Shifted convolution and the Titchmarsh divisor problem over $\mathbb{F}_q[t]$

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In this paper, we solve a function field analogue of classical problems in analytic number theory, concerning the autocorrelations of divisor functions, in the limit of a large finite field.

1. Introduction

The goal of this paper is to study a function field analogue of classical problems in analytic number theory, concerning the autocorrelations of divisor functions. First, we review the problems over the integers $\mathbb{Z}$ and then we proceed to investigate the same problems over the rational function field $\mathbb{F}_q(t)$.

(a) The additive divisor problem over $\mathbb{Z}$

Let $d_k(n)$ be the number of representations of $n$ as a product of $k$ positive integers ($d_2$ is the standard divisor function). Several authors have studied the additive divisor problem (other names are ‘shifted divisor’ and ‘shifted convolution’), which is to get bounds, or asymptotics, for the sum

$$D_k(x; h) := \sum_{n \leq x} d_k(n)d_k(n + h), \quad (1.1)$$

where $h \neq 0$ is fixed for this discussion.

The case $k = 2$ (the ordinary divisor function) has a long history: Ingham [1] computed the leading term, and Estermann [2] gave an asymptotic expansion

$$\sum_{n \leq x} d_2(n)d_2(n + h) = xP_2(\log x; h) + O(x^{11/12}(\log x)^3), \quad (1.2)$$
where
\[ P_2(u; h) = \frac{1}{\xi(2)} \sigma_{-1}(h)u^2 + a_1(h)u + a_2(h) \]  
(1.3)

with
\[ \sigma_w(h) = \sum_{d|h} d^w \]  
(1.4)

and \( a_1(h) \) and \( a_2(h) \) are very complicated coefficients.

The size of the remainder term has great importance in applications for various problems in analytic number theory, in particular, the dependence on \( h \). See Deshouillers & Iwaniec [3] and Heath-Brown [4] for an improvement of the remainder term.

The higher divisor problem \( k \geq 3 \) is also of importance, in particular, in relation to computing the moments of the Riemann \( \zeta \)-function on the critical line [5,6]. It is conjectured that
\[ D_k(x; h) \sim xP_{2(k-1)}(\log x; h) \]  
as \( x \to \infty \),  
(1.5)

where \( P_{2(k-1)}(u; h) \) is a polynomial in \( u \) of degree \( 2(k-1) \), whose coefficients depend on \( h \) (and \( k \)). We can get good upper bounds on the additive divisor problem from results in sieve theory on sums of multiplicative functions evaluated at polynomials, for instance, such as those by Nair & Tenenbaum [7]. The conclusion is that for \( h \neq 0 \)
\[ \sum_{n \leq X} d_k(n)d_k(n + h) \ll X(\log X)^{2(k-1)}, \]  
(1.6)

and we believe this is the right order of magnitude. But even a conjectural description of the polynomials \( P_{2(k-1)}(u; h) \) is difficult to obtain (see §7, [5,6]).

A variant of the problem about the autocorrelation of the divisor function is to determine an asymptotic for the more general sum given by
\[ D_{k,r}(x; h) := \sum_{n \leq X} d_k(n)d_r(n + h). \]  
(1.7)

Asymptotics are known for the case \( (k, r) = (k, 2) \) for any positive integer \( k \geq 2 \): Linnik [8] showed
\[ D_{k,2}(x; 1) = \sum_{n \leq X} d_k(n)d_2(n + 1) = \frac{1}{(k-1)!} \prod_p \left( 1 - \frac{1}{p} \right)^{k-1} \left( 1 - \frac{1}{p} \right)^{k-1} x(\log x)^k + O(\log x)^{2k} \]  
(1.8)

Motohashi [9–11] gave an asymptotic expansion
\[ D_{k,2}(x; h) = x \sum_{j=0}^k f_{k,j}(h)(\log x)^j + O(\log x)^{k-1}), \]  
(1.9)

for all \( \varepsilon > 0 \), where the coefficients \( f_{k,j}(h) \) can in principle be explicitly computed. For an improvement in the \( O \) term, see Fouvry & Tenenbaum [12].

