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2003 J. Phys. A: Math. Gen. 36 2919

(http://iopscience.iop.org/0305-4470/36/12/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 2919–2932 PII: S0305-4470(03)40052-8

Mock-Gaussian behaviour for linear statistics of classical compact groups

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Received 29 July 2002, in final form 25 November 2002
Published 12 March 2003
Online at stacks.iop.org/JPhysA/36/2919

Abstract
We consider the scaling limit of linear statistics for eigenphases of a matrix taken from one of the classical compact groups. We compute their moments and find that the first few moments are Gaussian, whereas the limiting distribution is not. The precise number of Gaussian moments depends upon the particular statistic considered.

PACS number: 02.20.Hj

1. Introduction

In this paper, we investigate the scaling limit of linear statistics for eigenphases of matrices in the classical groups. Given a unitary $N \times N$ matrix U with eigenvalues $e^{i\theta_n}$, $1 \leqslant n \leqslant N$, and a test function g which we assume is 2π-periodic, consider the linear statistic

$$\operatorname{Tr} g(U) := \sum_{n=1}^{N} g(\theta_n).$$

A number of authors have studied the limiting distribution as $N \to \infty$ of $\operatorname{Tr} g(U)$ as U varies over a family $G(N)$ of classical groups and have concluded that the distribution is Gaussian, see [1, 2, 4].

Soshnikov [9] showed that this result remains valid in the 'mesoscopic' regime, that is if one considers eigenphases θ_n in an interval of length about $1/L$ where $L = L_N \to \infty$ but $L/N \to 0$: for a Schwartz function f on the real line, define

$$F_L(\theta) := \sum_{j=-\infty}^{\infty} f\left(\frac{L}{2\pi}(\theta + 2\pi j)\right)$$

which is 2π-periodic and localized on a scale of $1/L$. Soshnikov [9] showed that as long as $L/N \to 0$, then the limiting distribution of $\mathrm{Tr}\, F_L(U)$ as U ranges over all unitary matrices in $\mathrm{U}(N)$, $N \to \infty$ is a Gaussian with mean

$$\frac{N}{L}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

and variance

$$\int_{-\infty}^{\infty} \hat{f}(t)^2|t|\,\mathrm{d}t$$

where the Fourier transform is defined as

$$\hat{f}(t) := \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}xt}\,\mathrm{d}x.$$

There are similar formulae for the other classical groups.

Our goal is to investigate these linear statistics in the *scaling limit*, that is to take $L = N$. Thus we set

$$Z_f(U) := \mathrm{Tr}\, F_N(U) = \sum_{n=1}^{N} F_N(\theta_n).$$

In [3] we proved

Theorem 1. *If* $\mathrm{supp}\hat{f} \subseteq [-2/m, 2/m]$ *then the first m moments of* $Z_f(U)$ *over the unitary group* $\mathrm{U}(N)$ *converge as* $N \to \infty$ *to the Gaussian moments with mean* $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ *and variance*

$$\int_{-\infty}^{\infty} \min(|u|, 1)|\hat{f}(u)|^2\,\mathrm{d}u.$$

We called this a 'mock-Gaussian' behaviour. It is worth remarking that in [3] we find the full distribution of Z_f, and it is not Gaussian; only the first few moments are.

The purpose of this paper is to demonstrate mock-Gaussian behaviour for linear statistics in other classical compact groups, the special orthogonal group $\mathrm{SO}(N)$ and the symplectic group $\mathrm{Sp}(N)$ (N must be even in the symplectic group). If $\mathrm{e}^{\mathrm{i}\theta}$ is an eigenvalue of a matrix U taken from one of these groups then $\mathrm{e}^{-\mathrm{i}\theta}$ is also an eigenvalue. This means 1 is always an eigenvalue of $U \in \mathrm{SO}(N)$ if N is odd.

Due to the pairing of eigenvalues, the function f must be even. Our results are

Theorem 2. *(i) If* $\mathrm{supp}\hat{f} \subseteq [-1/m, 1/m]$ *then the first m moments of* $Z_f(U)$ *over the symplectic group* $\mathrm{Sp}(N)$ *converge to the Gaussian moments with mean*

$$\hat{f}(0) - \int_0^1 \hat{f}(u)\,\mathrm{d}u$$

and variance

$$2\int_{-1/2}^{1/2} |u|\hat{f}(u)^2\,\mathrm{d}u.$$

(ii) If $\mathrm{supp}\hat{f} \subseteq (-1/m, 1/m)$ *then the first m moments of* $Z_f(U)$ *over the special orthogonal group* $U \in \mathrm{SO}(N)$ *converge to the Gaussian moments with mean*

$$\hat{f}(0) + \int_0^1 \hat{f}(u)\,\mathrm{d}u$$

and variance

$$2\int_{-1/2}^{1/2}|u|\hat{f}(u)^2\,\mathrm{d}u.$$

Remark. There exists f such that $\mathrm{supp}\hat{f} \subseteq [-1/m, 1/m]$ and whose $(m+1)$st moment is not Gaussian. To see this, observe that the mth cumulants can be calculated exactly, and it ceases to vanish once the support of $\hat{f}$ is no longer contained in the interval $[-1/m, 1/m]$.

1.1. Moments and cumulants

One approach to proving such results is to use the Fourier expansion $g(\theta) = \sum_n g_n\,\mathrm{e}^{\mathrm{i}n\theta}$ and expand $\mathrm{Tr}\,g(U)$ as a sum

$$\mathrm{Tr}\,g(U) = \sum_n g_n\,\mathrm{Tr}\,(U^n).$$

Computing moments of $\mathrm{Tr}\,g(U)$ then boils down to being able to compute integrals of products of $\mathrm{Tr}(U^n)$ over the classical group. Theorem 1 for the unitary group was proved in [3] using this approach by employing a result of Diaconis and Shahshahani [1, 2], concerning moments of traces of random unitary matrices. Their result is a consequence of Schur duality for representations of the unitary group and the symmetric group, and the second orthogonality relation for characters of the symmetric group.

