{f}_\chi = \prod_{v \in S} v^{n(\chi_v)}$, which is a nonzero ideal of \mathbb{Z}_K . Observe that $\prod_{v \in S} \mathcal{U}_v / (1 + \mathcal{M}_v^{n(\chi_v)}) \cong (\mathbb{Z}_K/\mathfrak{f}_\chi)^\times$, hence χ restricted to $\prod_{v \in S} \mathcal{U}_v$ induces a character $\xi = \xi(\chi)$ of $(\mathbb{Z}_K/\mathfrak{f}_\chi)^\times$. Moreover, since \mathfrak{f}_χ is the conductor of χ , ξ is a primitive character of $(\mathbb{Z}_K/\mathfrak{f}_\chi)^\times$ in the sense that it does not induce a character of $(\mathbb{Z}_K/\mathfrak{a})^\times$ for any ideal \mathfrak{a} of \mathbb{Z}_K properly containing \mathfrak{f}_χ . In conclusion, χ^∞ on the group of units $\prod_{v \text{ finite}} \mathcal{U}_v$ lifts a primitive character ξ on $(\mathbb{Z}_K/\mathfrak{f}_\chi)^\times$, where \mathfrak{f}_χ is the conductor of χ , and χ^∞ is determined by ξ and the values $\chi_v(\pi_v)$ for all finite places v .

Now assume χ has finite order. We discuss χ_∞ . If v is a real place of K , then $K_v = \mathbb{R}$ and χ_v is either trivial or the sign function on \mathbb{R}^\times ; while if v is a complex place, then $K_v = \mathbb{C}$ and χ_v is always trivial on \mathbb{C}^\times .

A character χ of I_K is called an idele class character of K if it is trivial on K^\times . This is a strong constraint on χ . For example, χ_∞ and $\xi(\chi)$ are related by $\chi(U_K) = 1$. Therefore, an idele class character χ of K is determined by χ_∞ , $\xi(\chi)$ and $\chi_v(\pi_v)$ for finitely many v representing the ideal class group of K . In particular, if K has class number one, then χ is determined by χ_∞ and $\xi(\chi)$.

The theorem below gives other occasions that an idele class character χ is determined by $\xi(\chi)$ and χ_∞ .

Theorem 3.1 *Let \mathfrak{a} be a nonzero ideal of the ring of integers \mathbb{Z}_K of a number field K . Let ξ be a primitive character of $(\mathbb{Z}_K/\mathfrak{a})^\times$ of even order r which lifts to a character χ_U on the group of units in $(I_K)^\infty$. Suppose that there is a character χ_∞ on $(I_K)_\infty$ with order ≤ 2 such that the character $\chi_J := \chi_\infty \times \chi_U$ on $J := (I_K)_\infty \prod_{v \in \Sigma(K) \text{ finite}} \mathcal{U}_v$ is trivial on the group of global units U_K . If the order r of ξ is coprime to the class number of K , then χ_J on J has a unique extension to a character χ on I_K of order r , conductor $\mathfrak{f}_\chi = \mathfrak{a}$, and trivial on K^\times .*

Proof At each finite place v of K , fix a uniformizer π_v . To extend χ_J to a character $\chi = \prod_{v \in \Sigma(K)} \chi_v$ on I_K , it remains to define $\chi^\infty(\pi_v) = \chi_v(\pi_v)$ so that χ is trivial on K^\times . It will be clear from the definition that the resulting χ has order r and conductor $\mathfrak{f}_\chi = \mathfrak{a}$. By assumption, the class number $h = h(K)$ is coprime to the order r of χ , so there is a positive integer e such that $eh \equiv 1 \pmod r$. By construction, χ_J has order r , hence $(\chi_J)^{he} = \chi_J$. We shall take advantage of the fact that each ideal of \mathbb{Z}_K raised to the h -th power is principal (since h is the class number of K) to facilitate our definition of $\chi_v(\pi_v)$.

For each finite place v of K , choose an element $\beta_v \in \mathbb{Z}_K$ which generates the ideal v^h , i.e., $v^h = (\beta_v)$. Note that β_v is a unit at all finite places $w \in \Sigma(K)$ outside v , and at v , $\beta_v = u_v \pi_v^h$ for some unit u_v in K_v . For $v \nmid \mathfrak{a}$, define

$$\chi_v(\pi_v) = \chi_\infty(\beta_v)^{-e} \prod_{w|\mathfrak{a}} \chi_w(\beta_v)^{-e}$$

so that $\chi_v(\pi_v)^h = \chi_\infty(\beta_v)^{-1} \prod_{w|\mathfrak{a}} \chi_w(\beta_v)^{-1}$ and

$$\chi(\beta_v) = \chi_\infty(\beta_v) \cdot \prod_{w|\mathfrak{a}} \chi_w(\beta_v) \cdot \chi_v(\beta_v) = \chi_\infty(\beta_v) \cdot \prod_{w|\mathfrak{a}} \chi_w(\beta_v) \cdot \chi_v(\pi_v)^h = 1;$$

while for $v|\mathfrak{a}$, define

$$\chi_v(\pi_v) = \chi_\infty(\beta_v)^{-e} \chi_v(u_v)^{-e} \prod_{w|\mathfrak{a}, w \neq v} \chi_w(\beta_v)^{-e}$$

so that $\chi_v(\pi_v)^h = \chi_\infty(\beta_v)^{-1} \chi_v(u_v)^{-1} \prod_{w|\mathfrak{a}, w \neq v} \chi_w(\beta_v)^{-1}$ and

$$\chi(\beta_v) = \chi_\infty(\beta_v) \prod_{w|\mathfrak{a}} \chi_w(\beta_v) = \chi_\infty(\beta_v) \chi_v(u_v \pi_v^h) \prod_{w|\mathfrak{a}, w \neq v} \chi_w(\beta_v) = 1.$$

Thus we have extended χ_J to a character χ of I_K trivial on U_K of order r , and with conductor $\mathfrak{f}_\chi = \mathfrak{a}$. Moreover χ^h is trivial on nonzero elements in \mathbb{Z}_K and hence on K^\times . Raising it to the e -th power shows that χ is trivial on K^\times , as desired.

To prove the uniqueness of χ , let χ' be another extension of χ_J with the desired properties. Then $\mu = \chi' \chi^{-1}$ is an idele class character of K which is trivial on J . Hence μ_∞ is trivial, μ is unramified everywhere and has order dividing r . Therefore, at a finite place v , we have $1 = \mu(\beta_v) = \mu_v(\beta_v) = \mu_v(\pi_v)^h$ since β_v is a unit outside v . Raising it to the e -th power gives $\mu_v(\pi_v) = 1$ for all finite v . Thus μ is trivial, in other words, $\chi = \chi'$.

We illustrate some constraints on $\xi(\chi)$ for an idele class character.

Proposition 3.2 *Let χ be an idele class character of K of finite order with conductor \mathfrak{f}_χ . Let $\xi = \xi(\chi)$ be the primitive character of $(\mathbb{Z}_K/\mathfrak{f}_\chi)^\times$ so that χ^∞ on units in $(I_K)^\infty$ is the lift of ξ . Then the following hold.*

- (i) $\xi(u) = \chi_\infty(u)$ for each $u \in U_K$, $\xi(U_K) \subseteq \langle -1 \rangle$, and ξ is trivial on units in U_K which are positive under all real imbeddings of K .
- (ii) If K is a CM field, then χ_∞ is trivial and $\xi(U_K) = 1$.
- (iii) For K real quadratic, if there is a fundamental unit ϵ_K with norm $N_{K/\mathbb{Q}}(\epsilon_K) = -1$, then χ_∞ is uniquely determined by $\xi(\epsilon_K)$ and $\xi(-1)$; if $N_{K/\mathbb{Q}}(U_K) = 1$, then χ_∞ is determined by $\xi(-1)$ and $\chi^\infty(\alpha)$ for any $\alpha \in K^\times$ with negative norm.

Proof Note that $\chi^\infty(u) = \xi(u)$ for any $u \in U_K$ since χ is unramified outside the support of \mathfrak{f}_χ .

(i) For a general K , χ_v at each infinite place v has order at most 2 so that χ_{∞}^2 is trivial. Thus $\xi(u) = \chi_{\infty}(u)^{-1} = \chi_{\infty}(u)$ for each $u \in U_K$. Hence $\chi_{\infty}(-1) = \xi(-1)$. Further, since χ_v at a real place v is either trivial or the sign function, $\chi_v(u) = 1$ if u is positive under the imbedding v . This shows that $\xi(u) = \chi_{\infty}(u) = 1$ if the unit u has positive images under all real imbeddings of K .

(ii) Suppose K is a CM field. Then all infinite places of K are complex and hence χ_{∞} is trivial. Given $u \in U_K$, from $\chi(u) = \chi_{\infty}(u)\chi^{\infty}(u) = 1$ we conclude $\chi^{\infty}(u) = \xi(u) = 1$. So $\xi(U_K) = 1$.

(iii) Now assume K is real quadratic. The group of global units $U_K = \langle -1 \rangle \times \langle \epsilon_K \rangle$, where ϵ_K is a fundamental unit of infinite order. The field K has two real imbeddings ∞_1 and ∞_2 , and each χ_{∞_i} is either trivial or the sign function on \mathbb{R}^{\times} . From $\chi_{\infty}(-1) = \xi(-1) = \pm 1$ we conclude $\chi_{\infty_1} \neq \chi_{\infty_2}$ if $\xi(-1) = -1$, and $\chi_{\infty_1} = \chi_{\infty_2}$ if $\xi(-1) = 1$. If $N_{K/\mathbb{Q}}(\epsilon_K) = \infty_1(\epsilon_K)\infty_2(\epsilon_K) = -1$, then exactly one of $\infty_1(\epsilon_K)$, $\infty_2(\epsilon_K)$, say, $\infty_1(\epsilon_K)$, is positive so that $\chi_{\infty_1}(\epsilon_K) = 1$ always holds, and χ_{∞_2} is determined by $\chi_{\infty_2}(\epsilon_K) = \chi_{\infty_1}(\epsilon_K)\chi_{\infty_2}(\epsilon_K) = \chi_{\infty}(\epsilon_K) = \xi(\epsilon_K)$. This in turn determines χ_{∞_1} so that $\chi_{\infty}(-1) = \xi(-1)$. Hence χ_{∞} is uniquely determined by $\xi(-1)$ and $\xi(\epsilon_K)$ in this case. On the other hand, if $N_{K/\mathbb{Q}}(U_K) = 1$, the information of ξ on U_K is not enough to determine χ_{∞} . Instead, we choose any $\alpha \in K^{\times}$ with negative norm. Then $\chi^{\infty}(\alpha) = \chi_{\infty}(\alpha) = \pm 1$ since $\chi(\alpha) = 1$. The above argument for ϵ_K with norm -1 is easily modified to pin down χ_{∞} . □

Suppose K is quadratic over \mathbb{Q} . In particular, given a primitive character ξ on $(\mathbb{Z}_K/\mathfrak{a})^{\times}$ satisfying the requirement (ii) for K imaginary quadratic, and requirement (i) for K real quadratic containing a fundamental unit with negative norm, then, by the proposition above, it determines a unique χ_{∞} on $(I_K)_{\infty}$ such that the character χ_J on J in Theorem 3.1 exists. If, in addition, the order of ξ is coprime to the class number of K , then we obtain a unique idele class character χ of K solely determined by ξ . We summarize this discussion in the corollary below, which will be used later.