(b) The Titchmarsh divisor problem over \( \mathbb{Z} \)

A different problem involving the mean value of the divisor function is the Titchmarsh divisor problem. The problem is to understand the average behaviour of the number of divisors of a shifted
prime, that is, the asymptotics of the sum over primes
\[ \sum_{p \leq x} d_2(p + a), \] (1.10)
where \( a \neq 0 \) is a fixed integer, and \( x \to \infty \). Assuming the generalized Riemann hypothesis (GRH), Titchmarsh [13] showed that
\[ \sum_{p \leq x} d_2(p + a) \sim C_1 x \] (1.11)
with
\[ C_1 = \frac{\zeta(2) \zeta(3) \zeta(6)}{\prod_{p \mid a} \left( 1 - \frac{p}{p^2 - p + 1} \right)} , \] (1.12)
and this was proved unconditionally by Linnik [8].

Fouvry [14] and Bombieri et al. [15] gave a secondary term,
\[ \sum_{p \leq x} d_2(p + a) = C_1 x + C_2 \text{Li}(x) + O \left( \frac{x (\log x)^A}{q} \right) , \] (1.13)
for all \( A > 1 \) and
\[ C_2 = C_1 \left( \frac{\gamma}{k} - \sum_p \frac{\log p}{p^2 - p + 1} + \sum_{p \mid a} \frac{p^2 \log p}{p - 1)(p^2 - p + 1)} \right) , \] (1.14)
with \( \gamma \) being the Euler–Mascheroni constant and \( \text{Li}(x) \) the logarithmic integral function.

In the following sections, we study the additive divisor problem and the Titchmarsh divisor problem over \( \mathbb{F}_q[t] \), obtaining definitive analogues of the conjectures described above.

(c) The additive divisor problem over \( \mathbb{F}_q[t] \)

We denote by \( \mathcal{M}_n \) the set of monic polynomials in \( \mathbb{F}_q[t] \) of degree \( n \). Note that \#\( \mathcal{M}_n = q^n \).

The divisor function \( d_k(f) \) is the number of ways to write a monic polynomial \( f \) as a product of \( k \) monic polynomials:
\[ d_k(f) = \#(a_1, \ldots, a_k), \quad f = a_1 \cdot a_2 \cdots a_k, \] (1.15)
where it is allowed to have \( a_i = 1 \).

The mean value of \( d_k(f) \) has an exact formula (see lemma 2.2):
\[ \frac{1}{q^n} \sum_{f \in \mathcal{M}_n} d_k(f) = \binom{n + k - 1}{k - 1} . \] (1.16)

Note that \( \binom{n + k - 1}{k - 1} \) is a polynomial in \( n \) of degree \( k - 1 \) and leading coefficient \( 1/(k - 1)! \)! Our first goal is to study the autocorrelation of \( d_k \) in the limit \( q \to \infty \). We show:

**Theorem 1.1.** Fix \( n > 1 \). Then
\[ \frac{1}{q^n} \sum_{f \in \mathcal{M}_n} d_k(f) d_k(f + h) = \binom{n + k - 1}{k - 1^2} + O(q^{-1/2}) , \] (1.17)
uniformly for all \( 0 \neq h \in \mathbb{F}_q[t] \) of degree \( \deg(h) < n \), as \( q \to \infty \).

In light of (1.16), theorem 1.1 may be interpreted as the statement that \( d_k(f) \) and \( d_k(f + h) \) become independent in the limit \( q \to \infty \) as long as \( \deg(h) < n \).

To compare with conjecture (1.5) over \( \mathbb{Z}_n \), we note that \( \binom{n + k - 1}{k - 1^2} \) is a polynomial in \( n \) of degree \( 2(k - 1) \) with leading coefficient \( 1/[(k - 1)!]^2 \), in agreement with the conjecture (see §7b).
The case \( h = 0 \): As an aside, we note that the case \( h = 0 \) is of course dramatically different. Indeed one can show that

\[
\lim_{q \to \infty} \frac{1}{q^n} \sum_{f \in \mathcal{M}_n} d_k(f)^2 = \left( \frac{n + k^2 - 1}{k^2 - 1} \right)
\]

is a polynomial of degree \( k^2 - 1 \) in \( n \), rather than degree \( 2(k - 1) \) for non-zero shifts.