The paper by Diaconis and Evans [1] (see also [2]) contains a corresponding result for moments of traces of random symplectic and orthogonal[3] matrices (which they deduce using the work of Ram [7] on Brauer algebras), which can be used to prove our theorems in *half* the range, that is the mth moment of Z_f is Gaussian if $\mathrm{supp}\hat{f}$ lies in the interval $(-1/2m, 1/2m)$. We wish to have the *full* range so as to compare with zeros of quadratic L-functions, where linear statistics show mock-Gaussian behaviour in the same full range (this can be deduced from the work of Rubinstein [8]). Using a result of Özlück and Snyder [6], if one assumes the generalized Riemann hypothesis, then the mean of a linear statistic for quadratic L-functions can be calculated so long as $\mathrm{supp}\hat{f} \subset (-2, 2)$. The case of Dirichlet L-functions, which correspond to the unitary group, was considered in [3].

To obtain the results we desire, we abandon moments and instead use the *cumulants* $C_\ell^{G(N)}(g)$ of $\mathrm{Tr}\,g(U)$. These are defined via the expansion

$$\log \mathbb{E}_{G(N)}(\mathrm{e}^{t\,\mathrm{Tr}\,g(U)}) = \sum_{\ell=1}^{\infty} C_\ell^{G(N)}(g)\frac{t^\ell}{\ell!}$$

where $\mathbb{E}_{G(N)}$ denotes the expectation with respect to Haar measure over the group $G(N)$. The cumulants have previously been considered in this context by Soshnikov [9] (interestingly, his results again only give half the required range), and it is his combinatorial approach that we adopt.

There is a natural decomposition for the cumulants on the symplectic and special orthogonal groups. For brevity, we will describe the situation for the symplectic group (so N, the matrix size, is assumed to be even). The cumulants can be written as

$$C_\ell^{\mathrm{Sp}(N)}(g) = 2^\ell C_{\ell,N+1}^{\mathrm{even}}(g) - 2^\ell C_{\ell,N+1}^{\mathrm{odd}}(g).$$

We show that the odd parts $C_{\ell,N+1}^{\mathrm{odd}}(g)$ of the cumulants vanish in a certain region, and in fact if $g_k = 0$ for $|k| > (N+1)/\ell$ then the ℓth cumulant vanishes.

[3] Note that Diaconis and Evans consider orthogonal matrices, whereas we are interested in the special orthogonal group.

For all g, the even summand equals half a unitary cumulant:

$$C^{\mathrm{even}}_{\ell,N+1}(g) = \tfrac{1}{2}C^{\mathrm{U}(N+1)}_{\ell}(g).$$

We may now employ the available results about the unitary group to deduce that $C^{\mathrm{even}}_{\ell,N+1}(g)$ also vanishes in a larger region. Setting $g = F_N$ we obtain theorem 2.

Since moments and cumulants give essentially equivalent information, we can now go back to computing averages of the product of traces on classical groups and resolve a problem raised in [1, remark 8.2], to show

Theorem 3. *Let Z_j be independent standard normal random variables, and let*

$$\eta_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

(i) If $a_j \in \{0, 1, 2, \ldots\}$ for $j = 1, 2, \ldots$ are such that $\sum ja_j \leqslant N+1$, where N is even, then

$$\mathbb{E}_{\mathrm{Sp}(N)}\left\{\prod(\mathrm{Tr}\,U^j)^{a_j}\right\} = E\left\{\prod\left(\sqrt{j}Z_j - \eta_j\right)^{a_j}\right\}.$$

(ii) If $a_j \in \{0, 1, 2, \ldots\}$ for $j = 1, 2, \ldots$ are such that $\sum ja_j \leqslant N-1$ then

$$\mathbb{E}_{\mathrm{SO}(N)}\left\{\prod(\mathrm{Tr}\,U^j)^{a_j}\right\} = E\left\{\prod\left(\sqrt{j}Z_j + \eta_j\right)^{a_j}\right\}.$$

Similar theorems have been proved by Diaconis and Evans [1], though only for half the range (that is, they require $\sum ja_j \leqslant N/2$).

2. Cumulants of linear statistics

In order to calculate $C^{\mathrm{Sp}(N)}_{\ell}(g)$ we need to know the moment generating function. Using Weyl's integration formula, one can write $\mathbb{E}_{\mathrm{Sp}(N)}\{\mathrm{e}^{t\,\mathrm{Tr}\,g(U)}\}$ as an integral over the $N/2$ independent eigenphases (recall that N must be even for a symplectic matrix to exist), see e.g. [5]. Thus, writing $N = 2M$,

$$\begin{aligned}\mathbb{E}_{\mathrm{Sp}(N)}\{\mathrm{e}^{t\,\mathrm{Tr}\,g(U)}\} &= \mathbb{E}_{\mathrm{Sp}(N)}\left\{\exp\left(2t\sum_{n=1}^{M} g(\theta_n)\right)\right\} \\ &= \int_{[0,\pi]^M} \mathrm{Det}\{Q^{\mathrm{Sp}(2M)}(\theta_i,\theta_j)\}_{1\leqslant i,j\leqslant M}\prod_{n=1}^{M}\mathrm{e}^{2tg(\theta_n)}\,\mathrm{d}\theta_n\end{aligned}$$

where the kernel is $Q^{\mathrm{Sp}(N)}(x, y) := S_{N+1}(x-y) - S_{N+1}(x+y)$ with

$$S_N(z) := \frac{1}{2\pi}\frac{\sin(Nz/2)}{\sin(z/2)}. \tag{1}$$

Now, it is a general fact that if

$$\mathbb{E}\left\{\exp\left(\sum_{n=1}^{M} tg(\theta_n)\right)\right\} = \int_{\mathbb{T}^M} \mathrm{Det}\{Q_M(\theta_i,\theta_j)\}_{1\leqslant i,j\leqslant M}\prod_{n=1}^{M}\mathrm{e}^{tg(\theta_n)}\,\mathrm{d}\theta_n$$

where $\mathbb{E}$ denotes averaging the θ_n over some real interval $\mathbb{T}$, then defining the ℓth cumulant of $\sum g(\theta_n)$, C_ℓ, by the expansion

$$\log\mathbb{E}\left\{\exp\left(t\sum_{n=1}^{M} g(\theta_n)\right)\right\} = \sum_{\ell=1}^{\infty}\frac{t^\ell}{\ell!}C_\ell$$

Table 1. Kernels for Haar measure over the classical compact groups.