Corollary 3.3 *Let K be a quadratic extension of \mathbb{Q} . Let \mathfrak{a} be a nonzero ideal of \mathbb{Z}_K and ξ a primitive character of $(\mathbb{Z}_K/\mathfrak{a})^{\times}$ of order 2^e for an integer $e \geq 1$. Suppose one of the following two conditions hold:*

- (a) K is imaginary with odd class number and $\xi(U_K) = 1$;
- (b) $K = \mathbb{Q}(\sqrt{d})$ for a prime $d \equiv 1 \pmod{4}$ or $d = 2$ is real quadratic, and $\xi(U_K) \subseteq \langle -1 \rangle$.

Then there is a unique idele class character χ of K with conductor $f_{\chi} = \mathfrak{a}$ and order 2^e such that χ^{∞} on units of $(I_K)_{\infty}$ lifts ξ .

Proof The conclusion will follow from Theorem 3.1 provided the assumptions there hold. This is obvious for K in (a) and $K = \mathbb{Q}(\sqrt{2})$ in (b). For K in (b), since $d \equiv 1 \pmod{4}$ is a prime, there is a fundamental unit ϵ_K of K with $N_{K/\mathbb{Q}}(\epsilon_K) = -1$ (see [2, Chapter XI, Theorem 3]), hence the class number $h(K)$ of K agrees with the narrow class number of K , whose 2-rank $r(K)$ is one less than the number of prime factors of the discriminant $d_K = d$ of K by Gauss’s genus theory (see [2, Chapter XIII, §3]). Therefore $r(K) = 1 - 1 = 0$, implying $h(K)$ is odd. Hence by Proposition 3.2, $\xi(-1)$ and $\xi(\epsilon_K)$ uniquely determine χ_{∞} so that all required conditions in Theorem 3.1 are satisfied.

3.2 Characters of Galois groups and idele class characters

Let χ be a character of the absolute Galois group G_K of K of finite order. Then the fixed field F of $\ker \chi$ is a finite cyclic extension of K with Galois group $\text{Gal}(F/K)$. Let S be

the set of finite places of K ramified in F . For each finite place v of K outside S , there is the associated Frobenius element Frob_v in $\text{Gal}(F/K)$. The Chebotarev density theorem says that these Frobenius elements are uniformly distributed among the elements in $\text{Gal}(F/K)$. By class field theory, the global Artin reciprocity map ψ_K gives rise to an isomorphism from the quotient $I_K/K^\times N_{F/K}(I_F)$ to $\text{Gal}(F/K)$ and χ on $\text{Gal}(F/K)$ is transported to the idele class character of K , also denoted by χ , with kernel $K^\times N_{F/K}(I_F)$ and $\chi_v(\pi_v) = \chi(\text{Frob}_v)$ for all finite v outside S . Moreover all finite order idele class characters of K arise this way. Note that at finite v outside S , \mathcal{U}_v is contained in $N_{F/K}(I_F)$ so that χ is unramified at v , while for $v \in S$, \mathcal{U}_v is not contained in $N_{F/K}(I_F)$ so that χ is ramified at v . Hence S is the support of the conductor f_χ of χ . By strong approximation theorem, the above information on χ uniquely determines χ_∞ and χ_v for $v \in S$.

Now we specify K to be a quadratic extension of \mathbb{Q} with $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$ as in Sect. 2. We reinterpret the discussion on characters of Galois groups in Sect. 2 in terms of idele class characters. By class field theory, the character χ of H and its conjugate χ^c as before correspond to the idele class characters χ and χ^c of K with respective conductors f_χ and f_{χ^c} , and, as discussed above, their respective restrictions to the group of units in $(I_K)^\infty$ lift primitive characters $\xi = \xi(\chi)$ of $(\mathbb{Z}_K/f_\chi)^\times$ and $\xi' = \xi(\chi^c)$ of $(\mathbb{Z}_K/f_{\chi^c})^\times$, respectively. The involution c on K swaps the two maximal ideals v, v^c above a prime p splitting in K and induces an isomorphism between K_v and K_{v^c} (both isomorphic to \mathbb{Q}_p), while it fixes the unique maximal ideal v above a prime p inert in K and induces the Frobenius automorphism on F_v over \mathbb{Q}_p . Consequently, c maps the conductor f_χ to $(f_\chi)^c = f_{\chi^c}$ so that $\xi = \xi' \circ c$. Since c is an involution, we also have $\xi^t = \xi \circ c$. In terms of the Frobenius conjugacy classes in H , c maps Frob_v to Frob_{v^c} . So at a finite v outside the support of $f_\chi f_{\chi^c}$, both χ and χ^c are unramified, and we have $(\chi^c)_v(\pi_v) = \chi_{v^c}(\pi_{v^c})$.

3.3 Reduction of order

In view of Corollary 2.3, our problem becomes

Question 1 Classify finite order idele class characters χ of K satisfying

$$\chi^c = \chi \cdot \delta \tag{6}$$

for some quadratic idele class character δ of K .

Recall the following well-known fact (cf. [6, Proposition 1]):

Proposition 3.4 *A finite order idele class character μ of a quadratic field K satisfies $\mu = \mu^c$ if and only if $\mu = v \circ N_{K/\mathbb{Q}}$ for an idele class character v of \mathbb{Q} . Equivalently, a finite order character μ of the Galois group G_K extends to a character v of $G_\mathbb{Q}$ if and only if $\mu = \mu^c$.*

Such μ is called the *base change* of v to K . Since the global Artin reciprocity map ψ_K is the product over places v of K of the local Artin reciprocity map ψ_{K_v} (see, for example [16, section 6]), Proposition 3.4 results from local base change for μ_v , which in turn follows from the functoriality of the local reciprocity map, as explained in [14, section 2.4]. Hence if χ satisfies (6), so does $\chi \cdot v \circ N_{K/\mathbb{Q}}$ for all idele class characters v of \mathbb{Q} . Thus it suffices to study χ up to multiplication by finite order characters arising from base change. Note that (6) implies $\delta^c = \delta$ so that δ is from base change.

Clearly χ and χ^c have the same order, say, r . The condition (6) implies $2|r$. Write $r = 2^e r'$, where $e \geq 1$ and r' is odd. Since 2^e and r' are coprime, there are integers a and b such that

$a2^e + br' = 1$. We have $\chi = \chi^{a2^e} \cdot \chi^{br'}$ as a product of two idele class characters χ^{a2^e} and $\chi^{br'}$ of K of order r' and 2^e , respectively. Squaring (6) yields

$$(\chi^2)^c = (\chi^c)^2 = \chi^2, \tag{7}$$

implying $(\chi^{a2^e})^c = \chi^{a2^e}$, which comes from base change by Proposition 3.4. This proves

Proposition 3.5 *Let K/\mathbb{Q} be a quadratic extension. Up to multiplication by a character from base change, an idele class character χ of K of finite order satisfying (6) has order a power of 2.*

Therefore we shall assume χ has order a power of 2 and study when it satisfies (6). The condition (7) says that its square comes from base change.

If χ is quadratic and not equal to χ^c , then they differ by a quadratic idele class character of K . This proves

Theorem 3.6 *An idele class character χ of K of order 2 satisfies $\chi^c = \chi \cdot \delta$ for some quadratic idele class character δ of K if and only if χ does not arise from base change.*

Such χ 's are not faithful on H .

3.4 Conductors of idele class characters with order a power of 2

Before going further, we explore restrictions on the conductor of an idele class character whose order is a power of 2.

Proposition 3.7 *Let K be a quadratic extension of \mathbb{Q} with $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Let χ be an idele class character of K with conductor \mathfrak{f} and order a power of 2. Let v be a finite place of K such that $\text{ord}_v \mathfrak{f} = m(v) \geq 1$. Suppose χ_v on \mathcal{U}_v has order 2^e . Denote by κ_v the residue field at v , and by π_v a uniformizer of K_v .*

- (i) *If v is above an odd prime p , then $m(v) = 1$ and 2^e divides $|\kappa_v| - 1$. Moreover, $\chi_v^c = \chi_v^p$ on \mathcal{U}_v has conductor v and the same order as χ_v if p is inert in K ; $\chi_v^c = \chi_v$ on \mathcal{U}_v if p ramifies in K ; and χ_v^c is a character on K_v^\times of conductor v^c and same order as χ_v if p splits in K .*
- (ii) *If v is above 2, then $m(v) \geq 2$. Further, more can be said according to the behavior of 2:*
 - (iia) *2 splits in K . If $m(v) = 2$, then $\mathcal{U}_v/(1 + \mathcal{M}_v^{m(v)}) = \langle -1 \rangle$ and $e = 1$. If $m(v) \geq 3$, then*

$$\mathcal{U}_v/(1 + \mathcal{M}_v^{m(v)}) = \langle -1 \rangle \times \langle 1 + \pi_v^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{m(v)-2}\mathbb{Z}$$

and $e = m(v) - 2$. Moreover χ_v^c is a character of K_v^\times with the same order as χ_v .

- (iib) *2 is inert in K . Choose $\pi_v = 2$. Then $\mathcal{U}_v = \langle \mu_3 \rangle(1 + \mathcal{M}_v)$ for a primitive cubic root of unity μ_3 and χ_v on \mathcal{U}_v is the trivial extension of χ_v on $V_v := (1 + \mathcal{M}_v)/(1 + \mathcal{M}_v^{m(v)})$. For $m(v) = 2$,*

$$V_v = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

and $e = 1$. For $m(v) \geq 3$,

$$V_v = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle \times \langle 1 - \mu_3 2^2 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{m(v)-1}\mathbb{Z}) \times (\mathbb{Z}/2^{m(v)-2}\mathbb{Z}).$$

We have $e = m(v) - 1$ if $\chi_v(1 + \mu_3 2)$ has order $2^{m(v)-1}$; otherwise $e = m(v) - 2$ and $\chi_v(1 - \mu_3 2^2)$ has order $2^{m(v)-2}$. Further, $(1 + \mu_3 2)^c = -(1 + \mu_3 2)$ and $(1 - \mu_3 2^2)^c = 1 - \mu_3^2 2^2$.

(iic) 2 ramifies in K . Then $\mathcal{U}_v = 1 + \mathcal{M}_v$ and $V_v := \mathcal{U}_v / (1 + \mathcal{M}_v^{m(v)})$ has order $2^{m(v)-1}$. We have $V_v = \langle 1 + \pi_v \rangle$ cyclic for $m(v) = 2$ or 3;

$$V_v = \langle 1 + \pi_v \rangle \times \langle 1 + \pi_v^3 \rangle \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

for $m(v) = 4$; and

$$V_v = \langle 1 + \pi_v \rangle \times \langle 1 + \pi_v^3 \rangle \times \langle 1 + \pi_v^4 \rangle \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

for $m(v) = 5$. Moreover, if $e = 1$, then either $m(v) = 2$ with $\chi_v(1 + \pi_v) = -1$, or $m(v) = 4$ with $\chi_v(1 + \pi_v^3) = -1$, or $m(v) = 5$ with $\chi_v(1 + \pi_v^4) = -1$. If $e \geq 2$, we have $m(v) \leq 2e + 3$.