Our method in fact gives the more general result:

**Theorem 1.2.** Let \( \mathbf{k} = (k_1, \ldots, k_s) \) be a tuple of positive integers and \( \mathbf{h} = (h_1, \ldots, h_s) \) a tuple of distinct polynomials in \( \mathbb{F}_q[t] \). We let

\[
D_k(n; \mathbf{h}) = \sum_{f \in \mathcal{M}_n} d_k(f + h_1) \cdots d_k(f + h_s).
\]

Then, for fixed \( n > 1 \),

\[
\frac{1}{q^n} D_k(n; \mathbf{h}) = \prod_{i=1}^s \left( \frac{n + k_i - 1}{k_i - 1} \right) + O(q^{-1/2}),
\]

uniformly on all tuples \( \mathbf{h} = (h_1, h_2, \ldots, h_s) \) of distinct polynomials in \( \mathbb{F}_q[t] \) of degrees \( \deg(h_i) < n \) as \( q \to \infty \).

In particular, for \( \mathbf{k} = (2, k) \) we get

\[
\lim_{q \to \infty} \frac{1}{q^n} D_{2,k}(n; h) = (n + 1) \left( \frac{n + k - 1}{k - 1} \right) = \frac{1}{(k-1)!} \left( n^k + \frac{k^2 - k + 2}{2} n^{k-1} + \cdots \right),
\]

in agreement with (1.8).

**d) The Titchmarsh divisor problem over \( \mathbb{F}_q[t] \)**

Let \( \mathcal{P}_n \) be the set of monic irreducible polynomials in \( \mathbb{F}_q[t] \) of degree \( n \). By the Prime Polynomial Theorem, we have

\[
\pi_q(n) := \# \mathcal{P}_n = \frac{q^n}{n} + O \left( \frac{q^{n/2}}{n} \right).
\]

Our next result is a solution of the Titchmarsh divisor problem over \( \mathbb{F}_q[t] \) in the limit of large finite field.

**Theorem 1.3.** Fix \( n > 1 \). Then

\[
\frac{1}{\pi_q(n)} \sum_{P \in \mathcal{P}_n} d_k(P + \alpha) = \left( \frac{n + k - 1}{k - 1} \right) + O(q^{-1/2}),
\]

uniformly over all \( 0 \neq \alpha \in \mathbb{F}_q[t] \) of degree \( \deg(\alpha) < n \).

For the standard divisor function \( (k = 2) \), we find

\[
\sum_{P \in \mathcal{P}_n} d_2(P + \alpha) = q^n + \frac{q^n}{n} + O(q^{n-1/2}),
\]

which is analogous to (1.13) under the correspondence \( x \leftrightarrow q^n \) and \( \log x \leftrightarrow n \).

**e) Independence of cycle structure of shifted polynomials**

We conclude the introduction with a discussion on the connection between shifted polynomials and random permutations and state a result that lies behind the results stated above.
The cycle structure of a permutation \( \sigma \) of \( n \) letters is the partition \( \lambda(\sigma) = (\lambda_1, \ldots, \lambda_n) \) of \( n \) if, in the decomposition of \( \sigma \) as a product of disjoint cycles, there are \( \lambda_j \) cycles of length \( j \). Note that \( \lambda(\sigma) \) is a partition of \( n \) in the sense that \( \lambda_j \geq 0 \) and \( \sum_j \lambda_j = n \). For example, \( \lambda_1 \) is the number of fixed points of \( \sigma \) and \( \lambda_n = 1 \) if and only if \( \sigma \) is an \( n \)-cycle.

For each partition \( \lambda \vdash n \), the probability that a random permutation on \( n \) letters has cycle structure \( \sigma \) is given by Cauchy’s formula [16, ch. 1]:

\[
p(\lambda) = \frac{\#\{\sigma \in S_n : \lambda(\sigma) = \lambda\}}{\#S_n} = \prod_{j=1}^{n} \frac{1}{j!^{\lambda_j} \cdot \lambda_j!}.
\]  

(1.22)

For \( f \in \mathbb{F}_q[t] \) of positive degree \( n \), we say its cycle structure is \( \lambda(f) = (\lambda_1, \ldots, \lambda_n) \) if, in the prime decomposition \( f = \prod_{i} P_i \) (we allow repetition), we have \( \#(i : \deg(P_i) = j) = \lambda_j \). Thus, we get a partition of \( n \). In analogy with permutation, \( \lambda_1(f) \) is the number of roots of \( f \) in \( \mathbb{F}_q \) (with multiplicity) and \( f \) is irreducible if and only if \( \lambda_n(f) = 1 \).