Group	$\mathrm{Tr}\, g(U)$	Kernel $Q_M(x, y)$	Range $\mathbb{T}$
$\mathrm{U}(N)$	$\sum_{n=1}^{N} g(\theta_n)$	$S_N(x, y)$	$(-\pi, \pi]$
$\mathrm{Sp}(N)$ $N = 2M$	$2\sum_{n=1}^{M} g(\theta_n)$	$S_{N+1}(x-y) - S_{N+1}(x+y)$	$[0, \pi]$
$\mathrm{SO}(N)$ $N = 2M$	$2\sum_{n=1}^{M} g(\theta_n)$	$S_{N-1}(x-y) + S_{N-1}(x+y)$	$[0, \pi]$
$\mathrm{SO}(N)$ $N = 2M+1$	$g(0) + 2\sum_{n=1}^{M} g(\theta_n)$	$S_{N-1}(x-y) - S_{N-1}(x+y)$	$[0, \pi]$

then [9, 10]

$$C_\ell = \sum_{m=1}^{\ell} \sum_{\sigma \in P(\ell,m)} (-1)^{m+1}(m-1)! \int_{\mathbb{T}^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j) Q_M(x_j, x_{j+1})\, \mathrm{d}x_j \tag{2}$$

where we identify x_{m+1} with x_1. Here $P(\ell, m)$ is the set of all partitions of ℓ objects into m non-empty blocks, where the jth block has $\lambda_j = \lambda_j(\sigma)$ elements (that is $\lambda_j := \#\{i : 1 \leqslant i \leqslant \ell, \sigma(i) = j\}$).

Thus,

$$C_\ell^{\mathrm{Sp}(N)}(g) = 2^\ell \sum_{m=1}^{\ell} \sum_{\sigma \in P(\ell,m)} (-1)^{m+1}(m-1)! \int_{[0,\pi]^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j) Q^{\mathrm{Sp}(N)}(x_j, x_{j+1})\, \mathrm{d}x_j.$$

Since $Q^{\mathrm{Sp}(N)}(x, y)$ is odd in both variables, $\prod_{j=1}^{m} Q^{\mathrm{Sp}(N)}(x_j, x_{j+1})$ is even in all variables, and so, since g is an even function, we may extend the integral to be over $[-\pi, \pi]$ and thus

$$\begin{aligned} C_\ell^{\mathrm{Sp}(N)}(g) = 2^\ell \sum_{m=1}^{\ell} \sum_{\sigma \in P(\ell,m)} & (-1)^{m+1}(m-1)! \\ & \times \frac{1}{2^m} \int_{[-\pi,\pi]^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j)(S_{N+1}(x_j - x_{j+1}) - S_{N+1}(x_j + x_{j+1}))\, \mathrm{d}x_j \end{aligned}$$

and on expanding out the middle product on the bottom line,

$$\begin{aligned} C_\ell^{\mathrm{Sp}(N)}(g) = 2^\ell \sum_{m=1}^{\ell} \sum_{\sigma \in P(\ell,m)} & (-1)^{m+1}(m-1)! \frac{1}{2^m} \\ & \times \sum_{\epsilon_1=\pm1,\ldots,\epsilon_m=\pm1} \int_{[-\pi,\pi]^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j) \epsilon_j S_{N+1}(x_j - \epsilon_j x_{j+1})\, \mathrm{d}x_j \\ = 2^\ell C_{\ell,N+1}^{\mathrm{even}}(g) - 2^\ell C_{\ell,N+1}^{\mathrm{odd}}(g) & \end{aligned}$$

where $C_{\ell,N+1}^{\mathrm{even}}(g)$ contains terms with $\prod_{j=1}^{m} \epsilon_j = +1$ and $C_{\ell,N+1}^{\mathrm{odd}}(g)$ contains terms with $\prod_{j=1}^{m} \epsilon_j = -1$.

Similarly one can calculate the other groups, using Weyl's calculation of Haar measure, which is summarized in table 1.

2.1. Summary

Put

$$C^{\text{even}}_{\ell,M}(g) = \sum_{m=1}^{\ell} \sum_{\sigma\in P(\ell,m)} (-1)^{m+1}(m-1)!\frac{1}{2^m} \times \sum_{\substack{\epsilon_1=\pm1,\ldots,\epsilon_m=\pm1\\ \prod\epsilon_j=+1}} \int_{[-\pi,\pi]^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j) S_M(x_j - \epsilon_j x_{j+1})\, \mathrm{d}x_j \tag{3}$$

and

$$C^{\text{odd}}_{\ell,M}(g) = \sum_{m=1}^{\ell} \sum_{\sigma\in P(\ell,m)} (-1)^{m+1}(m-1)!\frac{1}{2^m} \times \sum_{\substack{\epsilon_1=\pm1,\ldots,\epsilon_m=\pm1\\ \prod\epsilon_j=-1}} \int_{[-\pi,\pi]^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j) S_M(x_j - \epsilon_j x_{j+1})\, \mathrm{d}x_j \tag{4}$$

with S_M defined in (1):

