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On the Robin spectrum for the hemisphere

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Dedicated to Alexander Shnirelman on the occasion of his 75-th birthday

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Abstract
We study the spectrum of the Laplacian on the hemisphere with Robin boundary conditions. It is found that the eigenvalues fall into small clusters close to the Neumann spectrum, and satisfy a Szegő type limit theorem. Sharp upper and lower bounds for the gaps between the Robin and Neumann eigenvalues are derived, showing in particular that these are unbounded. Further, it is shown that except for a systematic double multiplicity, there are no multiplicities in the spectrum as soon as the Robin parameter is positive, unlike the Neumann case which is highly degenerate. Finally, the limiting spacing distribution of the desymmetrized spectrum is proved to be the delta function at the origin.

Keywords Robin boundary conditions · Robin–Neumann gaps · Laplacian · Hemisphere · Level spacing distribution

Mathematics Subject Classification Primary 35P20 · Secondary 37D50 · 58J51 · 81Q50

Résumé.
Nous étudions le spectre du laplacien sur une hémisphère sous la condition de Robin à la frontière. Nous démontrons que les valeurs propres se regroupent en grappes près des valeurs propres de Neumann et que leur distribution dans ces grappes satisfait un théorème limite de type Szegő. Nous obtenons des bornes supérieures et inférieures optimales pour les écarts entre les valeurs propres de Robin et celles de Neumann, en particulier nous démontrons que ces écarts ne sont pas bornés. De plus, nous démontrons qu'à l'exception de valeurs propres systématiquement doubles, le spectre n'exhibe pas de multiplicité dès que le paramètre de

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Robin est strictement positif, contrairement au spectre de Neumann qui est très dégénéré. Finalement, nous démontrons que la distribution limite des écarts du spectre désymétrisé est la fonction delta supportée à l'origine.

1 Introduction

1.1 The Robin problem

Let Ω be the upper unit hemisphere (Fig. 1), with its boundary $\partial\Omega$ the equator. Our goal is to study the Robin boundary problem on the hemisphere Ω:

$$\Delta F + \lambda F = 0, \quad \frac{\partial F}{\partial n} + \sigma F = 0$$

where $\partial/\partial n$ is the derivative in the direction of the outward pointing normal to the equator, and $\sigma \geq 0$ is a constant.

The cases of Neumann and Dirichlet boundary conditions ($\sigma = 0$ or $\sigma = \infty$) are classical [2, p. 243–244]: the eigenfunctions are restrictions to Ω of the eigenfunctions on the sphere (spherical harmonics), determined by the parity under reflection in the equator: the odd spherical harmonics give the Dirichlet eigenfunctions, the even ones give the Neumann eigenfunctions. The eigenvalues are thus of the form $\ell(\ell+1)$, where $\ell \geq 0$ is an integer, repeated with multiplicity $\ell + 1$ for the Neumann case, and ℓ for the Dirichlet case.

The Robin spectrum is significantly less understood, and it is the main object of our interest. So far, the main interest has been in the asymptotics of the Robin eigenvalues in the attractive regime $\sigma \to -\infty$, or a few low lying eigenvalues as $\sigma \to \infty$, for general domains Ω. We refer the reader to the survey [4] and the references therein. Our objective is in the statistics of the high Robin eigenvalues for fixed $\sigma > 0$ and their position relatively to the Neumann spectrum, for the particular case of the hemisphere, a natural counterpart to [4, Open problem 4.7].

For the hemisphere, the eigenvalue the problem admits separation of variables, and there is a basis of eigenfunctions in the form $f_{\nu,m} = e^{im\phi}\mathrm{P}_\nu^m(\cos\theta)$, $m \in \mathbb{Z}$, where $\mathrm{P}_\nu^m(x)$ is an associated Legendre function. For each m, the admissible ν's are determined by the boundary condition. Both $f_{\nu,m}$ and $f_{\nu,-m}$ share the same Laplace eigenvalue $\nu(\nu+1)$. Therefore the Robin spectrum admits a systematic double multiplicity, and we remove it beforehand by insisting that $m \geq 0$, resulting in a "desymmetrized spectrum". Let $\lambda_n(0)$ denote the ordered

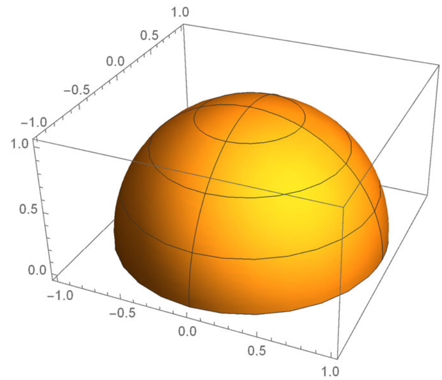

Fig. 1 The hemisphere

desymmetrized Neumann eigenvalues (repeated with appropriate multiplicity), and for $\sigma > 0$ we denote by $\lambda_n(\sigma)$ the ordered desymmetrized Robin eigenvalues, and define the Robin–Neumann (RN) gaps by

$$d_n(\sigma) := \lambda_n(\sigma) - \lambda_n(0).$$

These were recently investigated in [10] in the case of planar domains, and will be the main object of study here.

1.2 Clusters

In Sect. 3 (Theorem 3.1, Proposition 3.2) we will show that the desymmetrized Robin spectrum breaks up into small clusters $\mathcal{E}_\ell(\sigma)$ of size $\lfloor \ell/2 \rfloor + 1$, concentrated just above the Neumann eigenvalues $\ell(\ell+1)$:

For each eigenvalue $\nu(\nu+1)$, there is some $m \geq 0$, and a corresponding eigenfunction $e^{im\phi}\mathrm{P}_\nu^m(\cos\theta)$, so that the "degree" ν satisfies a secular equation

$$S_m(\nu) = \sigma, \tag{1.1}$$

where

$$S_m(\nu) = 2\tan\left(\frac{\pi(m+\nu)}{2}\right)\frac{\Gamma\left(\frac{\nu+m}{2}+1\right)\Gamma\left(\frac{\nu-m}{2}+1\right)}{\Gamma\left(\frac{\nu+m+1}{2}\right)\Gamma\left(\frac{\nu-m+1}{2}\right)}.$$

For any integer $\ell \geq m$ of the same parity ($\ell \equiv m \bmod 2$), there is a unique solution $\nu = \nu_{\ell,m}(\sigma)$ in the open interval $(\ell, \ell+1)$, and there are no other solutions, i.e. there are no solutions to (1.1) for ν in the same range with $\ell \not\equiv m \bmod 2$. Denote by $\Lambda_{\ell,m}(\sigma) = \nu_{\ell,m}(\sigma)(\nu_{\ell,m}(\sigma)+1)$ the resulting Laplace eigenvalue. Then the desymmetrized spectrum consists of $\Lambda_{\ell,m}(\sigma)$, with $0 \leq m \leq \ell$, and $m = \ell \bmod 2$, and is partitioned into disjoint clusters of size $\lfloor \ell/2 \rfloor + 1$:

$$\mathcal{E}_\ell(\sigma) = \{\Lambda_{\ell,m}(\sigma) : 0 \leq m \leq \ell, m = \ell \bmod 2\}.$$

We denote by $d_{\ell,m}(\sigma)$ the Robin–Neumann (RN) gaps in each cluster:

$$d_{\ell,m}(\sigma) = \Lambda_{\ell,m}(\sigma) - \ell(\ell+1).$$

We have an asymptotic formula:

Proposition 1.1 *Fix* $\sigma > 0$. *Let* $0 \leq m \leq \ell$, $m = \ell \bmod 2$. *If* $\ell - m \to \infty$ *then*

$$d_{\ell,m}(\sigma) \sim \frac{2\sigma}{\pi}\frac{2\ell+1}{\sqrt{\ell^2 - m^2}}. \tag{1.2}$$

An effective version of Proposition 1.1 will be proved in Sect. 5.2 below, see Corollary 5.2. We display a plot of these RN gaps in Fig. 2.