For a partition \( \lambda \vdash n \), we let \( \chi_\lambda \) be the characteristic function of \( f \in \mathcal{M}_n \) of cycle structure \( \lambda \):

\[
\chi_\lambda(f) = \begin{cases} 1, & \lambda(f) = \lambda \\ 0, & \text{otherwise.} \end{cases}
\]

(1.23)

The Prime Polynomial Theorem gives the mean values of \( \chi_\lambda \):

\[
\frac{1}{q^n} \sum_{f \in \mathcal{M}_n} \chi_\lambda(f) = p(\lambda) + O(q^{-1}),
\]

(1.24)
as \( q \to \infty \) (see lemma 2.1). We prove independence of cycle structure of shifted polynomials:

**Theorem 1.4.** For fixed positive integers \( n \) and \( s \) we have

\[
\frac{1}{q^n} \sum_{f \in \mathcal{M}_n} \chi_{\lambda_1}(f + h_1) \cdots \chi_{\lambda_s}(f + h_s) = p(\lambda_1) \cdots p(\lambda_s) + O(q^{-1/2}),
\]

uniformly for all \( h_1, \ldots, h_s \) distinct polynomials in \( \mathbb{F}_q[t] \) of degrees \( \deg(h_i) < n \) and on all partitions \( \lambda_1, \ldots, \lambda_s \vdash n \) as \( q \to \infty \).

**Remark.** In this theorem, \( \lambda_1, \ldots, \lambda_s \) are partitions of \( n \) and are not the same as the \( \lambda_1, \ldots, \lambda_n \) that appear in the definition of \( \lambda(f) \) or \( \lambda(\sigma) \) where in that case the \( \lambda_i \) are the number of parts of length \( i \).

We note that the statistic of theorem 1.4 is induced from the statistics of the cycle structure of tuples of elements in the direct product \( S_n^s \) of \( s \) copies of the symmetric group on \( n \) letters \( S_n \). This plays a role in the proof, where we use that a certain Galois group is \( S_n^s \) [17], and we derive the statistic from an explicit Chebotarev theorem. Since we have not found the exact formulation that we need in the literature, we provide a proof in the appendix.

### 2. Mean values

For the reader’s convenience, we prove in this section some results for which we did not find a good reference. We define the **norm** of a non-zero polynomial \( f \in \mathbb{F}_q[t] \) to be \( |f| = q^{\deg(f)} \) and set \( |0| = 0 \).

We start by proving (1.24):

**Lemma 2.1.** If \( \lambda \vdash n \) is a partition of \( n \) and \( n \) is a fixed number then

\[
\frac{1}{q^n} \#\{f \in \mathcal{M}_n : \lambda(f) = \lambda\} = p(\lambda)(1 + O(q^{-1})),
\]

(2.1)
as \( q \to \infty \).
Proof. To see this, note that to get a monic polynomial with cycle structure $\lambda$, we pick any $\lambda_1$ primes of degree 1, $\lambda_2$ primes of degree 2 (irrespective of the choice of ordering), and multiply them together. Thus
\[
\#\{f \in M_n : \lambda(f) = \lambda\} = \prod_{j=1}^{n} \frac{\pi_A(j)^{\lambda_j}}{\lambda_j!} \left(1 + O\left(\frac{1}{q}\right)\right),
\] (2.2)
where $\pi_A(j)$ is the number of primes of degree $j$ in $A = \mathbb{F}_q[t]$. By the Prime Polynomial Theorem, $\pi_A(j) = \frac{q^j}{j} + O(q^{j-1}/j)$ whenever $j \geq 2$ and $\pi_A(1) = q$. Hence $\pi_A(j) = \frac{q^j}{j} + O(q^{j-1}/j)$. So
\[
\#\{f \in M_n : \lambda(f) = \lambda\} = \prod_{j=1}^{n} \frac{1}{\lambda_j!} \left(\frac{q^j}{j} + O\left(\frac{q^{j-1}}{j}\right)\right)^{\lambda_j}
\]
which by (1.22) gives (2.1).