- For all ℓ,
$$C^{\text{Sp}(2M)}_{\ell}(g) = 2^{\ell} C^{\text{even}}_{\ell,2M+1}(g) - 2^{\ell} C^{\text{odd}}_{\ell,2M+1}(g).$$
- For all ℓ,
$$C^{\text{SO}(2M)}_{\ell}(g) = 2^{\ell} C^{\text{even}}_{\ell,2M-1}(g) + 2^{\ell} C^{\text{odd}}_{\ell,2M-1}(g).$$
- For $\ell = 1$,
$$C^{\text{SO}(2M+1)}_{1}(g) = 2C^{\text{even}}_{1,2M}(g) - 2C^{\text{odd}}_{1,2M}(g) + \sum_{k=-\infty}^{\infty} g_k.$$
- For all $\ell \geqslant 2$,
$$C^{\text{SO}(2M+1)}_{\ell}(g) = 2^{\ell} C^{\text{even}}_{\ell,2M}(g) - 2^{\ell} C^{\text{odd}}_{\ell,2M}(g).$$

In the next section, we will show that $C^{\text{even}}_{\ell,M}(g) = \frac{1}{2} C^{\text{U}(M)}_{\ell}(g)$, and then we will calculate $C^{\text{odd}}_{\ell,M}(g)$, first in the case when M is odd, and then in the case when M is even.

The results will show that

$$C^{G(N)}_{\ell}(g) = \sum_{k\in\mathbb{Z}^{\ell}} \mu^{G(N)}_{\ell}(k_1,\ldots,k_{\ell}) \prod_{j=1}^{\ell} g_{k_j} \tag{5}$$

where $\mu^{G(N)}_{\ell}(k_1,\ldots,k_{\ell})$ is invariant under permutations of its arguments.

Combining the results from the next section proves the following theorems:

Theorem 4.

$$C^{\text{Sp}(2M)}_{1}(g) = 2Mg_0 - 2\sum_{n=1}^{M} g_{2n}$$

$$C^{\text{Sp}(2M)}_{2}(g) = 4\sum_{n=1}^{\infty} \min(n, 2M+1) g_n^2 - 4\sum_{k=M+1}^{\infty} g_k^2 - 8\sum_{l=1}^{M}\sum_{k=M+1}^{\infty} g_{k+l}g_{k-l}$$

and for $\ell \geqslant 3$, $\mu^{\text{Sp}(N)}_{\ell}(k_1,\ldots,k_{\ell}) = 0$ *if* $\sum_{j=1}^{\ell} |k_j| \leqslant N+1$.

Theorem 5. *When averaged over the special orthogonal group, the mean of* $\mathrm{Tr}\, g(U)$ *is*

$$C_1^{\mathrm{SO}(2M)}(g) = 2Mg_0 + 2\sum_{n=1}^{M-1} g_{2n}$$

$$C_1^{\mathrm{SO}(2M+1)}(g) = (2M+1)g_0 + 2\sum_{n=1}^{M} g_{2n} + 2\sum_{n=2M+1}^{\infty} g_n$$

and the variance is

$$C_2^{\mathrm{SO}(2M)}(g) = 4\sum_{n=1}^{\infty} \min(n, 2M-1)g_n^2 + 4\sum_{k=M}^{\infty} g_k^2 + 8\sum_{l=1}^{M-1}\sum_{k=M}^{\infty} g_{k+l}g_{k-l}$$

$$C_2^{\mathrm{SO}(2M+1)}(g) = 4\sum_{n=1}^{\infty} \min(n, 2M)g_n^2 - 8\sum_{\substack{n=1\\ n\ \mathrm{odd}}}^{2M-1}\ \sum_{\substack{m=2M+1\\ m\ \mathrm{odd}}}^{\infty} g_{(m+n)/2}g_{(m-n)/2}$$

For $\ell \geqslant 3$, $\mu_\ell^{\mathrm{SO}(N)}(k_1, \ldots, k_\ell) = 0$ *if* $\sum_{j=1}^{\ell} |k_j| \leqslant N-1$.

3. The combinatorial calculations

3.1. The calculation of $C_{\ell,M}^{\mathrm{even}}(g)$

The following lemma was stated by Soshnikov in [9]:

Lemma 6. *For all* ℓ,

$$C_{\ell,M}^{\mathrm{even}}(g) = \tfrac{1}{2}C_\ell^{\mathrm{U}(M)}(g).$$

Proof. Symbolically, denote

$$\int_{[-\pi,\pi]^m} \prod_{j=1}^{m} g^{\lambda_j}(x_j) S_M(x_j - \epsilon_j x_{j+1})\, \mathrm{d}x_j \tag{6}$$

by $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$. If $\epsilon_1 = 1$ do nothing, but if $\epsilon_1 = -1$ then change variables to $x_2 \mapsto -x_2$, and note that since g and S_M are even functions, and the integral over x_2 is over $[-\pi, \pi]$, then (6) becomes $(+1, -\epsilon_2, \epsilon_3, \ldots, \epsilon_m)$.

Observe that this achieves the following: if the initial situation was $(-1, -1, \ldots)$ then it becomes $(+1, +1, \ldots)$ while if it was $(-1, +1, \ldots)$ it becomes $(+1, -1, \ldots)$. Therefore there is either the same number of -1 in the set of ϵ or there are two less -1.

Now repeat for the new ϵ_2, changing variables only if it is -1, and so on all the way up to ϵ_m. Each time the action either leaves the number of -1 unchanged or reduces it by 2. Since we started with an even number of -1 in the set of ϵ this algorithm will terminate with (6) equalling $(+1, +1, \ldots, +1)$, which is independent of ϵ. There are 2^{m-1} possible ϵ with an even number of -1, and so

$$C_{\ell,M}^{\mathrm{even}}(g) = \sum_{m=1}^{\ell}\sum_{\sigma\in P(\ell,m)} (-1)^{m+1}(m-1)!\frac{1}{2^m}2^{m-1}\int_{[-\pi,\pi]^m}\prod_{j=1}^{m} g^{\lambda_j}(x_j)S_M(x_j - x_{j+1})\,\mathrm{d}x_j$$

which we recognize as $\frac{1}{2}C_\ell^{\mathrm{U}(M)}(g)$. □

The cumulants of a random unitary matrix have previously been calculated, essentially by Soshnikov [9], but they can also be deduced from the work of Diaconis and Shahshahani [2] and of Diaconis and Evans [1].