1.3 A Szego type limit theorem

We show, using (1.2), that the RN gaps from each cluster have a limiting distribution, supported on the ray $[4\sigma/\pi, \infty)$:

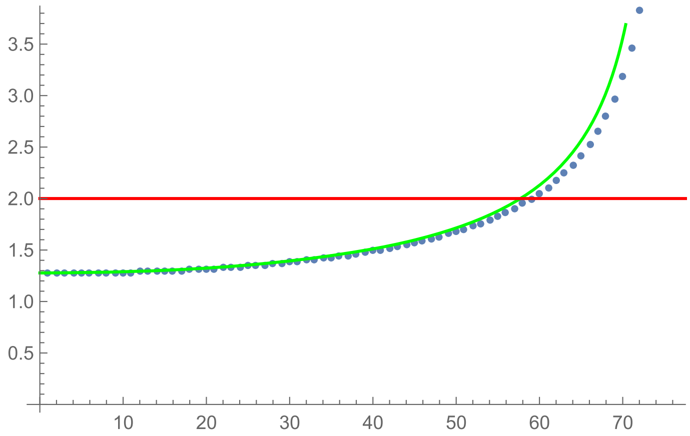

Fig. 2 The RN differences $d_{\ell,m}(\sigma)$ in the cluster $\mathcal{E}_\ell(\sigma)$ for $\ell = 150$ and $\sigma = 1$. The horizontal line (red) is their mean value 2. The solid curve (green) is the theoretical formula (1.2)

Corollary 1.2 *Fix $f \in C_c^\infty(0,\infty)$. As $\ell \to \infty$,*

$$\frac{1}{\#\mathcal{E}_\ell(\sigma)} \sum_{\lambda_n(\sigma)\in\mathcal{E}_\ell(\sigma)} f(d_n(\sigma)) = \int_{4\sigma/\pi}^{\infty} f(y) \frac{16\sigma^2 dy}{\pi^2 y^3 \sqrt{1-(\frac{4\sigma}{\pi y})^2}}.$$

Similarly, we can compute the mean value of the RN gaps within each cluster (Sect. 5.3):

$$\lim_{\ell\to\infty} \frac{1}{\#\mathcal{E}_\ell(\sigma)} \sum_{\lambda_n(\sigma)\in\mathcal{E}_\ell(\sigma)} d_n(\sigma) \sim 2\sigma. \tag{1.3}$$

Note that $2 = 2\,\mathrm{length}(\partial\Omega)/\,\mathrm{area}(\Omega)$, and the general theory[1] developed in [10] leads to (1.3) if we average over the entire spectrum. Finer than that, (1.3) asserts that for the hemisphere the same mean result holds in each cluster.

The cluster structure that we find is similar in nature to that found for the spectrum of operator $-\Delta + V$ on the unit sphere[2] S^2, for a smooth potential V [12,13]. The eigenvalues of $-\Delta + V$ fall into clusters C_ℓ of diameter $O(1)$, just above the eigenvalues $\ell(\ell+1)$ of the sphere (in our case, the clusters are bigger, of diameter $\approx \sqrt{\ell}$), and moreover the eigenvalues in each cluster C_ℓ become equidistributed with respect to a suitable measure. We observe that while for every $\sigma > 0$, the Robin eigenvalues interpolate between the Dirichlet and the Neumann ones, for a *fixed* $\sigma > 0$, *all* the Robin energies approach the corresponding Neumann energy as $\ell \to \infty$, whereas it takes a *large* σ for the Robin energies to tend to their Dirichlet counterparts, as asserted explicitly in Corollary 3.3.

1.4 RN gaps

We next examine the totality of the Robin–Neumann gaps $d_n(\sigma) := \lambda_n(\sigma) - \lambda_n(0)$.

[1] Strictly speaking, the results of [10] are only for planar domains.

[2] Similar results are available for the spectrum of the Laplace Beltrami operator on Zoll surfaces, which are spheres equipped with a Riemannian metric such that every geodesic is closed, and all geodesics have the same length.

Theorem 1.3 *There are constants* $0 < c < C$ *so that for each* $\sigma > 0$*,*

(a) *For all* n*,*

$$\lambda_n(\sigma) - \lambda_n(0) \le C\lambda_n(0)^{1/4} \cdot \sigma.$$

(b) *There are arbitrarily large* n *so that*

$$\lambda_n(\sigma) - \lambda_n(0) \ge c\lambda_n(0)^{1/4} \cdot \sigma.$$

In particular the Robin–Neumann gaps for the hemisphere are unbounded. We note that at this point, we do not know of any planar domain where the RN gaps are provably unbounded [10]. The upper bound is better than what is known for general smooth planar domains [10], which is $d_n(\sigma) \le C\lambda_n(0)^{1/3} \cdot \sigma$.

As a corollary to Theorem 1.3 we establish the limit *level spacing distribution* for the Robin spectrum, which is the distribution $P(s)$ (assuming it exists) of the nearest-neighbour gaps $\lambda_{n+1}(\sigma) - \lambda_n(\sigma)$, normalized to have mean unity (cf § 7). In the case of the Neumann spectrum on the hemisphere, most of the nearest neighbour gaps $\lambda_{n+1}(0) - \lambda_n(0)$ are zero and $P(s)$ is the delta function at the origin. We show that the Robin spectrum has the same level spacing distribution;

Corollary 1.4 *For every* $\sigma > 0$*, the level spacing distribution for the desymmetrized Robin spectrum on the hemisphere is a delta-function at the origin.*

However, unlike in the Neumann or Dirichlet case, the delta function is not a result of multiplicities, as there are none here:

Theorem 1.5 *Fix* $\sigma > 0$*. Then the desymmetrized Robin spectrum is simple:* $\lambda_m(\sigma) \ne \lambda_n(\sigma)$ *for all* $n \ne m$*.*

We note that there are few deterministic simplicity results available, unlike generic simplicity which is more common, e.g. the Dirichlet spectrum of generic triangles is simple [6]. For instance, simplicity of the desymmetrized Dirichlet spectrum on the disk was proved by Siegel in 1929 (Bourget's hypothesis) [11], and the same result holds for the Neumann spectrum [1]. However, there are arbitrarily small $\sigma > 0$ for which the Robin spectrum on the disk has multiplicities [14]. For the square, we have a result analogous to Theorem 1.5 for σ sufficiently small, but for rectangles with irrational squared aspect ratio, it fails for arbitrarily small σ [9].