Next, we prove (1.16):

Lemma 2.2. The mean value of $d_k(f)$ is
\[
\frac{1}{q^n} \sum_{f \in M_n} d_k(f) = \binom{n+k-1}{k-1}. \tag{2.4}
\]
Proof. The generating function for $d_k(f)$ is the $k$th power of the zeta function associated to the polynomial ring $\mathbb{F}_q[t]$:
\[
Z(u)^k = \sum_{f \text{ monic}} d_k(f) u^{\deg f} = \sum_{n=0}^{\infty} \sum_{f \in M_n} d_k(f) u^n. \tag{2.5}
\]
Here,
\[
Z(u) = \sum_{f \text{ monic}} u^{\deg f} = \sum_{n=0}^{\infty} q^n u^n = \frac{1}{1 - qu}. \tag{2.6}
\]
Using the Taylor expansion
\[
\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n \tag{2.7}
\]
and comparing the coefficients of $u^n$ in (2.5) gives
\[
q^n \binom{n+k-1}{k-1} = \sum_{f \in M_n} d_k(f), \tag{2.8}
\]
as needed.

3. Proof of theorem 1.4

In the course of the proof, we shall use the following explicit Chebotarev theorem, which is a special case of theorem A.4 of appendix A:

Theorem 3.1. Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of variables over $\mathbb{F}_q$, let $\mathcal{F}(t) \in \mathbb{F}_q[A][t]$ be monic, separable and of degree $m$ viewed as a polynomial in $t$, let $L$ be a splitting field of $\mathcal{F}$ over $K = \mathbb{F}_q(A)$, and let $G = \text{Gal}(\mathcal{F}, K) = \text{Gal}(L/K)$. Assume that $\mathbb{F}_q$ is algebraically closed in $L$. Then there exists a constant $c = c(n, \text{tot.deg}(\mathcal{F}))$ such that for every conjugacy class $C \subseteq G$ we have
\[
\left|\#\{a \in \mathbb{F}_q^n : \text{Fr}_a = C\} - \frac{|C|}{|G|} q^n\right| \leq cq^{n-1/2}.
\]
Here $\text{Fr}_a$ denotes the Frobenius conjugacy class $((S/R)/\phi)$ in $G$ associated to the homomorphism $\phi : R \to \mathbb{F}_q$ given by $A \mapsto a \in \mathbb{F}_q^n$, where $R = \mathbb{F}_q[A, \text{disc}F^{-1}]$ and $S$ is the integral closure of $R$ in the splitting field of $F$. See appendix A, in particular (A.11), for more details.

Let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of variables and set
\[
F_i(t) = T_i + A_1T_{i+1} + \cdots + A_n + h_i(T) \quad \text{and} \quad F = F_1 \cdots F_s,
\]
where the $h_i$ are distinct polynomials. Let $L$ be the splitting field of $F$ over $K = \mathbb{F}_q(A)$ and let $F$ be an algebraic closure of $\mathbb{F}_q$. By [17, Proposition 3.1],
\[
G := \text{Gal}(F, K) = \text{Gal}\left(\frac{L}{K}\right) = \text{Gal}\left(\frac{FL}{FK}\right) = S_n^r.
\]
In [17], it is assumed that $s$ is odd, but using [18] that restriction can now be removed for $n > 2$. This, in particular, implies that $L \cap F = \mathbb{F}_q$ (since the image of the restriction map $\text{Gal}(FL/\mathbb{F}_q) \to \text{Gal}(L/K)$ is $\text{Gal}(L/L \cap F)$, so by the above and Galois correspondence, $L \cap (\mathbb{F}K) = K$, and in particular $L \cap \mathbb{F} = K \cap \mathbb{F} = \mathbb{F}_q$). Hence, we may apply theorem 3.1 with the conjugacy class
\[
C = \{(\sigma_1, \ldots, \sigma_s) \in G : \lambda_i = \lambda_i\}
\]
to get that
\[
|\{a \in \mathbb{F}_q^n : \text{Fr}_a = C\} - |C|/|G| \cdot q^n| \leq c(s, n)q^{n-1}/2.
\]
Since $|C|/|G| = p(\lambda_1) \cdots p(\lambda_s)$ and since $|\{a \in \mathbb{F}_q^n : \text{disc}_F(a) = 0\} = O_{s,n}(q^{n-1})$, it remains to show that for $a \in \mathbb{F}_q^n$ with $\text{disc}_F(a) \neq 0$ we have $\text{Fr}_a = C$ if and only if $\lambda_1 = \lambda_i$ for all $i = 1, \ldots, s$.