Theorem 7 (Soshnikov). *Let $C_\ell^{\mathrm{U}(N)}$ be the ℓth cumulant of* $\operatorname{Tr} g(U)$*, averaged over all $N \times N$ unitary matrices with Haar measure. Then*

$$C_1^{\mathrm{U}(N)} = N g_0 \qquad C_2^{\mathrm{U}(N)} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \min(|n|, N) g_n g_{-n}$$

and for $\ell \geqslant 3$,

$$\left|C_\ell^{\mathrm{U}(N)}(g)\right| \leqslant \mathrm{const}_\ell \sum_{\substack{k_1+\cdots+k_\ell=0 \\ |k_1|+\cdots+|k_\ell|>2N}} |k_1||g_{k_1}| \ldots |g_{k_\ell}|.$$

Remark. The heart of the proof of this theorem is a deep combinatorial fact called the Hunt–Dyson formula.

Remark. Actually, the error term in [9] has the sum running over all $k_1 + \cdots + k_\ell = 0$ such that $|k_1| + \cdots + |k_\ell| > N$. But it is clear from equation (2.9) of [9] that there is no contribution to $C_\ell^{\mathrm{U}(N)}$ for $\ell \geqslant 3$ if $\sum k_i \mathbb{1}_{\{k_i>0\}} \leqslant N$ and if $\sum -k_i \mathbb{1}_{\{k_i<0\}} \leqslant N$. Since the k_i sum to zero, it must be that the sum over positive terms equals the sum over negative terms, and so this is the same as the condition $\sum |k_i| \leqslant 2N$, as we have it in the theorem.

3.2. The calculation of $C_{\ell,2M+1}^{\mathrm{odd}}(g)$

Observe from (1) that

$$S_{2M+1}(z) = \frac{1}{2\pi} \sum_{n=-M}^{M} \mathrm{e}^{-\mathrm{i}nz}. \tag{7}$$

Lemma 8. *One can calculate $C_{1,2M+1}^{\mathrm{odd}}(g)$ and $C_{2,2M+1}^{\mathrm{odd}}(g)$ exactly:*

$$C_{1,2M+1}^{\mathrm{odd}}(g) = \frac{1}{2} \sum_{n=-M}^{M} g_{2n} \qquad C_{2,2M+1}^{\mathrm{odd}}(g) = \frac{1}{2} \sum_{l=-M}^{M} \sum_{|k|>M} g_{l+k} g_{l-k}.$$

Proof. First of all, from (4) we have that

$$C_{1,2M+1}^{\mathrm{odd}}(g) = \frac{1}{2} \int_{-\pi}^{\pi} g(x) S_{2M+1}(2x)\, \mathrm{d}x$$

$$C_{2,2M+1}^{\mathrm{odd}}(g) = \frac{1}{2} \int_{-\pi}^{\pi} g^2(x) S_{2M+1}(2x)\, \mathrm{d}x - \frac{1}{4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(x) g(y) 2 S_{2M+1}(x+y) S_{2M+1}(x-y)\, \mathrm{d}x\, \mathrm{d}y$$

and using (7) we see that

$$C_{1,2M+1}^{\mathrm{odd}}(g) = \frac{1}{2} \sum_{n=-M}^{M} g_{2n}$$

and

$$C_{2,2M+1}^{\mathrm{odd}}(g) = \frac{1}{2} \sum_{l=-M}^{M} \sum_{k=-\infty}^{\infty} g_k g_{2l-k} - \frac{1}{2} \sum_{l=-M}^{M} \sum_{k=-M}^{M} g_{l+k} g_{l-k} = \frac{1}{2} \sum_{l=-M}^{M} \sum_{|k|>M} g_{l+k} g_{l-k}$$

as required. □

Lemma 9. *For* $\ell \geqslant 2$,

$$\left|C^{\mathrm{odd}}_{\ell,2M+1}(g)\right| \leqslant \mathrm{const}_\ell \sum_{\substack{\mathbf{k}\in\mathbb{Z}^\ell \\ |k_1|+\cdots+|k_\ell|>2M+1}} |g_{k_1}|\ldots|g_{k_\ell}|.$$

Proof. Fix $\sigma \in P(\ell, m)$, and for $\mathbf{k} = (k_1, \ldots, k_\ell) \in \mathbb{Z}^\ell$ set

$$K_1 = \sum_{l=1}^{\lambda_1} k_l$$
$$K_2 = \sum_{l=\lambda_1+1}^{\lambda_1+\lambda_2} k_l$$
$$\vdots$$
$$K_m = \sum_{l=\lambda_1+\cdots+\lambda_{m-1}+1}^{\ell} k_l$$

(recall that $\ell = \lambda_1 + \cdots + \lambda_m$). Therefore

$$\prod_{j=1}^m g^{\lambda_j}(x_j) = \sum_{\mathbf{k}\in\mathbb{Z}^\ell} \prod_{l=1}^{\ell} g_{k_l} \prod_{j=1}^m \mathrm{e}^{\mathrm{i}K_j x_j}.$$

Hence, the integral in (4)

$$\int_{[-\pi,\pi]^m} \prod_{j=1}^m g^{\lambda_j}(x_j) S_{2M+1}(x_j - \epsilon_j x_{j+1})\, \mathrm{d}x_j$$
$$= \sum_{-M\leqslant n_1,\ldots,n_m\leqslant M} \sum_{\mathbf{k}\in\mathbb{Z}^\ell} \prod_{l=1}^{\ell} g_{k_l} \int_{[-\pi,\pi]^m} \prod_{j=1}^m \mathrm{e}^{\mathrm{i}K_j x_j}\, \mathrm{e}^{\mathrm{i}n_j(x_j-\epsilon_j x_{j+1})} \frac{\mathrm{d}x_j}{2\pi}$$
$$= \sum_{\mathbf{k}\in\mathbb{Z}^\ell} \prod_{l=1}^{\ell} g_{k_l} \sum_{-M\leqslant n_1,\ldots,n_m\leqslant M} \int_{[-\pi,\pi]^m} \prod_{j=1}^m \exp(\mathrm{i}x_j(K_j + n_j - \epsilon_{j-1}n_{j-1})) \frac{\mathrm{d}x_j}{2\pi}$$

where we have used (7) to express $S_{2M+1}(x_j - \epsilon_j x_{j+1})$ in its Fourier representation, and we have defined $\epsilon_0 = \epsilon_m$, $n_0 = n_m$ (so all indices are cyclic).