Finally, we note that the theory developed here for the hemisphere is quite singular when compared to what we expect to hold for all other spherical caps. In that case we do not expect a cluster structure and moreover, we believe that the level spacing distribution will be Poissonian ($P(s) = \exp(-s)$), as is expected for most integrable systems [3,8], compare Fig. 5.

2 The Robin problem

2.1 Basics

Denote by Ω the upper hemisphere on the unit sphere, given in spherical coordinates as

$$\Omega = \left\{(\sin\theta\cos\varphi, \sin\theta\ \sin\varphi, \cos\theta) : 0 \le \phi < 2\pi,\ 0 \le \theta \le \pi/2\right\}$$

so that the north pole is at $\theta = 0$, and the equator, which is the boundary $\partial\Omega$, is at $\theta = \pi/2$.

We consider the Robin boundary problem on the hemisphere Ω:

$$\Delta F + \nu(\nu+1)F = 0, \quad \frac{\partial F}{\partial n} + \sigma F = 0$$

with $\nu > 0$, where $\partial/\partial n$ is the derivative in the direction of the outward pointing normal to the equator, and $\sigma > 0$. We will call ν the "degree", in keeping with the case of Dirichlet or Neumann boundary conditions, when the eigenfunctions are spherical harmonics of degree ℓ, with eigenvalue $\ell(\ell+1)$.

For $\sigma > 0$, all eigenvalues $\lambda = \nu(\nu+1)$ are positive, hence ν is real and $\nu > 0$ or $\nu < -1$. Since the two solutions of $\lambda = \nu(\nu+1)$ are ν and $-1-\nu$, we may assume that $\nu > 0$.

The Laplacian commutes with rotations, hence the problem admits a separation of variables, according to symmetry under rotations $\{R_\phi\}$ around the north-south pole, which defines "sectors" consisting of functions transforming as $F(R_\phi x) = e^{im\phi}F(x)$ (here $m \in \mathbb{Z}$). We write such a Robin eigenfunction as

$$F(\phi,\theta) = e^{im\phi} f_{\nu,m}(\cos\theta)$$

where $f(x)$ is a solution of ($x := \cos\theta$)

$$(1-x^2)f'' - 2xf' + \left(\nu(\nu+1) - \frac{m^2}{1-x^2}\right) f = 0. \tag{2.1}$$

The Robin boundary condition $\sigma F + \frac{\partial F}{\partial n} = 0$ is then translated to

$$\sigma f(0) - f'(0) = 0. \tag{2.2}$$

Indeed, the equator is $\theta = \pi/2$, or $x = 0$; and the normal derivative (outward normal) is

$$\frac{\partial}{\partial n}\Big|_{\theta=\pi/2} = -\frac{d}{dx}\Big|_{x=0}.$$

2.2 Desymmetrization

Since the equation (2.1) is independent of the sign of m, we see that the Robin spectrum has a systematic double multiplicity. We will remove it (desymmetrization) by insisting that $m \geq 0$. Note that this is equivalent to taking only eigenfunctions which are symmetric with respect to the reflection $(x, y, z) \mapsto (x, -y, z)$. We order the desymmetrized Neumann eigenvalues (including multiplicities) by

$$\lambda_0 = 0 < \lambda_1 = 2 < \lambda_2 = \lambda_3 = 6 < \cdots$$

2.3 The eigenfunctions

The solutions of the differential equation (2.1) which are nonsingular in $0 \leq x \leq 1$ form a one-dimensional space, all multiples of the associated Legendre functions (Ferrers functions)

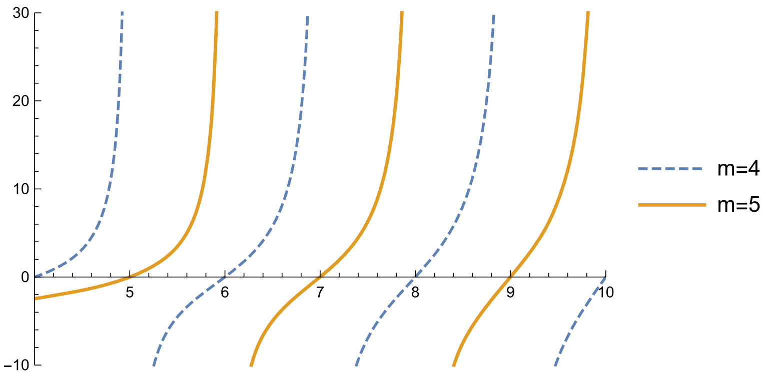

Fig. 3 $S_4(\nu)$ (dashed) and $S_5(\nu)$ (solid)

of the first kind P_ν^m [7, 14.3.4]

$$\begin{aligned}\mathrm{P}_\nu^m(x) &= (-1)^m \frac{\Gamma(\nu+m+1)}{2^m\Gamma(\nu-m+1)}(1-x^2)^{m/2}\mathrm{F}\left(\nu+m+1, m-\nu; m+1, \frac{1-x}{2}\right)\\ &= (-1)^m \frac{\Gamma(\nu+m+1)}{2^m\Gamma(\nu-m+1)}\left(\frac{1-x}{1+x}\right)^{m/2}\mathrm{F}\left(\nu+1, -\nu; m+1; \frac{1-x}{2}\right).\end{aligned} \tag{2.3}$$

Here $\mathrm{F}(a, b; c; z)$ is Olver's hypergeometric series

$$\mathrm{F}(a, b; c; z) = \sum_{s=0}^{\infty} \frac{(a)_s(b)_s}{\Gamma(c+s)s!} z^s, \quad |z| < 1$$

with $(a)_s = \Gamma(a+s)/\Gamma(a)$, so that (2.3) converges absolutely if $x \in (-1, 1]$, in particular in the range $x = \cos\theta \in [0, 1]$ which is relevant for the hemisphere.

3 The secular equation

For integer $m \geq 0$, we set

$$S_m(\nu) = 2\tan\left(\frac{\pi(m+\nu)}{2}\right)\frac{\Gamma\left(\frac{\nu+m}{2}+1\right)\Gamma\left(\frac{\nu-m}{2}+1\right)}{\Gamma\left(\frac{\nu+m+1}{2}\right)\Gamma\left(\frac{\nu-m+1}{2}\right)}. \tag{3.1}$$

Plots of $S_4(\nu)$ and $S_5(\nu)$ are displayed in Fig. 3.

Theorem 3.1 *Let* $\sigma > 0$.

(a) *For each* $m \geq 0$, *the degree* $\nu > 0$ *for which the boundary value problem* (2.1) *and* (2.2) *admits nonzero regular solutions satisfies the secular equation*

$$S_m(\nu) = \sigma.$$

(b) *The secular equation has no solutions in* $0 < \nu < m$.