Proof Since $\text{ord}_v \mathfrak{f} = m(v)$, χ_v on $\mathcal{U}_v / 1 + \mathcal{M}_v^{m(v)}$ is primitive.

(i) Assume v is above an odd prime p . If $m(v) > 1$, $\mathcal{U}_v / (1 + \mathcal{M}_v^{m(v)})$ contains the subgroup $(1 + \mathcal{M}_v) / (1 + \mathcal{M}_v^{m(v)})$ whose order is a power of p . Since p is coprime to 2^e , the order of χ_v on \mathcal{U}_v , so χ_v is trivial on $(1 + \mathcal{M}_v) / (1 + \mathcal{M}_v^{m(v)})$, which implies that χ_v comes from a character of $\mathcal{U}_v / (1 + \mathcal{M}_v)$, hence is not primitive on $\mathcal{U}_v / (1 + \mathcal{M}_v^{m(v)})$, a contradiction. This proves $m(v) = 1$. Note that $\mathcal{U}_v / (1 + \mathcal{M}_v) \cong \kappa_v^\times$, therefore the order of χ_v on \mathcal{U}_v , which is 2^e , divides $|\kappa_v| - 1$.

If p is inert in K , then κ_v is a quadratic extension of $\mathbb{Z}/p\mathbb{Z}$ and c induces the Frobenius automorphism on κ_v . So $\chi_v^c = \chi_v^p$ on \mathcal{U}_v . If p ramifies in K , then $\kappa_v \cong \mathbb{Z}/p\mathbb{Z}$ on which c acts trivially, and $\chi_v^c = \chi_v$ on \mathcal{U}_v . If p splits in K , then v and v^c are the two places of K above p , and c gives rise to the isomorphism $K_v \cong K_{v^c} (\cong \mathbb{Q}_p)$. Thus $\chi_v^c = \chi_v \circ c$ is a character on K_{v^c} with conductor v^c and the same order as χ_v .

(ii) Assume v is above 2. Let π_v be a uniformizer of K_v . We distinguish three cases.

(iia) 2 splits in K . Then $K_v \cong \mathbb{Q}_2$, $\mathcal{U}_v = 1 + \mathcal{M}_v$, and $\mathcal{U}_v / (1 + \mathcal{M}_v^m) \cong (\mathbb{Z}/2^m\mathbb{Z})^\times$, which is $\langle -1 \rangle$ for $m = 2$, and $\langle -1 \rangle \times \langle 1 + \pi_v^2 \rangle$ of order 2^{m-1} if $m \geq 3$. So $1 + \mathcal{M}_v = \langle -1 \rangle \times (1 + \mathcal{M}_v^2)$. As χ_v on \mathcal{U}_v has order 2^e , so χ_v is trivial on $(1 + \mathcal{M}_v^2)^{2^e} = 1 + \mathcal{M}_v^{e+2}$. Thus if $m(v) = 2$, then $e = 1$ and $\chi_v(-1) = -1$; if $m(v) \geq 3$, then $e = m(v) - 2$ and $\chi_v(1 + \pi_v^2)$ is a primitive 2^e th root of 1, while $\chi_v(-1) = \pm 1$. The involution c maps v to v^c , we choose $\pi_{v^c} = (\pi_v)^c$, and χ_v^c is a character on $K_{v^c}^\times$ with conductor equal to $v_c^{m(v)}$ such that $\chi_v^c(-1) = \chi_v(-1)$ and $\chi_v^c(1 + \pi_{v^c}^2) = \chi_v(1 + \pi_v^2)$.

(iib) 2 is inert in K . Then K_v is a quadratic unramified extension of \mathbb{Q}_2 with 2 as a uniformizer. It contains a primitive cubic root of unity μ_3 such that $\langle \mu_3 \rangle$ represents κ_v^\times . The involution c induces the Frobenius automorphism on K_v , which acts on $\langle \mu_3 \rangle$ by squaring. As χ_v has order a power 2, it is trivial on $\langle \mu_3 \rangle$. From $\mathcal{U}_v = \langle \mu_3 \rangle (1 + \mathcal{M}_v)$ we deduce that χ_v on $\mathcal{U}_v / (1 + \mathcal{M}_v^{m(v)})$ is a trivial extension of χ_v on $(1 + \mathcal{M}_v) / (1 + \mathcal{M}_v^{m(v)})$, a group of order $4^{m(v)-1}$.

The group $(1 + \mathcal{M}_v) / (1 + \mathcal{M}_v^2) = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and $(1 + \mathcal{M}_v) / (1 + \mathcal{M}_v^3) = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle \times \langle 1 - \mu_3 2^2 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ since $(1 + \mu_3 2)^2 = 1 - 2^2$. One checks inductively on $m \geq 3$ that $(1 + \mathcal{M}_v) / (1 + \mathcal{M}_v^m) = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle \times \langle 1 - \mu_3 2^2 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{m-1}\mathbb{Z}) \times (\mathbb{Z}/2^{m-2}\mathbb{Z})$. By assumption χ_v on $\mathcal{U}_v / (1 + \mathcal{M}_v^{m(v)})$ has order 2^e . Then $e = 1$ for $m(v) = 2$. When $m(v) \geq 3$, either $\chi_v(1 + \mu_3 2)$ has order $2^{m(v)-1}$ so that $e = m(v) - 1$ or $\chi_v(1 + \mu_3 2)$ has order $\leq 2^{m(v)-2}$ and $\chi_v(1 - \mu_3 2^2)$ has order $2^{m(v)-2}$ so that $e = m(v) - 2$. Note that $(1 + \mu_3 2)^c = 1 + \mu_3^2 2 = -(1 + \mu_3 2)$ and $(1 - \mu_3 2^2)^c = 1 - \mu_3^2 2^2$.

(iic) If 2 ramifies in K , then K_v is a totally ramified quadratic extension of \mathbb{Q}_2 with residue field $\kappa_v \cong \mathbb{Z}/2\mathbb{Z}$, group of units $\mathcal{U}_v = 1 + \mathcal{M}_v$ and $2 = u\pi_v^2$ for a unit $u \in \mathcal{U}_v$. Hence $\mathcal{U}_v/(1 + \mathcal{M}_v^m)$ has order 2^{m-1} . Note that $(1 + \mathcal{M}_v^2)^2 = 1 + \mathcal{M}_v^5$ and $(1 + \pi_v)^4 \in 1 + \mathcal{M}_v^5$. We find $\mathcal{U}_v/(1 + \mathcal{M}_v^m)$ is $\langle 1 + \pi_v \rangle \cong \mathbb{Z}/2^{m-1}\mathbb{Z}$ if $m \leq 3$, $\langle 1 + \pi_v \rangle \times \langle 1 + \pi_v^3 \rangle \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ if $m = 4$, and $\langle 1 + \pi_v \rangle \times \langle 1 + \pi_v^3 \rangle \times \langle 1 + \pi_v^4 \rangle \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ if $m = 5$. Further, if χ_v on \mathcal{U}_v has order 2, then there are three possible values for $m(v)$: $m(v) = 2$ with $\chi_v(1 + \pi_v) = -1$ (in which case $\chi_v(-1) = 1$), $m(v) = 4$ with $\chi_v(1 + \pi_v^3) = -1$ and $\chi_v(1 + \pi_v) = \pm 1$, and $m(v) = 5$ with $\chi_v(1 + \pi_v^4) = -1$, $\chi_v(1 + \pi_v^3) = \pm 1$ and $\chi_v(1 + \pi_v) = \pm 1$.

The structure of $\mathcal{U}_v/(1 + \mathcal{M}_v^m)$ for $m \geq 6$ will depend on the field K . For example, if the minimal polynomial of π_v over \mathbb{Q}_2 is $x^2 + 2x + 2$, then $1 + \pi_v$ has order 4, while if the minimal polynomial is $x^2 - 2$, then $1 + \pi_v$ has infinite order in \mathcal{U}_v . Since $(1 + \mathcal{M}_v^s)^2 = 1 + \mathcal{M}_v^{s+2}$ for $s \geq 3$, if χ_v on \mathcal{U}_v has order 2^e with $e \geq 2$, we get an upper bound $m(v) \leq 2e + 3$. The involution c on K gives the nontrivial automorphism on K_v over \mathbb{Q}_2 by sending π_v to the other root of its minimal polynomial.

4 Quadratic idele class characters of quadratic fields up to base change

In this section we classify quadratic idele class characters of a quadratic field K up to base change and determine the conductors of such characters. We distinguish the discussion according to K being imaginary or real.

4.1 Classification of quadratic idele class characters over imaginary quadratic fields up to base change

First consider the case that K is an imaginary quadratic extension of \mathbb{Q} with $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Let χ be a quadratic idele class character of K with conductor \mathfrak{f}_χ . Then for each v dividing \mathfrak{f}_χ , χ_v is nontrivial on the group of units \mathcal{U}_v , hence it has order 2 and thus is trivial on the squares in $\mathcal{U}_v/(1 + \mathcal{M}_v^{m(v)})$, where $m(v) = \text{ord}_v \mathfrak{f}_\chi$. Moreover, if v is above an odd prime p , then by Proposition 3.7(i), $m(v) = 1$ and χ_v on $\mathcal{U}_v/(1 + \mathcal{M}_v) \cong \kappa_v^\times$ is the unique quadratic character sending the squares in κ_v^\times to 1. If v is above 2, Proposition 3.7(ii) says that $m(v) = 2$ or 3 if 2 is not ramified in K , and $m(v) = 2, 4$, or 5 otherwise. Further, among these values of $m(v)$, only when $m(v) = 2$ and 2 is not inert in K , the group $\mathcal{U}_v/(1 + \mathcal{M}_v^{m(v)})$ is cyclic so that a quadratic character is unique.

We proceed to classify such χ up to multiplication by characters from base change.