The integral above will be 1 or 0 depending on whether $n_j - \epsilon_{j-1}n_{j-1} = -K_j$ or not, so defining

$$\mathcal{N}(M, \sigma, \mathbf{k}, \epsilon) = \#\{-M \leqslant n_1, \ldots, n_m \leqslant M : n_j - \epsilon_{j-1}n_{j-1} = -K_j,\ j = 1, \ldots, m\} \tag{8}$$

(the $K_1, \ldots, K_m$ depend on both $\mathbf{k}$ and σ, recall) we see that

$$C^{\mathrm{odd}}_{\ell,2M+1}(g) = \sum_{\mathbf{k}\in\mathbb{Z}^\ell} \prod_{l=1}^{\ell} g_{k_l} \sum_{m=1}^{\ell} \sum_{\sigma\in P(\ell,m)} \frac{(-1)^{m+1}(m-1)!}{2^m} \sum_{\substack{\epsilon_1,\ldots,\epsilon_m=\pm1 \\ \prod\epsilon_j=-1}} \mathcal{N}(M, \sigma, \mathbf{k}, \epsilon). \tag{9}$$

Lemma 10. *Let* $\prod_{j=1}^m \epsilon_j = -1$. *Then* $\mathcal{N}(M, \sigma, \mathbf{k}, \epsilon)$ *is either* 0 *or* 1*:*

- *If* $\sum_{l=1}^{\ell} k_l$ *is odd then* $\mathcal{N}(M, \sigma, \mathbf{k}, \epsilon) = 0$.
- *If* $\sum_{l=1}^{\ell} k_l$ *is even and* $\sum_{l=1}^{\ell} |k_l| \leqslant 2M$ *then* $\mathcal{N}(M, \sigma, \mathbf{k}, \epsilon) = 1$.

(Proof deferred until the end of this section.) Therefore, if $\sum_{l=1}^{\ell}|k_l| \leqslant 2M+1$ then

$$\sum_{m=1}^{\ell}\sum_{\sigma\in P(\ell,m)}(-1)^{m+1}(m-1)!\frac{1}{2^m}\sum_{\substack{\epsilon_1,\ldots,\epsilon_m=\pm1\\ \prod\epsilon_j=-1}}\mathcal{N}(M,\sigma,\mathbf{k},\epsilon)$$
$$=\frac{1}{2}\mathcal{M}(\mathbf{k})\sum_{m=1}^{\ell}\sum_{\sigma\in P(\ell,m)}(-1)^{m+1}(m-1)! \tag{10}$$

where

$$\mathcal{M}(\mathbf{k})=\begin{cases}1 & \text{if} \quad \sum_{l=1}^{\ell}k_l \text{ is even}\\ 0 & \text{otherwise.}\end{cases}$$

Using the fact that for $\ell \geqslant 2$

$$\sum_{m=1}^{\ell}\sum_{\sigma\in P(\ell,m)}(-1)^{m+1}(m-1)!=0$$

we see that (10) vanishes for $\sum_{l=1}^{\ell}|k_l| \leqslant 2M+1$ if $\ell \geqslant 2$. Inserting this into (9) and estimating the contribution from the terms with $\sum_{l=1}^{\ell}|k_l| \geqslant 2M+2$ we see that

$$\left|C_{\ell,2M+1}^{\text{odd}}(g)\right| \leqslant \text{const}_\ell \sum_{\substack{\mathbf{k}\in\mathbb{Z}^\ell\\ \sum_{l=1}^{\ell}|k_l|\geqslant 2M+2}}\prod_{l=1}^{\ell}|g_{k_l}|$$

where

$$\text{const}_\ell=\sum_{m=1}^{\ell}\tfrac{1}{2}(m-1)!\#\{\sigma\in P(\ell,m)\}$$

depends upon ℓ only. This completes the proof of lemma 9. □

Proof of lemma 10. We treat all indices as cyclic modulo m. So $n_0=n_m$ and $n_{m+1}=n_1$ etc.

We assume that $\prod_{j=1}^{m}\epsilon_j=-1$. Define the $m\times m$ matrix E to be such that

$$E_{i,j}=\begin{cases}\epsilon_{i-1} & \text{if} \quad j=i-1\\ 0 & \text{otherwise}\end{cases}$$

so that

$$(E\mathbf{n})_j=\epsilon_{j-1}n_{j-1}.$$

From the definition of $\mathcal{N}(M,\sigma,\mathbf{k},\epsilon)$ (which is given in (8)) we see that it is the number of solutions of $(I-E)\mathbf{n}=-\mathbf{K}$ subject to $-M\leqslant n_j\leqslant M$.

Now,

$$\begin{aligned}(E^k\mathbf{n})_j&=\epsilon_{j-1}(E^{k-1}\mathbf{n})_{j-1}\\ &=\epsilon_{j-1}\epsilon_{j-2}\ldots\epsilon_{j-k}n_{j-k}\end{aligned}$$

and so $E^m=\epsilon_1\ldots\epsilon_m I=-I$ by cyclicity of indices and the assumption that $\prod_{j=1}^{m}\epsilon_j=-1$.