Proof We saw that for all ν, there is a one-dimensional space of solutions of the ODE (2.1) which are regular for $x \in [-1, 1]$, spanned by the associated Legendre function $\mathrm{P}_\nu^m(x)$. The boundary condition (2.2) gives the secular equation

$$\frac{f'_{\nu,m}(0)}{f_{\nu,m}(0)} = \frac{\left(\frac{d\mathrm{P}_\nu^m}{dx}\right)(0)}{\mathrm{P}_\nu^m(0)} = \sigma.$$

The values at $x = 0$ of P_ν^m and its derivative are [7, Sect. 14.5 (i)]

$$\mathrm{P}_\nu^m(0) = \frac{2^m\sqrt{\pi}}{\Gamma\left(\frac{\nu-m}{2}+1\right)\Gamma\left(\frac{1-\nu-m}{2}\right)} = \frac{2^m}{\sqrt{\pi}}\cos\left(\frac{\pi(\nu+m)}{2}\right)\frac{\Gamma\left(\frac{\nu+m+1}{2}\right)}{\Gamma\left(\frac{\nu-m}{2}+1\right)}$$

and

$$\left(\frac{d\mathrm{P}_\nu^m}{dx}\right)(0) = -\frac{2^{m+1}\sqrt{\pi}}{\Gamma\left(\frac{\nu-m+1}{2}\right)\Gamma\left(-\frac{\nu+m}{2}\right)} = \frac{2^{m+1}}{\sqrt{\pi}}\sin\left(\frac{\pi(m+\nu)}{2}\right)\frac{\Gamma\left(\frac{\nu+m}{2}+1\right)}{\Gamma\left(\frac{\nu-m+1}{2}\right)}$$

and therefore

$$\frac{\left(\frac{d\mathrm{P}_\nu^m}{dx}\right)(0)}{\mathrm{P}_\nu^m(0)} = 2\tan\left(\frac{\pi(m+\nu)}{2}\right)\frac{\Gamma\left(\frac{\nu+m}{2}+1\right)\Gamma\left(\frac{\nu-m}{2}+1\right)}{\Gamma\left(\frac{\nu+m+1}{2}\right)\Gamma\left(\frac{\nu-m+1}{2}\right)}.$$

Hence we obtain the secular equation in the form $S_m(\nu) = \sigma$ with S_m as in (3.1).

We transform $S_m(\nu)$ by using Euler's reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ to convert

$$\begin{aligned}\frac{\Gamma(\frac{\nu-m}{2}+1)}{\Gamma(\frac{\nu-m+1}{2})} &= \left(\frac{\pi}{\sin(\frac{\pi(m-\nu)}{2})\Gamma(\frac{m-\nu}{2})}\right)\Big/\left(\frac{\pi}{\sin\frac{\pi(\nu-m+1)}{2}\Gamma(1-\frac{\nu-m+1}{2})}\right)\\ &= \frac{\Gamma(\frac{m-\nu+1}{2})}{\Gamma(\frac{m-\nu}{2})}\cdot\frac{\cos(\pi\frac{m-\nu}{2})}{\sin(\pi\frac{m-\nu}{2})} = \frac{\Gamma(\frac{m-\nu+1}{2})}{\Gamma(\frac{m-\nu}{2})}\cot\left(\pi\frac{m-\nu}{2}\right).\end{aligned}$$

Moreover, for integer m,

$$\tan\left(\frac{\pi(m+\nu)}{2}\right)\cdot\cot\left(\frac{\pi(m-\nu)}{2}\right) = -1.$$

Thus we obtain

$$S_m(\nu) = -2\frac{\Gamma(\frac{\nu+m}{2}+1)\Gamma(\frac{m-\nu+1}{2})}{\Gamma(\frac{m+\nu+1}{2})\Gamma(\frac{m-\nu}{2})}. \tag{3.2}$$

The expression (3.2) allows us to check that if $m \geq 1$, there is no solution for the secular equation if $0 < \nu < m$ (recall $\sigma > 0$), because the arguments of all the Gamma functions on the r.h.s. of (3.2) are positive if $0 < \nu < m$, hence so are the Gamma functions. Therefore $S_m(\nu)$ is *negative* for $\nu < m$. Thus for $0 < \nu < m$ there is no solution of the secular equation if $\sigma > 0$. □

Proposition 3.2 *Fix $\sigma > 0$. Then*

(a) *S_m vanishes at the points $m+2k$, with $k \geq 0$ integer, tends to infinity as $\nu \nearrow m+2k+1$, and $S_m(\nu)$ is negative for $m+2k-1 < \nu < m+2k$, positive in $m+2k < \nu < m+2k+1$ and increasing for $m+2k-1 < \nu < m+2k+1$.*
(b) *Let $\ell = m+2k$ with integer $k = 0, 1, 2, \ldots$. Then there is a unique solution $\nu_{\ell,m}(\sigma) \in (\ell, \ell+1)$ of the secular equation.*

(c) *Write* $\nu_{\ell,m}(\sigma) = \ell + \delta_{\ell,m}(\sigma)$*, with* $\delta = \delta_{\ell,m}(\sigma) \in (0,1)$*. Then*

$$\delta < \frac{\sqrt{\frac{2}{\pi}}\sigma}{\sqrt{\nu}}. \tag{3.3}$$

Proof We use S_m in the form

$$S_m(\nu) = 2\tan\left(\frac{\pi(m+\nu)}{2}\right) G(\nu+m)G(\nu-m)$$

where

$$G(s) := \frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{s+1}{2})}.$$

Note that $G(s)$ is positive for $s > 0$. We have for $s > 0$,

$$G'(s) = \frac{1}{2}G(s)\left(\psi\left(\frac{s}{2}+1\right) - \psi\left(\frac{s}{2}+\frac{1}{2}\right)\right)$$

with ψ the digamma function [7, 5.9.16]

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)} = -\gamma + \int_0^1 \frac{1-t^{s-1}}{1-t}dt, \quad \Re(s) > 0$$

so that

$$\frac{G'(s)}{G(s)} = \frac{1}{2}\int_0^1 \frac{(1-t^{s/2}) - (1-t^{(s-1)/2})}{1-t}dt = \frac{1}{2}\int_0^1 \frac{t^{(s-1)/2}}{1+\sqrt{t}}dt$$

is clearly positive for $s > 0$. Since $G(s) > 0$ we deduce that $G'(s) > 0$ for $s > 0$, so that $G(s)$ is increasing, and

$$0 < \frac{G'(s)}{G(s)} < \frac{1}{2}. \tag{3.4}$$

The function $S_m(\nu)$ is positive for $m + 2k < \nu < m + 2k + 1$ because both $G(\nu \pm m)$ are positive for $\nu > m$, and writing $\nu = m + 2k + \delta$ gives $\tan\frac{\pi}{2}(m+\nu) = \tan\frac{\pi}{2}\delta$ which is positive for $\delta \in (0,1)$, and negative for $\delta \in (-1,0)$.

The logarithmic derivative of S_m is

$$\frac{S_m'}{S_m}(\nu) = \frac{\pi}{\sin\pi\delta} + \frac{G'}{G}(\nu-m) + \frac{G'}{G}(\nu+m). \tag{3.5}$$

Since $G'/G > 0$, we find that if $\delta \in (0,1)$ then $S_m'/S_m(\nu) > 0$ and since $S_m(\nu) > 0$ for all $\upsilon > m$ we obtain that $S_m'(\nu) > 0$ for $\nu \in (m+2k, m+2k+1)$, so that S_m is increasing there. Otherwise, if $\nu \in (m+2k-1, m+2k)$, then $\delta \in (-1,0)$, and we already know that here $S_m(\nu) < 0$. Then, since in this range $\frac{\pi}{\sin\pi\delta} < -\pi$, the inequality (3.4) shows, with the use of the triangle inequality, that the r.h.s. of (3.5) is

$$\frac{\pi}{\sin\pi\delta} + \frac{G'}{G}(\nu-m) + \frac{G'}{G}(\nu+m) < -\pi + \frac{1}{2} + \frac{1}{2} < 0,$$

and so is the l.h.s. of (3.5), and then $S_m'(\nu) > 0$.