Proposition 4.1 *Let $K = \mathbb{Q}(\sqrt{-d})$ be imaginary quadratic, where $d > 0$ is squarefree, and denote $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Assume the class number of K is odd and the unit group is $U_K = \{\pm 1\}$ or $U_K = \langle \mu_6 \rangle$, where μ_n denotes a primitive n th root of unity. Then the following hold.*

- (1) $K = \mathbb{Q}(\sqrt{-d})$ satisfies the assumptions if and only if $d = 2$ or d is a prime $\equiv 3 \pmod{4}$. Therefore K/\mathbb{Q} is ramified at only one prime.
- (2) Let v be the place of K above an inert odd prime p . Then there is a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = v$.
- (3) Let v and v^c be the two places of K above a prime p split in K . Then there is a unique quadratic idele class character η of K with conductor \mathfrak{f}_η equal to vv^c for p odd, and

- $(vv^c)^2$ for $p = 2$. Further, for $p = 2$, there are two quadratic idele class characters η of K with conductor $\mathfrak{f}_\eta = (vv^c)^3$, determined by η_v on \mathcal{U}_v .
- (4) Let v be the place of K above 2 which is either inert or ramified in K . Then there is a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = v^2$. If 2 is inert, then there are two quadratic idele class characters η of K with conductor $\mathfrak{f}_\eta = v^3$.
 - (5) Let v be the place of $K = \mathbb{Q}(\sqrt{-2})$ above 2. Then there are no quadratic idele class characters η of K with conductor $\mathfrak{f}_\eta = v^4$, and there are two quadratic idele class characters η of K with conductor $\mathfrak{f}_\eta = v^5$, determined by $\eta_v(1 + \pi_v) = \pm 1$, $\eta_v(1 + \pi_v^3) = 1$ and $\eta_v(1 + \pi_v^4) = -1$, where $\pi_v = \sqrt{-2}$. These two characters differ by the quadratic idele class character of K of conductor v^2 described in (4).
 - (6) Let $K = \mathbb{Q}(\sqrt{-d})$ for a prime $d \equiv 3 \pmod{4}$ and v be the place of K above d . Then there are no quadratic idele class characters η of K ramified exactly at v .

Moreover, the characters η in (2)–(4) are from base change, but not (5).

Proof For a finite place v of K , denote by κ_v the residue field of K_v .

(1) It follows from Gauss’s genus theory (cf. [2, Chapter 8, Section 3, Theorem 4]) that an imaginary quadratic $K = \mathbb{Q}(\sqrt{-d})$ has odd class number if and only if its discriminant is divisible by only one prime, thus $d = 1, 2$ or a prime $\equiv 3 \pmod{4}$. The case $d = 1$ is ruled out because its group of units $\langle \mu_4 \rangle$ is not listed.

Since we are only concerned with quadratic characters η in the remaining statements, when $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\mu_6)$, it is automatic that $\eta(\mu_3) = 1$. Thus $\eta(U_K) = 1$ if and only if $\eta(-1) = 1$. Hence the argument below for $U_K = \{\pm 1\}$, i.e., $d > 3$, also applies to the case $U_K = \langle \mu_6 \rangle$, i.e., $d = 3$.

(2) If v is above an odd inert prime p , then κ_v^\times is cyclic of order $p^2 - 1$ so that -1 is a square in κ_v^\times . Thus the primitive quadratic character ξ on $(\mathbb{Z}_K/v)^\times \cong \kappa_v^\times$ is trivial on U_K . By Corollary 3.3 there is a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = v$ such that η^∞ lifts ξ . As ξ is the unique quadratic character of κ_v^\times , η is the unique quadratic idele class character of K with conductor v . Since v is the only place of K above such p , η^c has the same conductor and the same order as η , hence $\eta^c = \eta$ arises from base change.

(3) Suppose v and v^c are the two places above a split prime p . If p is odd, then κ_v^\times and $\kappa_{v^c}^\times$ are cyclic of order $p - 1$. Let ξ, ξ^c be the unique quadratic character of $\kappa_v^\times, \kappa_{v^c}^\times$, respectively. We have $\xi(-1) = \xi^c(-1) = \pm 1$. Then ξ and ξ^c determine a unique primitive character $\tilde{\xi}$ on $(\mathbb{Z}_K/vv^c)^\times \cong (\mathbb{Z}_K/v)^\times \times (\mathbb{Z}_K/v^c)^\times$ by Chinese remainder theorem. Note that $\tilde{\xi}(-1) = \xi(-1)\xi^c(-1) = 1$. By Corollary 3.3 there is a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = vv^c$ such that η^∞ lifts $\tilde{\xi}$ on $(\mathbb{Z}_K/vv^c)^\times$. Clearly $\eta^c = \eta$ is from base change.

If $p = 2$, then by Proposition 3.7(iia), $\mathcal{U}_v/(1 + \mathcal{M}_v^2) = \langle -1 \rangle$ is cyclic of order 2, the same argument as above shows the existence of a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = (vv^c)^2$. Further, by Proposition 3.7(iia), $\mathcal{U}_v/(1 + \mathcal{M}_v^3) = \langle -1 \rangle \times \langle 1 + \pi_v^2 \rangle$ is a Klein 4-group. Let η_v be a quadratic character on \mathcal{U}_v with conductor v^3 . Then $\eta_v(1 + \pi_v^2) = -1$ and $\eta_v(-1) = \pm 1$ so there are two possibilities. Notice that if η_v extends to a quadratic idele class character η of K with conductor $(vv^c)^3$, then the extension is unique. This is because η_{v^c} is a quadratic character on \mathcal{U}_{v^c} with conductor $(v^c)^3$. As such, $\eta_{v^c}(1 + \pi_{v^c}^2) = -1$. Moreover, $\eta_{v^c}(-1)$ must equal $\eta_v(-1)$ in order that η is trivial at $-1 \in K^\times$. It remains to prove the existence of an extension η .

Each η_v lifts a primitive character ξ on $(\mathbb{Z}_K/v^3)^\times \cong \mathcal{U}_v/(1 + \mathcal{M}_v^3)$. The Galois conjugate ξ^c is a primitive character on $(\mathbb{Z}_K/(v^c)^3)^\times$ satisfying $\xi^c(-1) = \xi(-1)$. By Chinese remainder theorem, ξ and ξ^c determine a unique primitive character $\tilde{\xi}$ on $(\mathbb{Z}_K/(vv^c)^3)^\times \cong$

$(\mathbb{Z}_K/v^3)^\times \times (\mathbb{Z}_K/(v^c)^3)^\times$. Note that $\tilde{\xi}(-1) = \xi(-1)\xi^c(-1) = 1$, hence $\tilde{\xi}(U_K) = 1$. We conclude from Corollary 3.3 the existence of a quadratic idele class character η of K extending the given η_v on \mathcal{U}_v with conductor $\mathfrak{f}_\eta = (vv^c)^3$. It follows from the construction that $\eta^c = \eta$ so that η is from base change.

(4) If v is above 2 which ramifies in K , then $\mathcal{U}_v/(1 + \mathcal{M}_v^2) = \langle 1 + \pi_v \rangle$ is cyclic of order 2 by Proposition 3.7(iic). There is a unique quadratic character ξ on $(\mathbb{Z}_K/v^2)^\times \cong \mathcal{U}_v/(1 + \mathcal{M}_v^2)$. Hence $\xi^c = \xi$. Moreover $\xi(-1) = 1$ since $-1 \in 1 + \mathcal{M}_v^2$. By Corollary 3.3 there is a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = v^2$ and $\eta^c = \eta$ is from base change.

Next assume 2 is inert in K . A quadratic idele class character η of K with conductor a power of v has to satisfy $\eta_v(-1) = 1$ and, as shown in Proposition 3.7(iib), possible conductors for η are v^2 and v^3 . We discuss each case. By Proposition 3.7(iib), $(1 + \mathcal{M}_v)/(1 + \mathcal{M}_v^2) = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle$ is a Klein 4-group and $(1 + \mu_3 2)^c = -(1 + \mu_3 2)$. Let ξ be a primitive quadratic character on $(\mathbb{Z}_K/v^2)^\times \cong \mathcal{U}_v/(1 + \mathcal{M}_v^2)$ satisfying $\xi(-1) = 1$, then $\xi^c = \xi$. In this case $\xi(1 + \mu_3 2) = -1$ in order to be primitive. By Corollary 3.3 there is a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = v^2$ and $\eta^c = \eta$ is from base change.

For conductor v^3 , from Proposition 3.7(iib) we know $(1 + \mathcal{M}_v)/(1 + \mathcal{M}_v^3) = \langle -1 \rangle \times \langle 1 + \mu_3 2 \rangle \times \langle 1 - \mu_3 2^2 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Our quadratic character η_v on \mathcal{U}_v with conductor v^3 has to satisfy $\eta_v(-1) = 1$ and $\eta_v(1 - \mu_3 2^2) = -1$, hence leaving two possibilities $\eta_v(1 + \mu_3 2) = \pm 1$. Note that $\eta_v^c(1 + \mu_3 2) = \eta_v(-(1 + \mu_3 2)) = \eta_v(1 + \mu_3 2)$ and $\eta_v^c(1 - \mu_3 2^2) = \eta_v(1 - \mu_3 2^2) = \eta_v(1 + (\mu_3 + 1)2^2) = \eta_v(1 + 2^2)\eta_v(1 + \mu_3 2^2) = \eta_v(1 - 2^2)\eta_v(1 - \mu_3 2^2) = \eta_v(1 - \mu_3 2^2)$ since $\eta_v(1 - 2^2) = \eta_v(1 + \mu_3 2)^2 = 1$. This shows that both choices of η_v on \mathcal{U}_v satisfy $\eta_v = \eta_v^c$. Now either choice of η_v lifts a primitive character ξ on $(\mathbb{Z}_K/v^3)^\times \cong \mathcal{U}_v/(1 + \mathcal{M}_v^3)$ satisfying $\xi(-1) = 1$ and $\xi^c = \xi$. By Corollary 3.3, either choice of η_v extends to a unique quadratic idele class character η of K with conductor $\mathfrak{f}_\eta = v^3$ and $\eta^c = \eta$ is from base change.

(5) Let v be the place of $K = \mathbb{Q}(\sqrt{-2})$ above 2. Choose $\pi_v = \sqrt{-2}$. Then $(1 + \pi_v)^2 = 1 + \pi_v^2 - \pi_v^3, -1 = 1 - 2 = 1 + \pi_v^2$. By Proposition 3.7(iic), we know $\mathcal{U}_v/(1 + \mathcal{M}_v^4) = \langle 1 + \pi_v \rangle \times \langle 1 + \pi_v^3 \rangle$. A quadratic character η_v on \mathcal{U}_v with conductor v^4 satisfies $\chi_v(1 + \pi_v^3) = -1$. Then $\chi_v(-1) = \chi_v((1 + \pi_v)^2(1 + \pi_v^3)) = \chi_v(1 + \pi_v^3) = -1$. Hence there are no quadratic idele class characters η of K with conductor v^4 since $\eta(-1) = \eta_v(-1) = -1$.