And indeed, extend the specialization $A \mapsto a$ to a homomorphism $\Phi$ of $\mathbb{F}_q[A, Y]$ to $\mathbb{F}$, where $Y = (Y_{ij})$, and $Y_{ij}$, $i = 1, 2, \ldots, s$ are the roots of $F_i$. Then $\text{Fr}_a$ is, by definition, the conjugacy class of the Frobenius element $\text{Fr}_\Phi$ in $G$, which is defined by
\[
\Phi(\text{Fr}_\Phi(Y_{ij})) = \Phi(Y_{ij})^q.
\]
Note that $\text{Fr}_\Phi$ permutes the roots of each $F_i$ and hence can be identified with an $s$-tuple of permutations $\text{Fr}_\Phi = (\sigma_1, \ldots, \sigma_s) \in G = S_n^r$. Since the $\Phi(Y_{ij})$ are distinct, the cycle structure of $\sigma_i$ equals the cycle structure of the $\Phi(Y_{ij}) \mapsto \Phi(Y_{ij})^q, i = 1, \ldots, n$ by (3.2), which in turn equals the cycle structure of the polynomial $F_i(a, T)$. Hence $\text{Fr}_\Phi \in C$ if and only if $\lambda_{F_i(a, T)} = \lambda_i$ for all $i$, as needed.

## 4. Proof of theorem 1.1

First, we need the following lemma:

**Lemma 4.1.** Let $f \in \mathcal{M}_n$ and $h \in \mathcal{F}[t]$ such that $\deg(h) < n$. Then we have that
\[
\#\{f \in \mathcal{M}_n : f + h \text{ are square-free}\} = q^n + O(q^{n-1}).
\]

**Proof.** The number of square-free $f \in \mathcal{M}_n$ is $q^n - q^{n-1}$ for $n \geq 2$ (for $n = 1$ it is $q$), and since $n > \deg(h)$, as $f$ runs over all monic polynomials of degree $n$ so does $f + h$, and hence the number of $f \in \mathcal{M}_n$ such that $f + h$ is square-free is also $q^n - q^{n-1}$. Therefore, there are at most $2q^{n-1}$ monic $f \in \mathcal{M}_n$ for which at least one of $f$ and $f + h$ is not square-free, as claimed.

We denote by $\langle A \rangle$ the mean value of an arithmetic function $A$ over $\mathcal{M}_n$:
\[
\langle A \rangle := \frac{1}{q^n} \sum_{f \in \mathcal{M}_n} A(f).
\]

For this, it follows that if $A$ is an arithmetic function on $\mathcal{M}_n$ that is bounded independently of $q$, then
\[
\langle A \rangle = \frac{1}{q^n} \sum_{f \in \mathcal{M}_n \text{ and } f + h \text{ square-free}} A(f) + O(q^{n-1}).
\]
Now for square-free $f$, the divisor function $d_k(f)$ depends only on the cycle structure of $f$, namely
\[ d_k(f) = k^{\lambda(f)}, \quad (4.4) \]
where for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $n$, we denote by $|\lambda| = \sum \lambda_j$ the number of parts of $\lambda$. Therefore, we may apply (4.3) with (4.4) to get
\[ \langle d_k(\bullet) d_k(\bullet + h) \rangle = \left( k^{\lambda(\bullet)} k^{\lambda(\bullet + h)} \right) + O(q^{-1}). \quad (4.5) \]
Since the function $k^{\lambda(f)}$ depends only on the cycle structure of $f$, it follows from theorem 1.4 that
\[ \left( k^{\lambda(\bullet)} k^{\lambda(\bullet + h)} \right) = \left( k^{\lambda(\bullet)} \right)^2 + O(q^{-1/2}). \quad (4.6) \]
Applying again (4.3) with (4.4) together with lemma 2.2, we conclude that
\[ \langle k^{\lambda(\bullet)} \rangle = \langle d_k(\bullet) \rangle + O(q^{-1}) = \left( n + k - 1 \right) / \left( k - 1 \right) + O(q^{-1}). \quad (4.7) \]
Combining (4.5), (4.6) and (4.7) then gives the desired result.