Since $G(s)$ is positive and increasing, for $\nu > m$ we get

$$G(\nu-m) \geq G(0) = \frac{1}{\sqrt{\pi}}.$$

By Stirling's formula $G(s) \sim \sqrt{\frac{s}{2}} + O(1/\sqrt{s})$ as $s \to \infty$, in fact [7, 5.6.4]

$$\sqrt{\frac{s}{2}} < G(s) < \sqrt{\frac{s}{2}+1}, \quad s > 0. \tag{3.6}$$

Also note

$$\tan\left(\frac{\pi(m+\nu)}{2}\right) = \tan\pi\left(m+k+\frac{\delta}{2}\right) = \tan\frac{\pi\delta}{2} \geq \frac{\pi\delta}{2}.$$

We obtain

$$\sigma = 2\tan\left(\frac{\pi(m+\nu)}{2}\right)G(\nu+m)G(\nu-m) > 2\frac{\pi\delta}{2}\sqrt{\frac{\nu+m}{2}}G(0) \geq \delta\sqrt{\frac{\pi}{2}}\sqrt{\nu}$$

so that $\delta < \sqrt{\frac{2}{\pi}}\sigma/\sqrt{\nu}$. □

Corollary 3.3 *Fix $\sigma > 0$. For $\ell \geq m \geq 0$, $\ell = m \bmod 2$, let $\nu = \nu_{\ell,m}(\sigma)$ be the unique solution of the secular equation $S_m(\nu) = \sigma$ with $\nu \in (\ell, \ell+1)$.*

Write $\nu = \ell + \delta$, with $\delta \in (0, 1)$. Then

a. *As $\sigma \to 0$, $\delta \to 0$,*
b. *As $\sigma \to \infty$, we have $\delta \to 1$.*

Consequently, as $\sigma \to 0$, $\nu \to \ell$, while as $\sigma \to \infty$, $\nu \to \ell+1$. Thus, as σ varies between 0 and $+\infty$, $\Lambda_{\ell,m}(\sigma) := \nu_{\ell,m}(\sigma)\cdot(\nu_{\ell,m}(\sigma)+1)$ interpolates between a Neumann eigenvalue $\ell(\ell+1)$ with ℓ of the same parity as m, and a Dirichlet eigenvalue $(\ell+1)(\ell+2)$ with same m and opposite parity between ℓ and m.

Proof That $\delta \to 0$ as $\sigma \to 0$ follows from (3.3). Using monotonicity of $G(s)$ we obtain

$$\sigma = S_m(\nu) \leq 2\tan\frac{\pi\delta}{2}G(2m+2k+1)G(2k+1) \ll_{m,k} \tan\frac{\pi\delta}{2}$$

so that as $\sigma \to \infty$, we have $\delta \to 1$. □

4 Multiplicity one

We have seen (Theorem 3.1) that the desymmetrized Robin spectrum of the hemisphere is given by the energies

$$\Lambda_{\ell,m}(\sigma) = \nu_{\ell,m}(\sigma)\cdot(\nu_{\ell,m}(\sigma)+1) \tag{4.1}$$

with $\ell \geq 0$, and $0 \leq m \leq \ell$ satisfying $m \equiv \ell \mod 2$, satisfying the secular equation $S_m(\nu) = \sigma$, with S_m given by (3.1):

$$S_m(\nu) = 2\tan\left(\frac{\pi(m+\nu)}{2}\right)\frac{\Gamma(\frac{\nu+m}{2}+1)\Gamma(\frac{\nu-m}{2}+1)}{\Gamma(\frac{\nu+m+1}{2})\Gamma(\frac{\nu-m+1}{2})}.$$

To show that there are no degeneracies in the desymmetrized spectrum (Theorem 1.5), it therefore suffices to prove:

Proposition 4.1 *Fix $\sigma > 0$. For all $\ell \geq 2$ and $0 \leq m \leq \ell - 2$ with $m \equiv \ell \bmod 2$,*

$$\nu_{\ell,m+2}(\sigma) > \nu_{\ell,m}(\sigma).$$

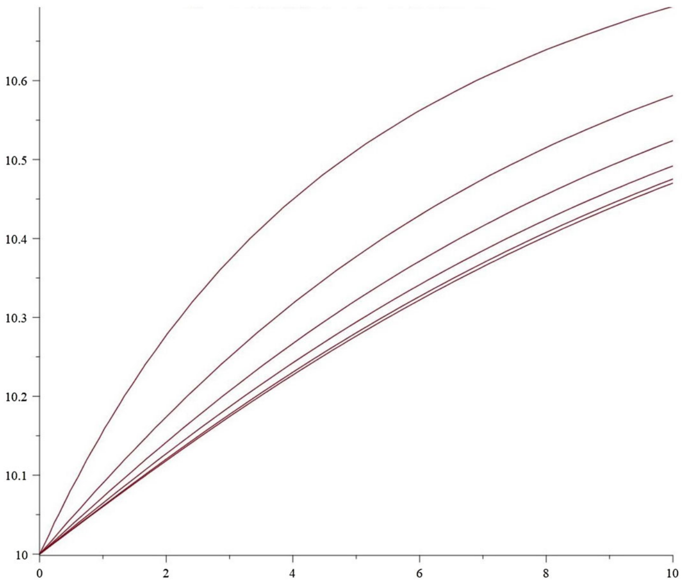

Fig. 4 Plots of $\nu_{10,m}$, $m = 0, 2, 4, 6, 8, 10$ on $[0, 10]$. As asserted by Proposition 4.1, higher curves correspond to larger value of m

The picture emerging for $\nu_{10,m}(\sigma)$ on $[0, 10]$, with all possible $0 \leq m \leq 10$, $m \equiv \ell \mod 2$, is displayed within Fig. 4. This clearly supports the statement of Proposition 4.1.

