Torsion points on curves and common divisors of $a^k - 1$ and $b^k - 1$

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1. Introduction. Let $a, b \neq \pm 1$ be nonzero integers. One of our goals in this paper is to study the common divisors of $a^k - 1$ and $b^k - 1$, specifically to understand small values of $\gcd(a^k - 1, b^k - 1)$. If $a = c^u$ and $b = c^v$ for some integer $c$ then clearly $c^k - 1$ divides $\gcd(a^k - 1, b^k - 1)$ and so for the purpose of understanding small values, we will assume that $a$ and $b$ are multiplicatively independent, that is, $a^r \neq b^s$ for $r, s \geq 1$. Further, since $\gcd(a - 1, b - 1)$ always divides $\gcd(a^k - 1, b^k - 1)$, we will assume that $a - 1$ and $b - 1$ are coprime.

Based on numerical experiments and other considerations, we conjecture:

**Conjecture A.** If $a, b$ are multiplicatively independent non-zero integers with $\gcd(a - 1, b - 1) = 1$, then there are infinitely many integers $k \geq 1$ such that

$$\gcd(a^k - 1, b^k - 1) = 1.$$ 

Note that the condition of multiplicative independence of $a$ and $b$ is not necessary, as the (trivial) example $b = -a$ shows (the gcd is 1 for odd $k$, and $a^k - 1$ for even $k$).

A recent result of Bugeaud, Corvaja and Zannier [BCZ] rules out large values of $\gcd(a^k - 1, b^k - 1)$. They show that if $a, b > 1$ are multiplicatively independent positive integers then for all $\varepsilon > 0$,

$$\gcd(a^k - 1, b^k - 1) \ll_{\varepsilon} e^{\varepsilon k}.$$ 

Their argument uses Diophantine approximation techniques and in particular Schmidt’s Subspace Theorem. They also indicate that there are arbitrarily large values of $k$ for which the upper bound (1) cannot be significantly improved.

In the function field case, when we replace integers by polynomials, we are able to prove a strong version of Conjecture A.

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Theorem 1. Let \( f, g \in \mathbb{C}[t] \) be nonconstant polynomials. If \( f \) and \( g \) are multiplicatively independent, then there exists a polynomial \( h \) such that

\[
\text{gcd}(f^k - 1, g^k - 1) | h
\]

for any \( k \geq 1 \). If, in addition, \( \text{gcd}(f - 1, g - 1) = 1 \), then there is a finite union of proper arithmetic progressions \( \bigcup d_i \mathbb{N}, d_i \geq 2 \), such that for \( k \) outside these progressions,

\[
\text{gcd}(f^k - 1, g^k - 1) = 1.
\]

Note that (2) is a strong form of (1). We derive Theorem 1 from a result proposed by Lang [L1] on the finiteness of torsion points on curves—see Section 2.

We next consider a generalization to the case of matrices. For an \( r \times r \) integer matrix \( A \in \text{Mat}_r(\mathbb{Z}) \), \( A \neq I \) (\( I \) being the identity matrix), we define \( \text{gcd}(A - I) \) as the greatest common divisor of the entries of \( A - I \). Equivalently, \( \text{gcd}(A - I) \) is the greatest integer \( N \geq 1 \) such that \( A \equiv I \mod N \). We say that \( A \) is primitive if \( \text{gcd}(A - I) = 1 \). Note that \( \text{gcd}(A - I) \) divides \( \text{gcd}(A^k - I) \) for all \( k \). A similar definition applies to the function field case \( A \in \text{Mat}_r(\mathbb{C}[t]) \). We will study the behaviour of \( \text{gcd}(A^k - I) \) as \( k \) varies for a fixed matrix \( A \) with coefficients in \( \mathbb{Z} \) or in \( \mathbb{C}[t] \). If \( \det A = 0 \) then trivially \( \text{gcd}(A^k - I) = 1 \) for all \( k \geq 1 \). So we will henceforth assume that \( A \) is nonsingular.

For the case of \( 2 \times 2 \) matrices, we will show in Section 3 that if \( A \in \text{SL}_2(\mathbb{Z}) \) is unimodular and hyperbolic, then \( \text{gcd}(A^k - I) \) grows exponentially as \( k \to \infty \). However, numerical experiments show that for other matrices, \( \text{gcd}(A^k - I) \) displays completely different behaviour. We formulate the following conjecture:

Conjecture B. Suppose \( r \geq 2 \) and \( A \in \text{Mat}_r(\mathbb{Z}) \) is nonsingular and primitive. Also assume that there is a pair of eigenvalues of \( A \) that are multiplicatively independent. Then \( A^k \) is primitive infinitely often.