Finally consider characters of conductor v^5 . We have $\mathcal{U}_v/(1 + \mathcal{M}_v^5) = \langle 1 + \pi_v \rangle \times \langle 1 + \pi_v^3 \rangle \times \langle 1 + \pi_v^4 \rangle$ by Proposition 3.7(iic). A quadratic character η_v on \mathcal{U}_v with conductor v^5 satisfies $\chi_v(1 + \pi_v^4) = -1$. Further, $\chi_v(-1) = \chi_v((1 + \pi_v)^2(1 + \pi_v^3)) = \chi_v(1 + \pi_v^3)$. In order that χ_v extends to an idele class character η of K , we need $\chi_v(-1) = \chi_v(1 + \pi_v^3) = 1$, which leaves two possibilities for η_v on \mathcal{U}_v , namely $\eta_v(1 + \pi_v) = \pm 1$. It follows from Corollary 3.3 that either choice of η_v extends to a quadratic idele class character η of K . To finish, we show that $\eta_v^c \neq \eta_v$. Note that $\pi_v^c = -\pi_v$ so that $(1 + \pi_v)^c = 1 - \pi_v = 1 + \pi_v + \pi_v^3$. Then $\eta_v^c(1 + \pi_v) = \eta_v(1 + \pi_v + \pi_v^3) = \eta_v((1 + \pi_v)(1 + \pi_v^3)(1 + \pi_v^4)) = -\eta_v(1 + \pi_v)$. Therefore $\eta^c \neq \eta$. Clearly these two characters differ by the quadratic idele class character of K of conductor v^2 discussed in (4).

(6) Let η be a quadratic idele class character of K ramified at v . As d is an odd prime, by Proposition 3.7(1), the conductor of η_v is v . Further, since $d \equiv 3 \pmod{4}$, -1 is not a square in κ_v , hence $\eta_v(-1) = -1$. In order that $\eta(U_K) = 1$, η has to ramify at least at two places. Therefore there are no quadratic idele class characters of K which ramify only at v .

Corollary 4.2 *Let K be as in Proposition 4.1. Let v be a place of K above 2. Let χ be a quadratic idele class character of K with conductor \mathfrak{f} . Then up to multiplication by a*

quadratic idele class character of K from base change with conductor supported at v , the following hold:

- (i) If 2 is inert in K , then $\text{ord}_v \mathfrak{f} = 0$ or 2. In the latter case χ_v satisfies $\chi_v(-1) = -1$ and $\chi_v(1 + \mu_3 2) = 1$ hence is unique;
- (ii) If 2 ramifies in K , then $\text{ord}_v \mathfrak{f} = 0, 4$ or 5. In the latter two cases, χ_v satisfies $\chi_v(1 + \pi_v) = 1$ so that it is unique if it has conductor v^4 .

Proof Suppose χ_v is ramified.

(i) 2 is inert in K . Then the conductor of χ_v is either v^2 or v^3 by Proposition 3.7(iib). If $\chi_v(-1) = 1$, then the character η in Proposition 4.1(4) is from base change and $\eta_v = \chi_v$ on the group of units \mathcal{U}_v so that v is coprime to the conductor of $\chi \cdot \eta^{-1}$.

Now assume $\chi_v(-1) = -1$. If it has conductor v^3 , then the proof of Proposition 4.1(4) above shows the existence of a quadratic idele class character η of K with conductor v^3 and is from base change such that $\eta_v(1 - \mu_3 2^2) = \chi_v(1 - \mu_3 2^2) = -1$ and $\eta_v(1 + \mu_3 2) = \chi_v(1 + \mu_3 2) = \pm 1$. In this case $\chi_v \cdot \eta_v^{-1}$ has conductor v^2 , and $\chi_v \cdot \eta_v^{-1}(-1) = -1$. If χ_v has conductor v^2 and $\chi_v(1 + \mu_3 2) = -1$, then the quadratic idele class character η of K with conductor v^2 as in Proposition 4.1(4) is from base change and it is such that $\chi_v \cdot \eta_v^{-1}(1 + \mu_3 2) = 1$ and it has conductor v^2 . So for the case $\chi_v(-1) = -1$, after multiplying by an idele class character of K arising from base change supported at v , we may assume $\text{ord}_v \mathfrak{f} = 2$, $\chi_v(-1) = -1$, and $\chi_v(1 + \mu_3 2) = 1$.

(ii) 2 ramifies in K . Then the conductor of χ_v is either v^2, v^4 or v^5 by Proposition 3.7(iic). If χ_v has conductor v^2 , then the character η in Proposition 4.1(4) with conductor v^2 is from base change and $\eta_v = \chi_v$ on the group of units \mathcal{U}_v , therefore v is coprime to the conductor of $\chi \cdot \eta^{-1}$. For the remaining cases, no such η with conductor v^4 or v^5 from base change exists by Proposition 4.1(5), hence $\text{ord}_v \mathfrak{f}$ is either 4 or 5. Finally, multiplying χ by the η from Proposition 4.1(4) with conductor v^2 if necessary, we may assume $\chi_v(1 + \pi_v) = 1$ without affecting the conductor of χ_v .

Given a quadratic idele class character χ of K with conductor \mathfrak{f} , Proposition 3.7 describes possible powers of each prime ideal dividing \mathfrak{f} . Using Proposition 4.1, by multiplying χ by suitable quadratic idele class characters of K obtained from base change from \mathbb{Q} , we can reduce the factors in \mathfrak{f} and limit the places occurring in \mathfrak{f} . More precisely, by Proposition 4.1(2) we can remove those places above an inert odd prime; for the two places above a split prime, by Proposition 4.1(3), we can either remove both of them, or make χ ramified at one of the designated place; Proposition 4.1(4) allows us to simplify the factors above 2 by either removing or putting further restrictions on χ ; while Proposition 4.1(5) says that if v is the place above 2 which ramifies in K , then no further reduction at v is possible if v^4 or v^5 divides \mathfrak{f} , and similarly Proposition 4.1(6) says that no further reduction at v above the (at most one) ramified odd prime in K is possible. The result at the place above 2 which does not split in K is summarized in Corollary 4.2. This proves the first statement of the following classification theorem for quadratic characters.

Theorem 4.3 *Let K be an imaginary quadratic extension of \mathbb{Q} . Assume the class number of K is odd and $U_K = \{\pm 1\}$ or $\langle \mu_6 \rangle$. For each prime p split in K , choose one place v of K above p and let S_K be the collection of these chosen places. Then*

- (I) *Up to multiplication by characters from base change, a quadratic idele class character χ of K with conductor \mathfrak{f} satisfies the following conditions:*
- (A)_i *No places above an odd inert prime divide \mathfrak{f} ;*

- (B)_i At most one place v above a ramified prime $\equiv 3 \pmod 4$ divides \mathfrak{f} , and $\text{ord}_v \mathfrak{f} = 1$;
 - (C)_i If a place v above a split prime p divides \mathfrak{f} to the power $m(v)$, then $v \in S_K$ and $m(v) = 1$ for p odd, and $m(v) = 2$ or 3 for $p = 2$;
 - (D)_i If a place v above 2 divides \mathfrak{f} to the power $m(v)$ and 2 does not split in K , then $m(v) = 4$ or 5 and $\chi_v(1 + \pi_v) = 1$ if 2 ramifies in K , and $m(v) = 2$ and $\chi_v(-1) = -1$ and $\chi_v(1 + \mu_3 2) = 1$ if 2 is inert in K .
- (II) Any quadratic idele class character χ of K with nontrivial conductor \mathfrak{f} satisfying (A)_i–(D)_i does not arise from base change. In other words, $\chi^c = \chi \cdot \delta$ for some quadratic idele class character δ of K .
- (III) No two distinct quadratic idele class characters of K satisfying (A)_i–(D)_i differ by multiplication by an idele class character of K from base change.
- (IV) Let $\mathfrak{f} = \prod_{v \text{ finite}} v^{m(v)}$ be an integral ideal of K with $m(v)$ satisfying (A)_i–(D)_i. Denote by $r(\mathfrak{f})$ the number of places v occurring in \mathfrak{f} such that v is above a prime $p \equiv 3 \pmod 4$. Then there is a quadratic idele class character χ of K with conductor \mathfrak{f} satisfying the conditions (A)_i–(D)_i if and only if $r(\mathfrak{f})$ is even if no $v \mid \mathfrak{f}$ is above 2 , and $r(\mathfrak{f})$ is odd if there is a prime $v \mid \mathfrak{f}$ above 2 with $m(v) = 2$ or 4 .

Proof It remains to prove assertions (II)–(IV).

(II) The second statement follows from Theorem 3.6. The first statement is equivalent to $\chi \neq \chi^c$ since both are quadratic idele class characters of K . This is obvious if some place v above a split prime p divides \mathfrak{f} because v^c does not divide \mathfrak{f} by condition (C)_i and it divides the conductor of χ^c . Now suppose no places in S_K divide \mathfrak{f} . Then the only places v dividing \mathfrak{f} are above primes p which are either ramified or inert in K . Condition (A)_i implies that p cannot be an odd inert prime. If $K = \mathbb{Q}(\sqrt{-2})$, this forces $p = 2$ and $\mathfrak{f} = v^4$ or v^5 by condition (D)_i. We conclude from Proposition 4.1(5) that $\mathfrak{f} = v^5$ and $\chi \neq \chi^c$. The remaining case is $K = \mathbb{Q}(\sqrt{-d})$ for $d > 3$ a prime $\equiv 3 \pmod 4$ by Proposition 4.1(1). So d is the only prime ramified in K . If v is above $p = d$, then $\chi_v(-1) = -1$ since -1 is not a square in the residue field κ_v . In order that $\chi(-1) = 1$, one place v' above 2 has to divide \mathfrak{f} . Therefore 2 is inert in K . Condition (D)_i implies that $\chi_{v'}(-1) = -1$, as it should, and v' divides \mathfrak{f} to the power 2 . Using Proposition 3.7(iib), we get $\chi_{v'}^c(1 + \mu_3 2) = \chi_{v'}(-1 + \mu_3 2) = -\chi_{v'}(1 + \mu_3 2)$, showing $\chi^c \neq \chi$.

(III) Let χ and η be two distinct quadratic idele class characters of K satisfying (A)_i–(D)_i. Then $\delta := \chi \eta^{-1} = \chi \eta$ is also a quadratic idele class character of K whose conductor \mathfrak{f}_δ divides the least common multiple of \mathfrak{f}_χ and \mathfrak{f}_η . It clearly satisfies conditions (A)_i–(C)_i. If (D)_i is also satisfied, then (III) will follow from (II). To check the condition (D), suppose v is a place of K above 2 dividing $\mathfrak{f}_\chi \mathfrak{f}_\eta$ and 2 does not split in K . We distinguish two cases.

Case (a) 2 is inert in K . If one of χ and η , say, η , is unramified at v , then on \mathcal{U}_v , we have $\delta_v = \chi_v$. If both χ and η are ramified at v , then, $\chi_v = \eta_v$ on \mathcal{U}_v by (D)_i so that δ_v is unramified at v . In both cases the condition (D)_i holds for δ_v .

Case (b) 2 ramifies in K . Then $\delta_v = \chi_v \eta_v$ has conductor at most v^5 . If it has conductor v^4 or v^5 , then $\delta_v(1 + \pi_v) = 1$ follows from $\chi_v(1 + \pi_v) = \eta_v(1 + \pi_v) = 1$. If δ_v has conductor less than v^4 , then it is trivial on \mathcal{U}_v since $\mathcal{U}_v / (1 + \mathcal{M}_v^3) = \langle 1 + \pi_v \rangle$ and $\chi_v(1 + \pi_v) = \eta_v(1 + \pi_v) = 1$. Hence (D)_i holds for δ_v in both situations.