Proof Recall that $\nu_{\ell,m}(\sigma) \in (\ell, \ell + 1)$, and that $\ell = m + 2k$, $k \geq 0$. By Proposition 3.2, both $S_m(\nu)$ and $S_{m+2}(\nu)$ are increasing and positive in $(\ell, \ell + 1)$. Using the recurrence $\Gamma(s + 1) = s\Gamma(s)$ we find

$$\frac{S_{m+2}(\nu)}{S_m(\nu)} = \frac{\frac{\nu+m}{2} + 1 \cdot \frac{\nu-m-1}{2}}{\frac{\nu+m+1}{2} \cdot \frac{\nu-m}{2}} = 1 - \frac{2(m+1)}{(\nu - m)(\nu + m + 1)} < 1.$$

Hence for $\nu \in (\ell, \ell + 1)$, where both $S_m(\nu)$ and $S_{m+2}(\nu)$ are positive, we must have $S_{m+2}(\nu) < S_m(\nu)$. Therefore

$$S_{m+2}(\nu_{\ell,m}(\sigma)) < S_m(\nu_{\ell,m}(\sigma)) = \sigma = S_{m+2}(\nu_{\ell,m+2}(\sigma)).$$

Since S_{m+2} is increasing in $(\ell, \ell + 1)$, we deduce that $\nu_{\ell,m}(\sigma) < \nu_{\ell,m+2}(\sigma)$ as claimed. □

5 Clusters and a Szegő type limit theorem

5.1 Cluster structure

Denote the cluster (a multiset) of desymmetrized multiple Neumann eigenvalues sharing a common value of $\ell(\ell + 1)$ by

$$\mathcal{E}_\ell(0) = \Big\{\ell(\ell + 1) : 0 \leq m \leq \ell, m = \ell \mod 2\Big\}.$$

This cluster has size $\#\mathcal{E}_\ell(0) = \lfloor \ell/2 \rfloor + 1$. We label the eigenvalues there by

$$\mathcal{E}_\ell(0) = \left\{ \lambda_L, \lambda_{L+1}, \ldots, \lambda_{L+\lfloor \ell/2 \rfloor} \right\}$$

where $L = L_\ell$ is given by

$$L = \#\left(\mathcal{E}_0(0) \cup \mathcal{E}_1(0) \cup \cdots \cup \mathcal{E}_{\ell-1}(0) \right) = \sum_{\ell'=0}^{\ell-1} \left\lfloor \frac{\ell'}{2} \right\rfloor + 1 = \frac{\ell^2}{4} + O(\ell).$$

The distance of the Neumann eigenvalue cluster $\mathcal{E}_\ell(0)$ to the closest other Neumann eigenvalue cluster, which for $\ell \geq 1$ is $\mathcal{E}_{\ell-1}(0)$ (in other words, the distance between distinct nearby Neumann eigenvalues), is

$$\min_{\ell':\ell' \neq \ell} \operatorname{dist}\left(\mathcal{E}_\ell(0), \mathcal{E}_{\ell'}(0) \right) = \ell(\ell+1) - (\ell-1)\ell = 2\ell. \tag{5.1}$$

We saw that the Robin eigenvalues are $\nu(\nu+1)$ where $\nu = \nu_{\ell,m}(\sigma) \in (\ell, \ell+1)$, $\ell = m \bmod 2$, is a solution of the secular equation $S_m(\nu) = \sigma$. Denote by

$$\mathcal{E}_\ell(\sigma) = \{ \Lambda_{\ell,m}(\sigma) : \ell \geq m \geq 0, \ell = m \bmod 2 \} \tag{5.2}$$

which is the evolution of the Neumann eigenvalue cluster $\mathcal{E}_\ell(0)$. Since $\ell < \nu_{\ell,m}(\sigma) < \ell+1$, the spectral cluster $\mathcal{E}_\ell(\sigma)$ is contained in the open interval $\left(\ell(\ell+1), (\ell+1)(\ell+2) \right)$, and in particular the evolved eigenvalue clusters $\mathcal{E}_\ell(\sigma)$ do not mix with each other.

5.2 Asymptotics of the Robin–Neumann gaps

Recall that we write $\nu_{\ell,m}(\sigma) = \ell + \delta_{\ell,m}(\sigma)$.

Lemma 5.1 *For fixed $\sigma > 0$, as $\ell \to \infty$, with $0 \leq m < \ell$, $\ell = m \bmod 2$,*

$$\delta_{\ell,m}(\sigma) = \frac{2\sigma}{\pi \sqrt{\ell^2 - m^2}} \left(1 + O\left(\frac{1}{\ell - m} \right) \right). \tag{5.3}$$

For $m = \ell$, we have

$$\delta_{\ell,\ell}(\sigma) \sim \frac{\sigma}{\sqrt{\pi \ell}}. \tag{5.4}$$

Proof For $0 < \ell - m = O(1)$, (5.3) is just the upper bound (3.3), so assume $\ell - m \to \infty$. The cluster $\mathcal{E}_\ell(\sigma)$ consists of $\lfloor \ell/2 \rfloor + 1$ eigenvalues $\Lambda_{\ell,m}(\sigma) = \nu_{\ell,m}(\nu_{\ell,m} + 1)$ with $m + 2k = \ell$, $m, k \geq 0$, and where $\nu_{\ell,m}(\sigma)$ is the unique solution of the secular equation $S_m(\nu) = \sigma$ in the interval $(\ell, \ell+1)$. We write

$$\nu = \nu_{\ell,m}(\sigma) = \ell + \delta = m + 2k + \delta, \qquad \delta = \delta_{\ell,m}(\sigma).$$

Recall that the S_m of the secular equation $S_m(\nu) = \sigma$ is given by

$$S_m(\nu) = 2 \tan\left(\frac{\pi(m+\nu)}{2} \right) G(\nu+m) G(\nu-m) \tag{5.5}$$

where $G(s) = \Gamma(\frac{s}{2}+1)/\Gamma(\frac{s+1}{2})$ satisfies (cf. (3.1))

$$G(s) = \sqrt{\frac{s}{2}} \left(1 + O\left(\frac{1}{s} \right) \right), \quad s \to \infty.$$

Since we assume that $\ell - m = 2k \to \infty$, both arguments of G in (5.5) tend to infinity, because $\nu + m = 2k + 2m + \delta = \ell + m + \delta$ and $\nu - m = 2k + \delta = \ell - m + \delta$. Moreover,

$$\tan \frac{\pi}{2}(\nu + m) = \tan \frac{\pi}{2}\delta$$

and we know (Proposition 3.2) that

$$\delta \ll \sigma/\sqrt{\ell} \to 0 \tag{5.6}$$

so that

$$\tan \frac{\pi}{2}(\nu + m) = \tan \frac{\pi}{2}\delta = \frac{\pi}{2}\delta + O\left(\frac{1}{\ell^{3/2}}\right).$$

Therefore we can write

$$\begin{aligned} S_m(\nu) &= 2\frac{\pi}{2}\delta\left(1 + O\left(\frac{1}{\ell}\right)\right) \cdot \sqrt{\frac{\nu - m}{2}}\left(1 + O\left(\frac{1}{\ell - m}\right)\right) \\ &\quad \cdot \sqrt{\frac{\nu + m}{2}}\left(1 + O\left(\frac{1}{\ell + m}\right)\right) \\ &= \pi\delta \cdot \sqrt{k + \frac{\delta}{2}}\sqrt{k + m + \frac{\delta}{2}}\left(1 + O\left(\frac{1}{\ell - m}\right)\right). \end{aligned} \tag{5.7}$$

Furthermore, since $2k = \ell - m$,

$$\sqrt{k + \frac{\delta}{2}} = \sqrt{k}\left(1 + O\left(\frac{\delta}{\ell - m}\right)\right) = \sqrt{\frac{\ell - m}{2}}\left(1 + O\left(\frac{1}{\sqrt{\ell}(\ell - m)}\right)\right)$$

and likewise since $k + m = (\ell + m)/2$

$$\sqrt{k + m + \frac{\delta}{2}} = \sqrt{\frac{\ell + m}{2}}\left(1 + O\left(\frac{1}{\ell^{3/2}}\right)\right).$$