Note that Conjecture B subsumes Conjecture A. It would be interesting to prove an analogue of the upper bound (1) in this setting.

In Section 4 we give an example where we can prove Conjecture B. To describe it, recall that one may obtain integer matrices by taking an algebraic integer \( u \) in a number field \( K \) and letting it act by multiplication on the ring of integers \( \mathcal{O}_K \) of \( K \). This is a linear map and a choice of integer basis of \( \mathcal{O}_K \) gives us an integer matrix \( A = A(u) \) whose determinant equals the norm of \( u \). We employ this method for the cyclotomic field \( \mathbb{Q}(\zeta_p) \) where \( p > 3 \) is prime and \( \zeta_p \) is a primitive \( p \)-th root of unity, and \( u \) is a nonreal unit. We show:
Theorem 2. Let $u$ be a nonreal unit in the extension $\mathbb{Q}(\zeta_p)$, and $A(u) \in \text{SL}_{p-1}(\mathbb{Z})$ be the corresponding matrix. Then $A(u)^k$ is primitive for all $k \neq 0 \mod p$.

In the function field case, we have a strong form of Conjecture B, which generalizes Theorem 1:

Theorem 3. Let $A$ be a nonsingular matrix in $\text{Mat}_r(\mathbb{C}[t])$. Assume that either

1. $A$ is not diagonalizable over the algebraic closure of $\mathbb{C}(t)$, or
2. $A$ has two eigenvalues that are multiplicatively independent.

Then there exists a polynomial $h$ such that $\gcd(A^k - I) | h$ for any $k$. If, in addition, $A$ is primitive, then $A^k$ is primitive for all $k$ outside a finite union of proper arithmetic progressions.

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2. Proof of Theorem 1. To prove the theorem, we will use a result which was conjectured by Serge Lang and proved by Ihara, Serre and Tate (see [L1] and [L2]), which states that the intersection of an irreducible curve in $\mathbb{C}^* \times \mathbb{C}^*$ with the roots of unity $\mu_\infty \times \mu_\infty$ is finite, unless the curve is of the form $X^nY^m - \zeta = 0$ or $X^m - \zeta Y^n = 0$ with $\zeta \in \mu_\infty$, that is, unless the curve is the translate of an algebraic subgroup by a torsion point of $\mathbb{C}^* \times \mathbb{C}^*$.

Applying this result to the rational curve $\{(f(t), g(t)) : t \in \mathbb{C}\}$, we conclude that only for finitely many $t$’s both $f(t)$ and $g(t)$ are roots of unity when $f$ and $g$ are multiplicatively independent.

Thus by Lang’s theorem there is only a finite set of points $S \subset \mathbb{C}$ such that for any $s \in S$ both $f(s)$ and $g(s)$ are roots of unity. So $\gcd(f^k - 1, g^k - 1)$ can only have linear factors from $\{t - s \mid s \in S\}$. Write

$$f^k - 1 = \prod_{j=0}^{k-1} (f - \zeta_j^k).$$

Any two factors on the right side are coprime, so $t - s$ can divide at most one of them with multiplicity at most $\deg(f)$, and a similar statement can be said for $g$. Therefore the required polynomial $h$ can be chosen as

$$h(t) = \prod_{s \in S} (t - s)^{\min(\deg(f), \deg(g))}.$$
For the second part of Theorem 1, let \( s \in S \) and let \( d_s \) be the least positive integer such that
\[
 t - s \mid \gcd(f(t)^{d_s} - 1, g(t)^{d_s} - 1).
\]
Then \( d_s > 1 \) because \( \gcd(f - 1, g - 1) = 1 \), and clearly for \( k \notin d_s \mathbb{N} \),
\[
 t - s \nmid \gcd(f(t)^k - 1, g(t)^k - 1).
\]
Then \( \bigcup_{s \in S} d_s \mathbb{N} \) is the required finite union of proper arithmetic progressions outside which \( \gcd(f_k - 1, g_k - 1) = 1 \).

Note that Theorem 3 implies Theorem 1. We have chosen to give the proof of Theorem 1 separately to illustrate the ideas in a simple context.

### 3. 2 × 2 matrices

Let \( A \in \text{SL}_2(\mathbb{Z}) \) be a 2 × 2 unimodular matrix which is hyperbolic, that is, \( A \) has two distinct real eigenvalues. We show:

**Proposition 4.** Let \( A \in \text{SL}_2(\mathbb{Z}) \) be a hyperbolic matrix with eigenvalues \( \varepsilon, \varepsilon^{-1} \), where \( |\varepsilon| > 1 \). Then \( \gcd(A^k - I) \gg |\varepsilon|^{k/2} \).