(IV) First assume χ exists. Since K is imaginary, the local component χ_∞ is the trivial character of \mathbb{C}^\times . Therefore $\chi^\infty(-1) = \prod_{v \mid \mathfrak{f}} \chi_v(-1) = 1$. Suppose $v \mid \mathfrak{f}$ is above the prime p . If p is odd, then $m(v) = 1$ and $\chi_v(-1) = -1$ if and only if $p \equiv 3 \pmod 4$. If $p = 2$ and $m(v) = 2$, then $\chi_v(-1) = -1$ by Proposition 3.7, (iia) for 2 splits, and (D)_i for 2 inert. Further if $p = 2$ and $m(v) = 4$ (hence 2 ramifies), we have $\chi_v(-1) = -1$ as shown in the proof of Proposition 4.1(5). Hence $1 = (-1)^{r(\mathfrak{f})}$ if no prime above 2 divides \mathfrak{f} , and

$1 = (-1)^{r(f)+1}$ if there is a prime v above 2 divides f and $m(v) = 2$ or 4. For the remaining case that some $v|f$ with $m(v) = 3$ or 5 (hence $p = 2$), there are two choices of χ_v satisfying $(D)_i$ so that $\chi_v(-1) = \pm 1$. With given $r(f)$ there is a unique choice to make $\chi^\infty(-1) = 1$. This proves the necessity.

Conversely, assume f and $r(f)$ satisfy the hypotheses, we proceed to prove the existence of a quadratic idele class character χ of K as described. For each v occurring in f , let χ_v be a quadratic character of \mathcal{U}_v with conductor $v^{m(v)}$ and satisfying $(D)_i$ if v is above 2 which does not split in K . Such χ_v is unique except for v above 2 which either splits in K with $m(v) = 3$ or ramifies in K with $m(v) = 5$. In each case there are two choices for χ_v and we choose the one so that $\prod_{v|f} \chi_v(-1) = 1$. When there is no choice, this identity follows from the assumption on $r(f)$.

Since $(\mathbb{Z}_K/f)^\times \simeq \prod_{v \text{ finite}} \mathcal{U}_v / (1 + \mathcal{M}_v^{m(v)})$, the product $\prod_{v|f} \chi_v$ lifts a unique quadratic primitive character ξ of $(\mathbb{Z}_K/f)^\times$ such that $\xi(U_K) = 1$. By Corollary 3.3, ξ lifts to a quadratic idele class character χ of K satisfying $(A)_i$ - $(D)_i$.

The remaining imaginary quadratic K over \mathbb{Q} with odd class number is $K = \mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(\mu_4)$. Its group of units is $U_K = \langle \mu_4 \rangle$. We go over the statements in Proposition 4.1 one by one. Let v be a place v of K above a prime p . For p inert in K , i.e. $p \equiv 3 \pmod{4}$, since μ_4 is a square in κ_v^\times which is cyclic of order $p^2 - 1$, divisible by 8, statement (2) holds. For p split in K , i.e., $p \equiv 1 \pmod{4}$, the argument in the proof for (3) goes through with -1 replaced by μ_4 , so (3) also holds. Finally for $p = 2$ which ramifies in K , choose $\pi_v = \mu_4 - 1$ so that $1 + \pi_v = \mu_4$. Then $\mu_4^c = -\mu_4$ and $\pi_v^c = -\mu_4 - 1 = \mu_4\pi_v$. By Proposition 3.7 (iic), a quadratic character η_v on \mathcal{U}_v has conductor $v^2, v^4, \text{ or } v^5$. If η_v has conductor v^2 , then $\eta_v(\mu_4) = -1$ and it cannot be extended to an idele class character of K with conductor v^2 . If η_v has conductor v^4 , then $\eta_v(1 + \pi_v^3) = -1$ and $\eta_v(\mu_4) = 1$ is the only such character which can be extended to a quadratic idele class character η of K with conductor v^4 by Corollary 3.3. In this case $\eta^c = \eta$ so that it is from base change. Similarly, there are two quadratic idele class characters η of K with conductor v^5 , given by $\eta_v(\mu_4) = 1, \eta_v(1 + \pi_v^3) = \pm 1$, and $\eta_v(1 + \pi_v^4) = -1$. We check that $\eta_v^c(1 + \pi_v^3) = \eta_v(1 + \mu_4^3\pi_v^3) = \eta_v(1 + (1 + \pi_v)^3\pi_v^3) = \eta_v(1 + \pi_v^3 + \pi_v^4) = \eta_v((1 + \pi_v^3)(1 + \pi_v^4)) = -\eta_v(1 + \pi_v^3)$ since η_v has conductor \mathcal{M}_v^3 . So none of the characters η with conductor v^5 is from base change. The order of the conductor f at v of a quadratic idele class character χ of K is among 0, 2, 4, 5. Multiplying it by the above idele class character η of conductor v^4 from base change if necessary, we may assume that χ satisfies $\chi_v(1 + \pi_v^3) = 1$ and $\text{ord}_v f = 5, 2$ or 0.

Similar arguments as before with the unit -1 replaced by μ_4 prove the following classification theorem for $K = \mathbb{Q}(\sqrt{-1})$. Note that condition $(B)_i$ is automatically satisfied since no odd primes ramify in $\mathbb{Q}(\sqrt{-1})$.

Theorem 4.4 *Let $K = \mathbb{Q}(\sqrt{-1})$. For each prime p split in K , choose one place v of K above p and let S_K be the collection of these chosen places. Then*

(I) *Up to multiplication by characters from base change, a quadratic idele class character χ of K with conductor f satisfies the conditions $(A)_i$ - $(C)_i$ in Theorem 4.3 and*

(D)'_i *If a place v above 2 divides f to the power $m(v)$, then $m(v) = 2$ or 5. Further $\chi_v(\mu_4) = -1$ if $m(v) = 2$, and $\chi_v(1 + \pi_v^3) = 1$ if $m(v) = 5$. Here $\pi_v = \mu_4 - 1$.*

(IV) *Let $f = \prod_{v \text{ finite}} v^{m(v)}$ be an integral ideal of K with $m(v)$ satisfying $(A)_i$ - $(C)_i$ and $(D)'_i$. Denote by $r(f)$ the number of places v occurring in f such that v is above a prime $p \equiv 5 \pmod{8}$. Then there is a quadratic idele class character χ of K with conductor f satisfying the conditions $(A)_i$ - $(C)_i$ and $(D)'_i$ if and only if $r(f)$ is even if no $v|f$ is above 2, and $r(f)$ is odd if there is a prime $v|f$ above 2 with $m(v) = 2$.*

Moreover, (II) and (III) in Theorem 4.3 hold with $(D)_i$ replaced by $(D)'_i$.

4.2 Classification of quadratic idele class characters over real quadratic fields up to base change

Next we consider the case for real quadratic $K = \mathbb{Q}(\sqrt{d})$ with $d > 0$ square-free integer along the same vein. Gauss' genus theory says that the narrow class number $h^+(K)$ of K is odd if and only if its discriminant d_K has one prime factor, which holds precisely when d is a prime $\equiv 1 \pmod{4}$ or $d = 2$. In this case K contains a unit with norm -1 , hence the class number $h(K) = h^+(K)$. The remaining case for $h(K)$ odd is when $h^+(K) = 2m$ with m odd and there is no unit of negative norm so that $h(K) = h^+(K)/2 = m$ is odd. In this case d_K has two prime factors, hence either $d_K = 4d$ for a prime $d \equiv 3 \pmod{4}$ or $d = p_1 p_2$ with two distinct primes $p_1 \equiv p_2 \pmod{4}$ since $d_K \equiv 1 \pmod{4}$. In the former case d a prime, the field K does not contain a unit with norm -1 , hence $h(K)$ is odd. In the latter case $d = p_1 p_2$, if $p_1 \equiv p_2 \equiv 3 \pmod{4}$, again K has no unit with negative norm so that $h(K)$ is odd. If $p_1 \equiv p_2 \equiv 1 \pmod{4}$, then the situation is more complicated. Dirichlet (1834) showed that if the Legendre symbol $\left(\frac{p_1}{p_2}\right) = -1$, then there is a unit of negative norm, so that $h(K) = h^+(K)$ is even. But if $\left(\frac{p_1}{p_2}\right) = 1$ and the product of the quartic residue symbols $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$ then there is no unit of negative norm, so $h(K)$ is odd.

To illustrate the key points, we consider the simplest case of real quadratic fields and state conclusions similar to the case of imaginary quadratic fields studied above.

Proposition 4.5 *Let $K = \mathbb{Q}(\sqrt{d})$, where $d = 2$ or a prime $\equiv 1 \pmod{4}$, be real quadratic with Galois group $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Then the following hold.*

- (1) *d is the only prime ramified in K . Further, the class number of K is odd and there is a unit in U_K with norm -1 .*
- (2) *Let v be the place of K above an inert or ramified odd prime p . Then there is a unique quadratic idele class character η of K with conductor v . It is from base change.*
- (3) *Let v be a place of K above a prime p split in K . Then there is a unique quadratic idele class character η of K with conductor equal to v for p odd, and v^2 for $p = 2$. Further, for $p = 2$, there are two quadratic idele class characters η of K with conductor v^3 , determined by $\eta_v(-1) = \pm 1$ and $\eta_v(1 + \pi_v^2) = -1$. None of these characters η are from base change, but all $\eta\eta^c$ are.*
- (4) *Let v be the place of K above 2 which is inert in K . Then there are three quadratic idele class character η of K with conductor v^2 . Exactly one of them, satisfying $\eta_v(-1) = 1$ and $\eta_v(1 + \mu_3 2) = -1$, is from base change. There are four quadratic idele class characters η of K with conductor v^3 . Exactly two of them, satisfying $\eta_v(-1) = 1$, $\eta_v(1 + \mu_3 2) = \pm 1$ and $\eta_v(1 - \mu_3 2^2) = -1$, are from base change.*
- (5) *Let v be the place of $K = \mathbb{Q}(\sqrt{2})$ above 2. There is one quadratic idele class character $\eta(2)$ of K of conductor v^2 , satisfying $\eta_v(1 + \pi_v) = -1$, and it is from base change. Up to multiplication by $\eta(2)$, there is one quadratic idele class character η of K with conductor v^4 , satisfying $\eta_v(1 + \pi_v) = 1$ and $\eta_v(1 + \pi_v^3) = -1$, and it is not from base change. Up to multiplication by $\eta(2)$, there are two quadratic idele class characters η of K with conductor v^5 . Exactly one of them, satisfying $\eta_v(1 + \pi_v) = 1$, $\eta_v(1 + \pi_v^3) = -1$ and $\eta_v(1 + \pi_v^4) = -1$, is from base change. Here $\pi_v = \sqrt{2}$.*

Proof (1) is already discussed in the paragraph before the proposition. In view of Corollary 3.3 and Theorem 3.1, for our K , any quadratic character on the group of units in I_K^∞ has a unique

extension to a quadratic idele class character of I_K . This makes the proof of the remaining statements simpler than their counterparts in Proposition 4.1 because there are no constraints for the character on U_K .