Inserting (5.7) into the secular equation $S_m(\nu) = \sigma$ gives, when $\ell - m \to \infty$, that

$$\delta_{\ell,m}(\sigma) = \frac{2\sigma}{\pi\sqrt{\ell^2 - m^2}}\left(1 + O\left(\frac{1}{\ell - m}\right)\right).$$

When $m = \ell$, we use $\delta = \delta_{\ell,\ell}(\sigma) \ll 1/\sqrt{\ell} \to 0$ and $G(0) = 1/\sqrt{\pi}$ to obtain

$$\sigma = S_\ell(\sigma) = 2\tan\left(\frac{\pi}{2}\delta\right)G(2\ell + \delta)G(\delta) \sim \pi\delta G(2\ell)G(0) \sim \pi\delta\sqrt{\ell}\frac{1}{\sqrt{\pi}}$$

as $\ell \to \infty$, which gives (5.4). □

We derive an asymptotic for the RN gaps $d_{\ell,m}(\sigma) = \Lambda_{\ell,m}(\sigma) - \ell(\ell(+1)$ in each cluster, an effective version of Proposition 1.1:

Corollary 5.2 *For fixed $\sigma > 0$, as $\ell \to \infty$, for all $0 \le m < \ell$ with $m = \ell \bmod 2$, the Robin–Neumann gaps satisfy*

$$d_{\ell,m}(\sigma) = \frac{2\sigma}{\pi} \cdot \frac{2\ell + 1}{\sqrt{\ell^2 - m^2}} + O\left(\frac{\sqrt{\ell}}{(\ell - m)^{3/2}}\right). \tag{5.8}$$

For $m = \ell$ we have

$$d_{\ell,\ell}(\sigma) \sim \frac{2\sigma}{\sqrt{\pi}}\sqrt{\ell}. \tag{5.9}$$

Proof We have

$$d_{\ell,m}(\sigma) = \Lambda_{\ell,m}(\sigma) - \Lambda_{\ell,m}(0) = (\nu-\ell)(\nu+\ell+1) = \delta_{\ell,m}(2\ell+1+\delta_{\ell,m})$$
$$= (2\ell+1)\delta_{\ell,m} + \delta_{\ell,m}^2 = (2\ell+1)\delta_{\ell,m} + O\left(\frac{1}{\ell}\right)$$

where we have used (5.6). Moreover, for $m < \ell$ we have the asymptotic formula (5.3) for $\delta_{\ell,m}$, and hence

$$d_{\ell,m}(\sigma) = \frac{2(2\ell+1)\sigma}{\pi\sqrt{\ell^2-m^2}}\left(1+O\left(\frac{1}{\ell-m}\right)\right) + O\left(\frac{1}{\ell}\right)$$
$$= \frac{2(2\ell+1)\sigma}{\pi\sqrt{\ell^2-m^2}} + O\left(\frac{\sqrt{\ell}}{(\ell-m)^{3/2}} + \frac{1}{\ell}\right)$$
$$= \frac{2(2\ell+1)\sigma}{\pi\sqrt{\ell^2-m^2}} + O\left(\frac{\sqrt{\ell}}{(\ell-m)^{3/2}}\right).$$

For the case $m = \ell$, (5.9) similarly follows from (5.4). □

5.3 Equidistribution of gaps in the cluster

We can now deduce the equidistribution of gaps in each cluster (Corollary 1.2) and compute the average gap in a cluster as asserted in (1.3). The average gap formally corresponds to the test function $f(x) = x$, which is unbounded. In both cases the arguments are similar, except that for the case of Corollary 1.2, where we use a compactly supported smooth test function f, the derivatives of f are invoked to control the error terms. In what follows we only give the detailed proof for the average gap.

Corollary 5.3 *For fixed* $\sigma > 0$, *as* $\ell \to \infty$,

$$\frac{1}{\#\mathcal{E}_\ell(\sigma)} \sum_{\lambda_n(\sigma)\in\mathcal{E}_\ell(\sigma)} d_n(\sigma) \sim 2\sigma.$$

Proof Using $d_{\ell,m} = (2\ell+1)\delta_{\ell,m} + \delta_{\ell,m}^2 \ll \sqrt{\ell}$ by (3.3), we see that we may restrict the average to $m \le \ell - 1$ with an error of $O(\ell^{-1/2})$:

$$\frac{1}{\#\mathcal{E}_\ell(\sigma)} \sum_{\lambda_n(\sigma)\in\mathcal{E}_\ell(\sigma)} d_n(\sigma) = \frac{1}{\ell/2+O(1)} \sum_{\substack{0\le m\le \ell-1 \\ m=\ell \bmod 2}} d_{\ell,m} + O(\ell^{-1/2}).$$

Then we use (5.8) to obtain

$$\frac{1}{\ell/2+O(1)} \sum_{\substack{0\le m\le \ell-1 \\ m=\ell \bmod 2}} d_{\ell,m} = \frac{1}{\ell/2} \sum_{\substack{0\le m\le \ell-1 \\ m=\ell \bmod 2}} \frac{2(2\ell+1)\sigma}{\pi\sqrt{\ell^2-m^2}} + O\left(\frac{1}{\ell^{1/2}}\right).$$

Moreover, using standard bounds for the rate of convergence of Riemann sums gives

$$\frac{1}{\ell/2} \sum_{\substack{0\le m\le \ell-1 \\ m=\ell \bmod 2}} \frac{2(2\ell+1)\sigma}{\pi\sqrt{\ell^2-m^2}} = \left(\frac{4\sigma}{\pi} + O\left(\frac{1}{\ell}\right)\right)\left(\int_0^1 \frac{dx}{\sqrt{1-x^2}} + O\left(\frac{1}{\ell^{1/2}}\right)\right)$$
$$= 2\sigma + O\left(\frac{1}{\ell^{1/2}}\right).$$

Altogether, we obtain

$$\frac{1}{\#\mathcal{E}_\ell(\sigma)}\sum_{\lambda_n(\sigma)\in\mathcal{E}_\ell(\sigma)} d_n(\sigma) = 2\sigma + O\left(\frac{1}{\ell^{1/2}}\right) \sim 2\sigma$$

as claimed. □

6 Bounds for the RN gaps: Proof of Theorem 1.3

Proof Using (3.3) shows that for $\ell \gg 1$ and $0 \le m \le \ell$ with $m \equiv \ell \mod 2$,

$$\Lambda_{\ell,m}(\sigma) - \ell(\ell+1) = (\nu_{l,m}(\sigma) - \ell)(\nu_{\ell,m}(\sigma) + \ell + 1) \ll \sigma\sqrt{\ell}$$

so that

$$\max\left\{|\lambda - \ell(\ell+1)| : \lambda \in \mathcal{E}_\ell(\sigma)\right\} \ll \sigma\sqrt{\ell}.$$

Therefore, for all n, we have

$$\lambda_n(\sigma) - \lambda_n(0) \ll \sigma\lambda_n(0)^{1/4}. \tag{6.1}$$

This proves Theorem 1.3(a).