**Proof (1).** Let \( K \) be the real quadratic field \( \mathbb{Q}(\varepsilon) \) and \( \mathcal{O}_K \) its ring of integers. We may diagonalize the matrix \( A \) over \( K \), that is, write \( A = P \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} P^{-1} \) with \( P \) a 2 × 2 matrix having entries in \( K \). Since \( P \) is only determined up to a scalar multiple, we may, after multiplying \( P \) by an algebraic integer of \( \mathcal{O}_K \), assume that \( P \) has entries in \( \mathcal{O}_K \). Then \( P^{-1} = (1/\det(P))P^\text{ad} \) where \( P^\text{ad} \) also has entries in \( \mathcal{O}_K \).

Thus we have
\[
 A^k - I = \frac{1}{\det(P)} P \begin{pmatrix} \varepsilon^k - 1 & 0 \\ 0 & \varepsilon^{-k} - 1 \end{pmatrix} P^\text{ad}.
\]

The entries of \( A^k - I \) are thus \( \mathcal{O}_K \)-linear combinations of \( (\varepsilon^k - 1)/\det(P) \) and of \( (\varepsilon^{-k} - 1)/\det(P) \). We now note that
\[
 \varepsilon^{-k} - 1 = -\varepsilon^{-k}(\varepsilon^k - 1)
\]
and thus the entries of \( A^k - I \) are all \( \mathcal{O}_K \)-multiples of \( (\varepsilon^k - 1)/\det(P) \). In particular, \( \gcd(A^k - I) \), which is a \( \mathbb{Z} \)-linear combination of the entries of \( A^k - I \), can be written as
\[
 \gcd(A^k - I) = \frac{\varepsilon^k - 1}{\det(P)} \gamma_k
\]
with \( \gamma_k \in \mathcal{O}_K \).

Now taking norms from \( K \) to \( \mathbb{Q} \) we see that
\[
 |\gcd(A^k - I)|^2 = \frac{|N(\varepsilon^k - 1)|}{|N(\det(P))|} |N(\gamma_k)|.
\]

(1) We thank the referee for suggesting this proof, which replaces our original, more complicated, version.
Since $\gamma_k \neq 0$, we have $|\mathcal{N}(\gamma_k)| \geq 1$ and thus

$$|\gcd(A^k - I)|^2 \geq \frac{|\mathcal{N}(\epsilon^k - 1)|}{|\mathcal{N}(\det P)|} \gg \epsilon^k,$$

which gives $|\gcd(A^k - I)| \gg \epsilon^{k/2}$. □

A special case of this proposition appeared as a problem in the 54th W. L. Putnam Mathematical Competition, 1994 (see [An, pp. 82, 242]).

4. Cyclotomic fields. A standard construction of unimodular matrices is to take a unit $u$ of norm one in a number field $K$ and let it act by multiplication on the ring of integers $\mathcal{O}_K$ of $K$. This gives a linear map, and a choice of integer basis of $\mathcal{O}_K$ gives us an integer matrix whose determinant equals the norm of $u$ and is thus unimodular. We employ this method for the case when $u$ is a nonreal unit to give a construction of matrices $A$ with the property that $A^k$ is primitive infinitely often.

We recall some basic facts on units in a cyclotomic field. Let $p > 3$ be a prime, $\zeta_p$ a primitive $p$th root of unity, and $K = \mathbb{Q}(\zeta_p)$ the cyclotomic extension of the rationals. It is a field of degree $p - 1$. The ring of integers $\mathcal{O}_K$ of this field is $\mathbb{Z}[\zeta_p]$. Since $K$ is purely imaginary, it follows that the norm function is positive, and the norm of a unit $u$ is always 1. Also note that the structure of the unit group $E_p$ of $\mathcal{O}_K$ is

$$E_p = W_p E_p^+,$$

where $W_p$ are the roots of unity in $K$ and $E_p^+$ is the group of the real units in $\mathcal{O}_K$. A proof of this fact can be found, for example, in [L3, Theorem 4.1].

4.1. Proof of Theorem 2. We now prove Theorem 2, that is, show that if $u \in E_p \setminus E_p^+$ is a nonreal unit and $k \not\equiv 0 \mod p$ then the matrix corresponding to $u^k$ is primitive.