(2) For v being the unique place of K above an inert or ramified odd prime p , there is a unique quadratic idele class character η of K with conductor v . Since c stabilizes v , we have $\eta = \eta^c$.

(3) For v a place of K above a split prime p , the existence of quadratic idele class characters η of K as stated follows from Proposition 3.7(i) and (ii). As $v^c \neq v$, η is not from base change, but the character $\eta\eta^c$ is.

(4) For v the only place above a split prime 2, Proposition 3.7(ii) describes the three (resp. four) quadratic idele class characters of K of conductor v^2 (resp. v^3). The same argument as in the proof of Proposition 4.1(4) shows that $\eta = \eta^c$ if and only if $\eta_v(-1) = 1$ in both cases.

(5) Choose $\pi_v = \sqrt{2}$ so that $\pi_v^c = -\pi_v = \pi_v - \pi_v^3$. By Proposition 3.7(iic), there is one quadratic idele class character $\eta(2)$ of K of conductor v^2 determined by $\eta(2)_v(1 + \pi_v) = -1$. Since $\eta(2)_v^c(1 + \pi_v) = \eta(2)_v(1 - \pi_v) = \eta(2)_v(1 + \pi_v)$, we have $\eta(2) = \eta(2)^c$, hence is from base change.

Up to multiplication by $\eta(2)$, for quadratic idele class characters η of K of conductor v^r with $r \geq 3$, we may assume $\eta_v(1 + \pi_v) = 1$. By Proposition 3.7(iic), there is one quadratic character η of K with conductor v^4 , satisfying $\eta_v(1 + \pi_v^3) = -1$ and $\eta_v(1 + \pi_v) = 1$. We show that $\eta \neq \eta^c$ so that η is not a base change from \mathbb{Q} . This is because $\eta_v^c(1 + \pi_v^3) = \eta_v(1 - \pi_v^3) = \eta_v(1 + \pi_v^3)$ since η_v has conductor v^4 , and $\eta_v^c(1 + \pi_v) = \eta_v(1 - \pi_v) = \eta_v(1 + \pi_v - \pi_v^3) = \eta_v(1 + \pi_v)\eta_v(1 - \pi_v^3) = -\eta_v(1 + \pi_v)$.

Similarly, by Proposition 3.7(iic), up to multiplication by $\eta(2)$, there are two quadratic characters η of K with conductor v^5 , satisfying $\eta_v(1 + \pi_v) = 1$, $\eta_v(1 + \pi_v^4) = -1$ and $\eta(1 + \pi_v^3) = \pm 1$. We check η^c . Indeed, we have $\eta_v^c(1 + \pi_v^4) = \eta_v(1 + \pi_v^4)$, $\eta_v^c(1 + \pi_v^3) = \eta_v(1 - \pi_v^3) = \eta_v(1 + \pi_v^3)$, and $\eta_v^c(1 + \pi_v) = \eta_v(1 - \pi_v) = \eta_v(1 + \pi_v - \pi_v^3) = \eta_v((1 + \pi_v)(1 - \pi_v^3)(1 + \pi_v^4)) = -\eta_v(1 + \pi_v)\eta_v(1 + \pi_v^3)$, which is equal to $\eta_v(1 + \pi_v) = 1$ if and only if $\eta_v(1 + \pi_v^3) = -1$.

The conclusion below follows immediately from Proposition 4.5 above by an argument similar to the proof of Corollary 4.2.

Corollary 4.6 *Let K be as in Proposition 4.5. Let v be a place of K above 2. Let χ be a quadratic idele class character of K with conductor \mathfrak{f} . Then up to multiplication by a quadratic idele class character of K from base change with conductor supported at v , the following hold:*

- (i) *If 2 is inert in K , then $\text{ord}_v \mathfrak{f} = 0$ or 2. In the latter case χ_v satisfies $\chi_v(-1) = -1$ and $\chi_v(1 + \mu_3 2) = 1$, hence is unique;*
- (ii) *If 2 ramifies in K , then $\text{ord}_v \mathfrak{f} = 0$ or 4. In the latter case, χ_v satisfies $\chi_v(1 + \pi_v) = 1$ and $\chi_v(1 + \pi_v^3) = -1$, hence is unique.*

Now we are ready to state the classification of quadratic idele class characters for real quadratic fields parallel to its counterpart Theorem 4.3 for imaginary quadratic fields. The proof is similar and hence is omitted.

Theorem 4.7 *Let $K = \mathbb{Q}(\sqrt{d})$ where $d = 2$ or a prime $\equiv 1 \pmod{4}$ be a real quadratic extension of \mathbb{Q} . For each prime p split in K , choose one place v of K above p and let S_K be the collection of these chosen places. Then*

- (I) *Up to multiplication by characters from base change, a quadratic idele class character χ of K with conductor \mathfrak{f} satisfies the following conditions:*

- (A)_r No places above an odd inert or ramified prime divide \mathfrak{f} ;
 - (C)_r If a place v above a split prime p divides \mathfrak{f} to the power $m(v)$, then $v \in S_K$ and $m(v) = 1$ for p odd, and $m(v) = 2$ or 3 for $p = 2$;
 - (D)_r If a place v above 2 divides \mathfrak{f} to the power $m(v)$ and 2 does not split in K , then $m(v) = 4$ and $\chi_v(1 + \pi_v) = 1$ and $\chi_v(1 + \pi_v^3) = -1$ if 2 ramifies in K , and $m(v) = 2$ and $\chi_v(-1) = -1$ and $\chi_v(1 + \mu_3 2) = 1$ if 2 is inert in K .
- (II) Any quadratic idele class character χ of K with nontrivial conductor \mathfrak{f} satisfying (A)_r, (C)_r, (D)_r does not arise from base change. In other words, $\chi^c = \chi \cdot \delta$ for some quadratic idele class character δ of K .
- (III) No two distinct quadratic idele class characters of K satisfying (A)_r, (C)_r, (D)_r differ by multiplication by an idele class character of K from base change.
- (IV) Given an integral ideal $\mathfrak{f} = \prod_v \text{finite } v^{m(v)}$ of K with $m(v)$ satisfying (A)_r, (C)_r and (D)_r, there is a quadratic idele class character χ of K with conductor \mathfrak{f} so that the conditions (A)_r, (C)_r and (D)_r hold.

Statement (IV) above is simpler than its counterpart because in the parallel construction, the character ξ is quadratic so that $\xi(U_K) \subset \{\pm 1\}$ automatically holds and Corollary 3.3 applies.

5 Examples of arithmetically equivalent pairs with quadratic characters

In this section we construct two families of examples of arithmetically equivalent pairs with quadratic characters, which give rise to families of holomorphic weight one cusp forms and Maass cusp forms arising from characters of two different fields, respectively.

5.1 Holomorphic weight one cusp forms arising from two different fields

Let $K = \mathbb{Q}(\sqrt{-1})$. Denote by c the complex conjugation so that $\text{Gal}(K/\mathbb{Q}) = \langle c \rangle$. Let q be a prime $\equiv 1 \pmod 8$, then q splits in K . Let Q be a place of K above q with residue field $\kappa_Q \simeq \mathbb{Z}_K/Q \simeq \mathbb{Z}/(q)$. Since $|\kappa_Q^\times| = q - 1$ is divisible by 8 , $\sqrt{-1}$ is a square in κ_Q . The quadratic character ξ of $(\mathbb{Z}_K/Q)^\times$ is trivial on the group of global units $U_K = \langle \sqrt{-1} \rangle$. By Corollary 3.3, there is a unique quadratic idele class character χ of K with conductor $\mathfrak{f}_\chi = Q$ lifting ξ . Then $\chi \neq \chi^c$ and hence is not from base change.

We take a closer look at $\chi_v(\pi_v)$ for $v \neq Q$. Since K has class number 1 , every maximal ideal v in \mathbb{Z}_K is principal, that is, $v = (\pi_v)$ for some element $\pi_v \in \mathbb{Z}_K$. For $v \neq Q$, the value $\chi_v(\pi_v)$ is given by ξ evaluated at $\pi_v \pmod Q$. Specifically, the image of a prime number $p \neq q$ in $\mathbb{Z}[i]/Q$ is the same as its image in $\mathbb{Z}/(q)$. So p is a quadratic residue in κ_Q if and only if p is a quadratic residue in $\mathbb{Z}/(q)$, i.e., the Legendre symbol $\left(\frac{p}{q}\right) = 1$, which holds if and only if p is a quadratic residue in κ_{Q^c} .

If v is above $p \equiv 3 \pmod 4$, then

$$\chi_v(\pi_v) = (\chi^c)_v(\pi_v) = \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$$

since $q \equiv 1 \pmod 8$. If v is above $p \equiv 1 \pmod 4$, then $(p) = (\pi_v)(\pi_v)^c = (\pi_v)(\pi_v^c)$, so

$$\chi_v(\pi_v)(\chi^c)_v(\pi_v) = \chi_v(\pi_v)\chi_{v^c}(\pi_v^c) = \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

We have shown that $\chi^c = \chi \cdot \delta_{K/M/K}$ for $M = \mathbb{Q}(\sqrt{q})$. By Theorem 2.2, there is an idele class character η of M , not self-conjugate, such that $\rho_\chi := \text{Ind}_K^{\mathbb{Q}} \chi = \text{Ind}_M^{\mathbb{Q}} \eta =: \rho_\eta$ and hence

$$L(s, \chi, K) = L(s, \eta, M).$$

The above computation gives the following description of $L(s, \chi, K)$ for $\Re(s) > 1$:

$$\begin{aligned} L(s, \chi, K) &= \frac{1}{1 - \chi_{(1-i)}(\pi_{1-i})2^{-s}} \cdot \frac{1}{1 - \chi_{Q^c}(\pi_{Q^c})q^{-s}} \\ &\times \prod_{p \equiv 1 \pmod{4}, \left(\frac{p}{q}\right)=1, \text{ any } v|p} \frac{1}{(1 - \chi_v(\pi_v)p^{-s})^2} \\ &\times \prod_{p \equiv 1 \pmod{4}, \left(\frac{p}{q}\right)=-1} \frac{1}{1 + \left(\frac{p}{q}\right)p^{-2s}} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 - \left(\frac{p}{q}\right)p^{-2s}} \\ &= \sum_{n \geq 1} a_\chi(n)n^{-s}. \end{aligned}$$

The representation ρ_χ has conductor $4q$ (where 4 comes from the discriminant of K over \mathbb{Q} and q comes from the norm of the conductor of χ). The field M has odd class number and a fundamental unit with norm -1 . It follows from $q \equiv 1 \pmod{8}$ that 2 splits in M . In order that ρ_η has conductor $4q$, and η not self-conjugate, η has conductor $f_\eta = R^2$ for a place R of M above 2. By Corollary 3.3, η is the unique quadratic idele class character of M lifting the unique quadratic character of $(\mathbb{Z}_M/R^2)^\times$. At R , η_R takes value -1 (resp. 1) on units congruent to -1 (resp. 1) mod R^2 . In particular $\eta_R(-1) = -1$, which implies that the two local components of η at the two real places of M take opposite values at -1 . This shows that $\rho_\eta = \rho_\chi$ is odd. Therefore the cusp form $g_\chi = g_\eta$ with $L(s, g_\chi) = L(s, \chi, K)$ is a holomorphic weight 1 cusp form with level $4q$ and character $(\frac{-4q}{\cdot})$; its Fourier expansion is

$$g_\chi(z) = \sum_{n \geq 1} a_\chi(n)e^{2\pi inz}.$$

As q varies, this gives a family of examples of holomorphic weight 1 cusp forms arising from idele class characters of two different quadratic fields.