Hence $2I=I-E^m$. But $I-E^m$ factorizes as

$$I-E^m=(I-E)(I+E+\cdots+E^{m-2}+E^{m-1})$$

and therefore

$$(I-E)^{-1}=\tfrac{1}{2}(I+E+\cdots+E^{m-2}+E^{m-1}).$$

If we ignore the restriction that $-M \leqslant n_j \leqslant M$ then, over the reals, there is exactly one solution to $(I - E)\mathbf{n} = -\mathbf{K}$ which is

$$n_j = -\tfrac{1}{2}(K_j + \epsilon_{j-1}K_{j-1} + \epsilon_{j-1}\epsilon_{j-2}K_{j-2} + \cdots + \epsilon_{j-1}\epsilon_{j-2}\ldots\epsilon_{j-m+1}K_{j-m+1}). \tag{11}$$

This is a solution over the integers if n_j is an integer, which will be the case when the term inside the brackets is even. Since $\epsilon_j \equiv 1 \pmod 2$ for all j, the term inside the brackets is even when

$$K_j + K_{j-1} + \cdots + K_{j-m+1} = \sum_{i=1}^{m} K_i = \sum_{l=1}^{\ell} k_l$$

is even. There are no solutions over the integers when this is odd. (Note that the parity is independent of ϵ and the partition σ).

Finally, one must check that the condition $-M \leqslant n_j \leqslant M$ holds. From (11) we see that

$$|n_j| \leqslant \frac{1}{2}\sum_{i=1}^{m}|K_i| \leqslant \frac{1}{2}\sum_{l=1}^{\ell}|k_l|$$

and so if we assume that $\sum_{l=1}^{\ell}|k_l| \leqslant 2M$, then the condition holds.

Thus $\mathcal{N}(M,\sigma,\mathbf{k},\epsilon) = 0$ if $\sum_{l=1}^{\ell} k_l$ is odd, and $\mathcal{N}(M,\sigma,\mathbf{k},\epsilon) = 1$ if $\sum_{l=1}^{\ell} k_l$ is even and $\sum_{l=1}^{\ell}|k_l| \leqslant 2M$.

This proves lemma 10. □

3.3. The calculation of $C^{\mathrm{odd}}_{\ell,2M}(g)$

Basically, this section is like the previous, with the essential change being that

$$S_{2M}(z) = \frac{1}{2\pi}\sum_{\substack{n=-(2M-1)\\ n\ \mathrm{odd}}}^{2M-1} \mathrm{e}^{-\mathrm{i}nz/2}$$

as opposed to (7) which states

$$S_{2M+1}(z) = \frac{1}{2\pi}\sum_{\substack{n=-2M\\ n\ \mathrm{even}}}^{2M} \mathrm{e}^{-\mathrm{i}nz/2}.$$

Lemma 11. *One can calculate* $C^{\mathrm{odd}}_{1,2M}(g)$ *and* $C^{\mathrm{odd}}_{2,2M}(g)$ *exactly:*

$$C^{\mathrm{odd}}_{1,2M}(g) = \frac{1}{2}\sum_{n=-(M-1)}^{M} g_{2n-1} \qquad C^{\mathrm{odd}}_{2,2M}(g) = \frac{1}{2}\sum_{\substack{n=-(2M-1)\\ n\ \mathrm{odd}}}^{2M-1}\ \sum_{\substack{|m|\geqslant 2M+1\\ m\ \mathrm{odd}}} g_{\frac{1}{2}(n+m)}g_{\frac{1}{2}(n-m)}.$$

Lemma 12. *For* $\ell \geqslant 2$,

$$\left|C^{\mathrm{odd}}_{\ell,2M}(g)\right| \leqslant \mathrm{const}_\ell \sum_{\substack{\mathbf{k}\in\mathbb{Z}^\ell\\ |k_1|+\cdots+|k_\ell|>2M}} |g_{k_1}|\ldots|g_{k_\ell}|.$$

The proof goes through the same steps as before, with equation (8) becoming

$$\mathcal{N}_{\mathrm{odd}}(M,\sigma,\mathbf{k},\epsilon) = \#\big\{-(2M-1) \leqslant n_j \leqslant 2M-1,\ n_j\ \mathrm{odd} : \tfrac{1}{2}n_j - \epsilon_{j-1}\tfrac{1}{2}n_{j-1} = -K_j,\ j = 1,\ldots,m\big\}.$$

Rewriting equation (11) we see the solution requested by $\mathcal{N}_{\mathrm{odd}}(M, \sigma, \mathbf{k}, \epsilon)$ is

$$n_j = -(K_j + \epsilon_{j-1} K_{j-1} + \epsilon_{j-1}\epsilon_{j-2} K_{j-2} + \cdots + \epsilon_{j-1}\epsilon_{j-2} \ldots \epsilon_{j-m+1} K_{j-m+1})$$

so long as n_j is odd and $-(2M-1) \leqslant n_j \leqslant 2M-1$ (and there is no solution otherwise). Therefore lemma 10 becomes

Lemma 13. *Let* $\prod_{j=1}^{m} \epsilon_j = -1$. *Then* $\mathcal{N}_{\mathrm{odd}}(M, \sigma, \mathbf{k}, \epsilon)$ *is either* 0 *or* 1*:*

- *If* $\sum_{l=1}^{\ell} k_l$ *is even then* $\mathcal{N}_{\mathrm{odd}}(M, \sigma, \mathbf{k}, \epsilon) = 0$.
- *If* $\sum_{l=1}^{\ell} k_l$ *is odd and* $\sum_{l=1}^{\ell} |k_l| \leqslant 2M-1$ *then* $\mathcal{N}_{\mathrm{odd}}(M, \sigma, \mathbf{k}, \epsilon) = 1$.

4. Moments of traces

We will now use theorem 5 to prove the second part of theorem 3. (The proof of the first part from theorem 4 being analogous.)