The method we will use is that if we choose a basis $\omega_0 = 1, \omega_1, \ldots, \omega_{p-2}$ of $\mathbb{Z}[\zeta_p]$ and take a unit $U$ in $\mathbb{Z}[\zeta_p]$, then we get a matrix $A(U) = (a_{i,j})$ whose entries are determined by

$$U \omega_i = \sum_{j=0}^{p-2} a_{j,i} \omega_j.$$

In particular if we find that in the expansion of

$$U = U \omega_0 = \sum_{j=0}^{p-2} a_{j,0} \omega_j$$

we have an index $j \neq 0$ so that $a_{j,0} = a_{0,0}$, then in the matrix $A(U) - I$ corresponding to $U - 1$, the first column will contain the entries $a_{0,0} - 1$ and $a_{j,0} = a_{0,0}$ which are clearly coprime, and thus the matrix $A(U)$ is primitive.
Another option is to have \( a_{0,0} = 0 \), in which case in the matrix of \( U - 1 \), the \((0,0)\) entry is \(-1\), and thus again \( A(U) \) is primitive. We will apply this method to the case that \( U = u^k \) is a power of a nonreal unit \( u \) and \( k \neq 0 \) mod \( p \).

Let \( u \in E_p \setminus E_p^+ \) be a nonreal unit. By (3), we can write

\[
u = \zeta_p^x u^+, \]

where \( u^+ \) is a real unit and \( x \) is an integer not congruent to 0 mod \( p \). Therefore, \( u^k = \zeta_p^x (u^+)^k \) and \( \zeta_p^{-x} u^k = (u^+)^k \) is real. Hence it can be represented as an integer combination of \( \zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1} \) as follows:

\[
\zeta_p^{-xk} u^k = \sum_{j=1}^{p-1} \alpha_j \zeta_p^j,
\]

where \( \alpha_j = \alpha_{p-j} \) for each \( j \). For convenience we will set \( \alpha_0 := 0 \).

Multiplying by \( \zeta_p^{xk} \), we find

\[
u^k = \sum_{j=0}^{p-1} \alpha_j \zeta_p^{j+xk}
\]

and changing the summation variable,

\[
u^k = \sum_{i=0}^{p-1} \alpha_i^{xk} \zeta_p^i,
\]

where the index of \( \alpha \) is calculated mod \( p \). Using the relation

\[
\zeta_p^{p-1} = -1 - \zeta_p - \ldots - \zeta_p^{p-2}
\]

we find that in terms of the integer basis \( \omega_j = \zeta_p^j, j = 0, \ldots, p-2 \), we have

\[
u^k = \sum_{i=0}^{p-2} (\alpha_i^{xk} - \alpha_{p-1-xk}) \omega_i.
\]

If \( k \neq 0 \) mod \( p \) then \( 2xk \neq 0 \) mod \( p \) since \( x \neq 0 \) mod \( p \). If \( 2xk \neq -1 \) mod \( p \) then the coefficients of \( \omega_0 \) and \( \omega_{2xk} \) are equal. Therefore \( u^k \) is primitive. If \( 2xk \) is congruent to \(-1\) mod \( p \), then the coefficient of \( \omega_0 \) vanishes and thus in this case as well, \( u^k \) is primitive.

Thus we have found that if \( k \neq 0 \) mod \( p \), the matrix corresponding to \( u^k \) is primitive. \( \blacksquare \)

Note that by virtue of (3), the eigenvalues of \( A(u) \) come in complex conjugate pairs whose ratios are \( p \)th roots of unity. This is somewhat similar to the trivial scalar example described in the introduction, namely \( b = \pm a \).
5. Proof of Theorem 3. We extend the idea of the proof of Theorem 1 to cover the matrix case. We first show that there is only a finite set $S$ of points $s \in \mathbb{C}$ such that $t - s$ divides $\gcd(A^k - I)$ for some $k$.

Let $M$ be a matrix such that $MAM^{-1}$ is in Jordan form. The elements of $M$ are meromorphic functions on the Riemann surface $R$ corresponding to some finite extension of $\mathbb{C}(t)$. Denote by $pr : R \to \mathbb{P}^1$ the associated projection of $R$ to the projective line. Let $S_0$ be the finite collection of poles of these functions.

Assume first that $A$ is not diagonalizable over the algebraic closure of $\mathbb{C}(t)$. Thus for any $t_0 \in R \setminus S_0$, $A(t_0)$ is not diagonalizable, and therefore $A(t_0)^k - I \neq 0$ for all $k$ (recall that a matrix of finite order ($A^m = I$) is automatically diagonalizable), in other words, $t - t_0$ does not divide $\gcd(A^k - I)$. Thus only the finitely many linear forms $t - s$, where $s \in pr(S_0)$ is the projection of some point in $S_0$, can divide $\gcd(A^k - I)$.