5.2 Maass cusp forms arising from two different fields

Consider two real quadratic fields $K = \mathbb{Q}(\sqrt{t})$ and $M = \mathbb{Q}(\sqrt{q})$, where t and q are two distinct primes $\equiv 1 \pmod{4}$ such that the Legendre symbol $(\frac{q}{t}) = (\frac{t}{q}) = 1$. Then K has odd class number, and it has a fundamental unit ϵ_K with norm -1 . The same holds for M . Denote by σ the generator of the Galois group $\text{Gal}(K/\mathbb{Q})$ and τ that of $\text{Gal}(M/\mathbb{Q})$. By choice, q splits in K , that is, $(q) = QQ^\sigma$ in \mathbb{Z}_K and t splits in M , namely $(t) = TT^\tau$ in \mathbb{Z}_M .

By Corollary 3.3 there is a unique quadratic idele class character χ of K with conductor Q . It lifts the quadratic character of $(\mathbb{Z}_K/Q)^\times \cong (\mathbb{Z}/q\mathbb{Z})^\times$. Further, since $\chi_Q(-1) = 1$, the local components of χ at the two infinite places ∞_1 and ∞_2 of K agree, hence the induced representation $\rho_\chi := \text{Ind}_{G_K}^{G_\mathbb{Q}} \chi$ is even. In fact, at the complex conjugation c in $G_\mathbb{Q}$, we have $\rho_\chi(c) = \pm Id$ with the sign given by the value $\chi_{\infty_1}(-1) = \chi_{\infty_2}(-1) = \chi_Q(\epsilon_K)$, where ϵ_K is a fundamental unit of K with norm -1 . Therefore, the sign is $+$ if and only if ϵ_K is a square in the residue field \mathbb{Z}_K/Q at Q .

We compare χ and its conjugate χ^σ .

Proposition 5.1 $\chi\chi^\sigma = \delta_{KM/K}$.

Proof Denote by $h(K)$ the class number of K . Let v be a place of K above a prime p . If p splits in K , that is, $(p) = vv^\sigma$ in \mathbb{Z}_K , we have $(p)^{h(K)} = v^{h(K)}(v^\sigma)^{h(K)} = (\beta_v)(\beta_{v^\sigma})$ and $(\beta_{v^\sigma}) = ((\beta_v)^\sigma)$. Recall from the definition of χ in the proof of Theorem 3.1 that, for $p \neq q$, we have

$$\begin{aligned} \chi_v(\pi_v)\chi_v^\sigma(\pi_v) &= \chi_v(\beta_v)\chi_v^\sigma(\beta_v) = \chi_v(\beta_v)\chi_v((\beta_v)^\sigma) \\ &= \chi_v(p)^{h(K)} = \chi_v(p) = \left(\frac{q}{p}\right) = \delta_{M/\mathbb{Q}}(p) = \delta_{KM/K}(\pi_v) \end{aligned}$$

since p splits completely in K and is unramified in M , so it splits completely in KM (i.e. v splits in KM) if and only if p splits in M . If v is inert in K , then $v = (p)$ and

$$\chi_v(\pi_v)\chi_v^\sigma(\pi_v) = \chi_v(p)^2 = 1 = \delta_{KM/K}(\pi_v)$$

since such v is unramified in KM and its residue field already has p^2 elements, it has to split in KM or else KM would have a place with p^4 elements in its residue field, which is impossible because $Gal(KM/\mathbb{Q})$ is a Klein 4 group. Now both $\chi\chi^\sigma$ and $\delta_{KM/K}$ are idele class characters of K which agree at all but finitely many places, they agree.

It then follows from the proof of Theorem 2.2 that there is an idele class character η of M such that $L(s, \chi, K) = L(s, \eta, M)$. This in turn implies that η is quadratic and ramified exactly at one of the places above t , say, T , and it has conductor T . Thus η is unique by Corollary 3.3 and it is not self-conjugate.

Let $g = g_\chi = g_\eta$ be the Maass cusp form with associated L-function $L(s, g) = L(s, \chi, K) = L(s, \eta, M) = \sum_{n \geq 1} a_\chi(n)n^{-s}$. There are two possible explicit Fourier expansions for $g(z)$, depending on the sign of $\rho_\chi(c) = \pm Id$ given by $\chi_Q(\epsilon_K)$:

$$g_\chi(z) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})\sqrt{y}K_0(2\pi N(\mathfrak{a})y)2 \begin{cases} \cos(2\pi N(\mathfrak{a})x), & \text{if } \rho_\chi(c) = Id; \\ \sin(2\pi N(\mathfrak{a})x), & \text{if } \rho_\chi(c) = -Id, \end{cases}$$

where K_0 is the K -Bessel function (see [1, Theorem 1.9.1]). We see from the two examples below that both signs can occur.

Example 1 $t = 5$ and $q = 29$ so that $K = \mathbb{Q}(\sqrt{5})$, $M = \mathbb{Q}(\sqrt{29})$ and $(\frac{29}{5}) = 1$. We know that \mathbb{Z}_K has class number 1 (in fact, it is a Euclidean domain) and

$$\epsilon_K = \frac{1 + \sqrt{5}}{2}$$

is a fundamental unit of norm -1 . The ideal (29) factors as $Q \cdot Q^\sigma$ with

$$Q = (7 + 2\sqrt{5}).$$

To compute $\chi_Q(\epsilon_K)$, we use Euler’s criterion: If Q is a prime ideal and $\alpha \in \mathbb{Z}_K$ is coprime to Q , then

$$\chi_Q(\alpha) \equiv \alpha^{(N(Q)-1)/2} \pmod{Q}.$$

As our Q has norm $N(Q) = 29$, we compute by using the Euclidean division algorithm in \mathbb{Z}_K

$$\epsilon_K^{14} = \frac{843 + 377\sqrt{5}}{2} = (7 + 2\sqrt{5}) \cdot \left(20 + 33\frac{1 + \sqrt{5}}{2}\right) + 1$$

so that

$$\epsilon_K^{(29-1)/2} = +1 \pmod{Q}$$

which shows that

$$\chi_Q(\epsilon_K) = +1.$$

Therefore $\rho_\chi(c) = Id$ and the Maass form g_χ has Fourier expansion

$$g_\chi(x + iy) = \sum_{n \geq 1} a_\chi(n) \sqrt{y} K_0(2\pi ny) 2 \cos(2\pi nx),$$

where $a_\chi(n) = \sum_{N(\mathfrak{a})=n} \chi(\mathfrak{a})$ is the coefficient of n^{-s} in $L(s, \chi, K)$.

Example 2 $t = 5$ and $q = 41$ so that $K = \mathbb{Q}(\sqrt{5})$, $M = \mathbb{Q}(\sqrt{41})$ and $\left(\frac{41}{5}\right) = 1$. The ideal (41) factors as $Q \cdot Q^\sigma$ with

$$Q = \left(6 + \frac{1 + \sqrt{5}}{2}\right)$$

and $N(Q) = 41$ in this case. Then

$$\epsilon_K^{(41-1)/2} = \frac{1}{2} (15127 + 6765\sqrt{5}) = \left(6 + \frac{1 + \sqrt{5}}{2}\right) \cdot \left(549 + 888 \frac{1 + \sqrt{5}}{2}\right) - 1$$

so that

$$\chi_Q(\epsilon_K) \equiv \epsilon_K^{(41-1)/2} = -1 \pmod{Q}.$$

Therefore $\rho_\chi(c) = -Id$ and g_χ is a Maass form with Fourier expansion

$$g_\chi(x + iy) = \sum_{n \geq 1} a_\chi(n) \sqrt{y} K_0(2\pi ny) 2 \sin(2\pi nx).$$

References

1. Bump, D.: Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics, vol. 55. Cambridge University Press, Cambridge (1997)
2. Cohn, H.: Advanced Number Theory. Reprint of a Second Course in Number Theory, 1962. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York (1980)
3. Cornelissen, G.: Curves, dynamical systems, and weighted point counting. Proc. Natl. Acad. Sci. USA **110**(24), 9669–9673 (2013)
4. Cornelissen, G., de Smit, B., Li, X., Marcolli, M., Smit, H.: Characterization of global fields by Dirichlet L-series. Res. Number Theory **5**, 7 (2019)
5. Gassmann, F.: Bemerkungen zu der vorstehenden Arbeit von Hurwitz. Math. Z. **25**, 124–143 (1926)
6. Gérardin, P., Labesse, J.-P.: The solution to a base change problem for $GL(2)$ (following Langlands, Saito, Shintani). In: Automorphic Forms, Representations, and L-Functions. Proceedings of Symposia in Pure Mathematics, vol. 33, part 2, pp. 115–133. American Mathematical Society, Providence (1979)
7. Klüners, J., Nicolae, F.: Are number fields determined by Artin L-functions? J. Number Theory **167**, 161–168 (2016)
8. Mantilla-Soler, G.: On a question of Perlis and Stuart regarding arithmetic equivalence. N. Y. J. Math. **25**, 558–573 (2019)
9. Perlis, R.: On the equation $\zeta_K(s) = \zeta_{K'}(s)$. J. Number Theory **9**(3), 342–360 (1977)
10. Prasad, D.: A refined notion of arithmetically equivalent number fields, and curves with isomorphic Jacobians. Adv. Math. **312**, 198–208 (2017)

11. Rohrlich, D.E.: Artin representations of \mathbb{Q} of dihedral type. *Math. Res. Lett.* **22**, 1767–1789 (2015)
12. Rohrlich, D.E.: Almost abelian Artin representations of \mathbb{Q} . *Mich. Math. J.* **68**(1), 127–145 (2019)
13. Serre, J.-P.: Modular forms of weight one and Galois representations. In: *Algebraic Number Fields: L-Functions and Galois Properties* (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 193–268. Academic Press, London (1977)
14. Serre, J.-P.: Local class field theory. In: Cassels, J.W.S., Fröhlich, A. (eds.) *Algebraic Number Theory*, pp. 128–161. Academic Press, London (1967)
15. Sutherland, A.: Arithmetic Equivalence and Isospectrality. *Lecture Notes from a Minicourse in Topics in Algebra* (18.708), Spring (2018). <https://math.mit.edu/~drew/ArithmeticEquivalenceLectureNotes.pdf>
16. Tate, J.: Global class field theory. In: Cassels, J.W.S., Fröhlich, A. (eds.) *Algebraic Number Theory*, pp. 162–203. Academic Press, London (1967)

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