Recall from (5) that

$$C_\ell^{\mathrm{SO}(N)}(g) = \sum_{n \in \mathbb{Z}^\ell} \mu_\ell^{\mathrm{SO}(N)}(n_1, \ldots, n_\ell) \prod_{j=1}^{\ell} g_{n_j}$$

where $\mu_\ell^{\mathrm{SO}(N)}(n_1, \ldots, n_\ell)$ is invariant under permutations of its arguments. Assuming $g_0 = 0$, we have the following:

- If $|n_1| < N$ then
$$\mu_1^{\mathrm{SO}(N)}(n_1) = \begin{cases} 1 & \text{if} \quad n_1 \neq 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$
- If $|n_1| + |n_2| < N$ then
$$\mu_2^{\mathrm{SO}(N)}(n_1, n_2) = \begin{cases} |n_1| & \text{if} \quad |n_1| = |n_2| \\ 0 & \text{otherwise.} \end{cases}$$
- If $\ell \geqslant 3$ and $\sum_{j=1}^{\ell} |n_j| < N$ then $\mu_\ell^{\mathrm{SO}(N)}(n_1, \ldots, n_\ell) = 0$.

It is also true that if $g_0 = 0$,

$$\mathbb{E}_G\left\{\left(\operatorname{Tr} g(U) - C_1^G(g)\right)^m\right\}$$

$$= 2^m \sum_{n \in \mathbb{N}^m} \mathbb{E}_G\left\{\left(\operatorname{Tr} U^{n_1} - \mu_1^G(n_1)\right) \ldots \left(\operatorname{Tr} U^{n_m} - \mu_1^G(n_m)\right)\right\} \prod_{j=1}^{m} g_{n_j} \tag{12}$$

$$= \sum \left(\frac{C_2^G(g)}{2!}\right)^{k_2} \left(\frac{C_3^G(g)}{3!}\right)^{k_3} \cdots \left(\frac{C_m^G(g)}{m!}\right)^{k_m} \frac{m!}{k_2! k_3! \ldots k_m!} \tag{13}$$

where the second sum runs over all values of $k_j \geqslant 0$ such that $\sum_{j=2}^{m} j k_j = m$ (it is simply writing the mth moment in terms of its cumulants, having subtracted the mean).

Let $a_j \in \{0, 1, 2, \ldots\}$ for $j = 1, 2, \ldots$ such that $\sum j a_j < N$. Define

$$\eta_j = \begin{cases} 1 & \text{for even } j \\ 0 & \text{for odd } j \end{cases}$$

so that $\mu_1^{\mathrm{SO}(N)}(j) = \eta_j$ for $|j| < N$.

Putting $m = \sum a_j$, we will evaluate the coefficient of $\prod (g_j)^{a_j}$ in (12) and (13), the two being equal to each other.

Consider first equation (12). The coefficient of $\prod(g_j)^{a_j}$ in

$$2^m \sum_{n\in\mathbb{N}^m} \mathbb{E}_{\mathrm{SO}(N)} \left\{ \left(\operatorname{Tr} U^{n_1} - \eta_{n_1}\right) \dots \left(\operatorname{Tr} U^{n_m} - \eta_{n_m}\right) \right\} \prod_{j=1}^{m} g_{n_j}$$

equals

$$\frac{2^m m!}{\prod(a_j)!} \mathbb{E}_{\mathrm{SO}(N)} \left\{ \prod (\operatorname{Tr} U^j - \eta_j)^{a_j} \right\}. \tag{14}$$

Consider next equation (13). Note that the restriction on the a_j means that there is no contribution to the coefficient of $\prod(g_j)^{a_j}$ from $C_\ell^{\mathrm{SO}(N)}(g)$ for all $\ell \geqslant 3$. Therefore the coefficient in (13) is 0 if m is odd and is the coefficient of $\prod(g_j)^{a_j}$ in

$$\frac{m!}{2^{m/2}(m/2)!}\left(C_2^{\mathrm{SO}(N)}(g)\right)^{m/2} = \frac{m!}{2^{m/2}(m/2)!} 2^m \sum_{n\in\mathbb{N}^m} \prod_{j=1}^{m/2} \mu_2^{\mathrm{SO}(N)}(n_{2j-1}, n_{2j}) \prod_{j=1}^{m} g_{n_j}$$

if m is even. This coefficient is zero unless all the a_j are even, in which case it is

$$\frac{m!}{(m/2)!} 2^{m/2} \frac{(m/2)!}{\prod(a_j/2)!} \prod j^{a_j/2} \tag{15}$$

(to see this, note that the structure of $\mu_2^{\mathrm{SO}(N)}$ means that n_{2j} must equal n_{2j-1} for $j = 1, \dots, m/2$. The second pre-factor is just the number of ways of picking $m/2$ integers such that $a_1/2$ of them equal 1, $a_2/2$ of them equal 2 etc).

Setting (14) equal to (15) and recalling that $m = \sum a_j$, we have

$$\mathbb{E}_{\mathrm{SO}(N)} \left\{ \prod (\operatorname{Tr} U^j - \eta_j)^{a_j} \right\} = \begin{cases} \prod j^{a_j/2} \frac{(a_j)!}{2^{a_j/2}(a_j/2)!} & \text{if all the } a_j \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$
$$= E\left\{ \prod \left(\sqrt{j} Z_j\right)^{a_j} \right\}$$

where Z_j are iid normal random variables with mean 0 and variance 1.

Observe that this can all be rewritten as

$$\mathbb{E}_{\mathrm{SO}(N)} \left\{ \prod (\operatorname{Tr} U^j)^{a_j} \right\} = E\left\{ \prod \left(\sqrt{j} Z_j + \eta_j\right)^{a_j} \right\}$$

and is valid so long as $\sum j a_j < N$.

Acknowledgment

This work was supported in part by the EC TMR network 'Mathematical aspects of quantum chaos', EC-contract no HPRN-CT-2000-00103.

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