To show that we can actually attain the upper bound in (6.1), note that Proposition 4.1 demonstrates that to get the largest possible Robin–Neumann gaps, it is worth, given $\ell \ge 0$, to take $m = \ell$. We then use (5.9) to obtain

$$d_{\ell,\ell}(\sigma) \sim \frac{2\sigma}{\sqrt{\pi}}\sqrt{\ell} \sim \frac{2}{\sqrt{\pi}}\Lambda_{\ell,\ell}(0)^{1/4}\sigma,$$

which proves Theorem 1.3(b). □

We note that $\Lambda_{\ell,\ell}(0) \in \mathcal{E}_\ell(0)$, and therefore for each $\ell \gg 1$, we have found $n = \ell^2/4 + O(\ell)$ for which

$$\lambda_n(\sigma) - \lambda_n(0) \gg \lambda_n(0)^{1/4} \cdot \sigma,$$

and in particular that the Robin–Neumann gaps are unbounded.

7 Level spacings

In this section, we show that the level spacing distribution of the desymmetrized Robin spectrum on the hemisphere is a delta function at the origin, as is the case with Neumann or Dirichlet boundary conditions. We note that for other spherical caps (cf [5] for background), we expect that the level spacing distribution is Poissonian. A numerical plot for the desymmetrized Dirichlet spectrum on the cap with opening angle $\theta_0 = \pi/3$ (the hemisphere has $\theta_0 = \pi/2$) is displayed in Fig. 5.

Proof of Corollary 1.4 The statement of Corollary 1.4 is equivalent to the fact that for every $y > 0$,

$$\lim_{N\to\infty}\frac{1}{N}\#\{n \le N : \lambda_{n+1}^\sigma - \lambda_n^\sigma > y\} = 0. \tag{7.1}$$

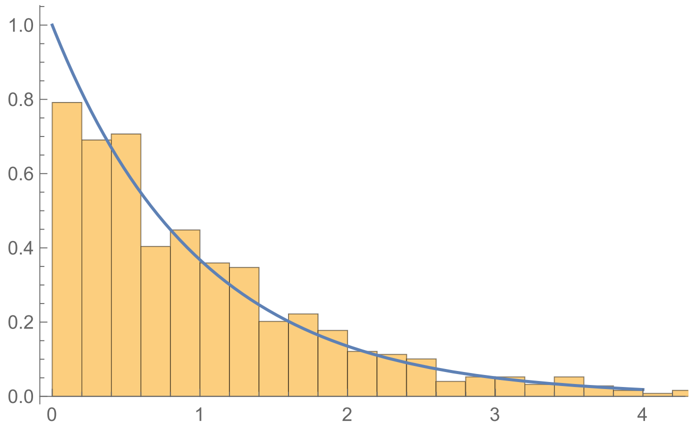

Fig. 5 The level spacing distribution $P(s)$ for all 1258 desymmetrized Dirichlet eigenvalues $\nu(\nu+1)$ with $\nu < 100$ for the spherical cap with opening angle $\theta_0 = \pi/3$. The solid curve is the Poisson result $\exp(-s)$

Recall that we divided the ordered desymmetrized Robin eigenvalues $\{\lambda_n^\sigma\}_{n\geq 0}$ into disjoint clusters $\mathcal{E}_\ell(\sigma)$ (see (5.2)), each at distance $O(\sqrt{\ell})$ from the Neumann eigenvalues $\ell(\ell+1)$, so diam $\mathcal{E}_\ell(\sigma) \ll \sqrt{\ell}$ (Theorem 1.3(a)), and hence of distance $2\ell + O(\sqrt{\ell})$ from the closest other cluster, and of size $\#\mathcal{E}_\ell(\sigma) = \lfloor \ell/2 \rfloor + 1 = \ell/2 + O(1)$.

For $N \gg 1$, denote by L the index of the cluster to which $\lambda_N(\sigma)$ belongs, so that

$$\bigcup_{\ell\leq L-1} \mathcal{E}_\ell(\sigma) \subset \{\lambda_n(\sigma) : n \leq N\} \subseteq \bigcup_{\ell \leq L} \mathcal{E}_\ell(\sigma)$$

and therefore

$$N = \sum_{\ell\leq L-1} \#\mathcal{E}_\ell(\sigma) + O(L) = \frac{L^2}{4} + O(L)$$

so that $L = O(\sqrt{N})$. Then

$$\#\{n \leq N : \lambda_{n+1}^\sigma - \lambda_n^\sigma > y\} \leq \sum_{\ell=0}^{L} \sum_{\substack{\lambda_{n+1}^\sigma - \lambda_n^\sigma > y \\ \lambda_n^\sigma \in \mathcal{E}_\ell(\sigma)}} 1. \tag{7.2}$$

Denote by n_+ the maximal index of an eigenvalue in $\mathcal{E}_\ell(\sigma)$, and by n_- the minimal index. Then the gaps corresponding to the cluster $\mathcal{E}_\ell(\sigma)$ are firstly those with $\lambda_{n+1}(\sigma) - \lambda_n(\sigma)$ with $n_- \leq n \leq n_+ - 1$ and secondly, the last gap $\lambda_{n_++1}(\sigma) - \lambda_{n_+}(\sigma)$. The number of those gaps of the second kind is at most $L + 1 = O(\sqrt{N})$.

For the gaps $> y$ of the first kind, we have in each cluster

$$\begin{aligned}\sum_{\substack{\lambda_{n+1}^\sigma - \lambda_n^\sigma > y \\ \lambda_n^\sigma \in \mathcal{E}_\ell(\sigma) \\ n<n_+}} 1 &< \sum_{\substack{\lambda_{n+1}^\sigma - \lambda_n^\sigma > y \\ n_- \leq n < n_+}} \frac{\lambda_{n+1}(\sigma) - \lambda_n(\sigma)}{y} \\ &\leq \sum_{n_-\leq n<n_+} \frac{\lambda_{n+1}(\sigma) - \lambda_n(\sigma)}{y} = \frac{\lambda_{n_+} - \lambda_{n_-}}{y}.\end{aligned}$$

Now $\lambda_{n_+} - \lambda_{n_-} = \operatorname{diam} \mathcal{E}_\ell(\sigma) \ll \sqrt{\ell}$, and so we find that

$$\sum_{\substack{\lambda^\sigma_{n+1} - \lambda^\sigma_n > y \\ \lambda^\sigma_n \in \mathcal{E}_\ell(\sigma) \\ n < n_+}} 1 \ll \frac{\sqrt{\ell}}{y}. \tag{7.3}$$

Summing the inequality (7.3) over $\ell \leq L = O(\sqrt{N})$ gives

$$\sum_{\ell=0}^{L} \sum_{\substack{\lambda^\sigma_{n+1} - \lambda^\sigma_n > y \\ \lambda^\sigma_n \in \mathcal{E}_\ell(\sigma) \\ n < n_+}} 1 \ll \sum_{\ell \leq L} \frac{\sqrt{\ell}}{y} \ll \frac{L^{3/2}}{y} \ll \frac{N^{3/4}}{y}.$$

Altogether, substituting this into (7.2), and upon taking into account the gaps of the second kind, we find that for $N \gg_y 1$,

$$\#\{n \leq N : \lambda^\sigma_{n+1} - \lambda^\sigma_n > y\} \ll \frac{N^{3/4}}{y} + \sqrt{N},$$

which proves (7.1). □

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