We denote by $\lambda_i(t)$ the eigenvalues of $A$ which are multivalued functions of $t$, that is, meromorphic functions on the Riemann surface. Assume now that $\lambda_1$ and $\lambda_2$ are multiplicatively independent, and that $A$ is diagonalizable. Suppose that $(t - t_0)|\gcd(A^k - I)$ for some $k > 1$ and $t_0 \in R \setminus S_0$. Then $A^k - I$ evaluated at $t_0$ is the zero matrix, and also

$$M(t_0)(A(t_0)^k - I)M(t_0)^{-1} = 0,$$

and we deduce that

$$\lambda_1(t_0)^k - 1 = \lambda_2(t_0)^k - 1 = 0.$$

In particular, $\lambda_1(t_0)$ and $\lambda_2(t_0)$ are roots of unity. Thus, we have reduced our task to proving that $\lambda_1$ and $\lambda_2$ can be simultaneous roots of unity only at a finite set of points.

To prove this, we want to use Lang’s theorem for the curve in $\mathbb{C}^2$ parameterized by $(\lambda_1(t), \lambda_2(t))$. Denote by $Y$ the Zariski closure of the image of the map $(\lambda_1, \lambda_2) : R \setminus S_0 \to \mathbb{C}^2$. Then $Y$ is an irreducible algebraic curve in $\mathbb{C}^2$. If $Y$ is of dimension 0, then it is a point, so $\lambda_1(t)$ and $\lambda_2(t)$ are constants, and since they are multiplicatively independent none of them can be a root of unity. Otherwise, we may apply Lang’s theorem for this curve to conclude that unless the curve $Y$ is of the form $F^m - \zeta G^n = 0$ or $F^mG^n = \zeta$ with $\zeta$ a root of unity (which is not the case when $\lambda_1$ and $\lambda_2$ are multiplicatively independent), it has only finitely many torsion points. In other words, there can only be finitely many points of the form $(\zeta_1, \zeta_2)$ on $Y$, where $\zeta_1$ and $\zeta_2$ are roots of unity.

We now prove that there is a polynomial $h$ such that $\gcd(A^k - I)$ divides $h$ for all $k$. Since there is a finite set $S$ of possible zeros of $\gcd(A^k - I)$, it suffices to show that the multiplicity of a zero of $\gcd(A^k - I)$ is bounded.

Write $B = MAM^{-1}$, so $B$ is in Jordan form. Denote by $v_{t_0}(f)$ the multiplicity of the zero of $f$ at $t_0 \in R$. So clearly, for any $t_0 \in R$ there exists
c(t_0) \in \mathbb{N} \text{ such that}
\begin{align*}
v_{t_0}(\gcd(A^k - I)) \leq c(t_0) + v_{t_0}(\gcd(B^k - I)),
\end{align*}
and for all \( t_0 \) outside the finite set \( S_0 \) of poles of entries of \( M \), \( c(t_0) = 0 \). So it suffices to prove that \( v_{t_0}(\gcd(B^k - I)) \) is bounded. Clearly,
\begin{align*}
gcd(B^k - I) | \det(B^k - I) = \prod_{j=0}^{k-1} \det(B - \zeta_k^j I),
\end{align*}
where \( \zeta_k \) is a primitive \( k \)th root of unity. Denoting the diagonal elements of \( B - I \) by \( b_1, \ldots, b_r \), we see that
\begin{align*}
\det(B^k - I) = \prod_{d=1}^r \prod_{j=0}^{k-1} (b_d - \zeta_k^j).
\end{align*}

Because a meromorphic function on a Riemann surface has a finite degree, reasoning as in the proof of Theorem 1 we see that for any \( t_0 \in R \), \( v_{t_0}(\prod_{j=1}^k (b_d - \zeta_k^j)) \) is bounded, for all \( k \). Therefore \( v_{t_0}(\det(B^k - I)) \) is bounded for all \( k \).

Now assume in addition that \( A \) is primitive: \( \gcd(A - I) = 1 \). For any \( s \in S \), the set of \( k \)'s such that \( A(s)^k = I \), i.e. \((t-s) | \gcd(A^k - I)\), is an arithmetic progression \( ds \mathbb{Z} \) which is proper since it does not contain 1. Therefore the set of \( k \)'s with \( \gcd(A^k - I) \neq 1 \) is a finite union of proper arithmetic progressions, and hence for \( k \) outside this union, we have \( \gcd(A^k - I) = 1 \). 

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