5. Proof of theorem 1.2
We argue as in §4:
\[ \left( \prod_{i=1}^{s} d_{k_i}(\bullet + h_i) \right) = \left( \prod_{i=1}^{s} k_i^{\lambda_i(\bullet + h_i)} \right) + O(q^{-1}) \]
\[ = \prod_{i=1}^{s} \left( k_i^{\lambda_i(\bullet)} \right) + O(q^{-1/2}) \]
\[ = \prod_{i=1}^{s} \left( n + k_i - 1 \right) / \left( k_i - 1 \right) + O(q^{-1/2}). \]
(Here the first passage uses (4.3) with (4.4), the last also uses lemma 2.2, and the middle passage is done by invoking theorem 1.4.)

6. Proof of theorem 1.3
Let $1_{\mathcal{P}}$ be the characteristic function of the primes of degree $n$, i.e.
\[ 1_{\mathcal{P}}(f) = \chi(0,0,\ldots,0,1)(f) = \begin{cases} 1, & \text{if } f \in \mathcal{P}_n, \\ 0, & \text{otherwise.} \end{cases} \quad (6.1) \]
The Prime Polynomial Theorem gives that $\langle 1_{\mathcal{P}} \rangle = 1/n + O(q^{-1})$ and we have calculated in §4 that $\langle k^{\lambda(\bullet)} \rangle = \left( n + k - 1 \right) / \left( k - 1 \right) + O(q^{-1})$. Since these two functions clearly depend only on cycle structures (recall that $\alpha \neq 0$), theorem 1.4 gives
\[ \langle 1_{\mathcal{P}}(\bullet) \cdot k^{\lambda(\bullet)} \rangle = \langle 1_{\mathcal{P}}(\bullet) \rangle \langle k^{\lambda(\bullet)} \rangle = \frac{1}{n} \left( n + k - 1 \right) / \left( k - 1 \right) + O(q^{-1/2}). \quad (6.2) \]
Therefore,
\[
\frac{n}{q^n} \sum_{P \in \mathcal{P}_n} d_k(P + \alpha) = n \{1 + (k|\lambda|)^2\} = \left( \frac{n + k - 1}{k - 1} \right) + O(q^{-1/2}),
\]
as needed.

7. Comparing conjectures and our results

In this section, we check the compatibility of the theorems presented in §1c with the known results over the integers.

(a) Estermann’s theorem for $\mathbb{F}_q [t]$

First, we prove the function field analogue of Estermann’s result (1.2). For simplicity, we carry it out for $h = 1$.

**Theorem 7.1.** Assume that $n \geq 1$. Then
\[
\frac{1}{q^n} \sum_{f \in M_n} d_2(f)d_2(f + 1) = (n + 1)^2 - \frac{1}{q}(n - 1)^2.
\]

(Note that $q$ is fixed in this theorem).

We need two auxiliary lemmas before proving theorem 7.1.

Let $A, B \in \mathbb{F}_q [t]$ be monic polynomials. We want to count the number of monic polynomial solutions $(u, v) \in \mathbb{F}_q [t]^2$ of the linear Diophantine equation
\[
Au - Bv = 1, \quad \deg(Au) = n = \deg(Bv).
\]

As follows from the Euclidean algorithm, a necessary and sufficient condition for the equation $Au - Bv = 1$ to be solvable in $\mathbb{F}_q [t]$ is $\gcd(A, B) = 1$.

**Lemma 7.2.** Given monic polynomials $A, B \in \mathbb{F}_q [t]$, $\gcd(A, B) = 1$ and
\[
n \geq \deg(A) + \deg(B),
\]
then the set of monic solutions $(u, v)$ of (7.2) forms a non-empty affine subspace of dimension $n - \deg(A) - \deg(B)$, hence the number of solutions is exactly $q^n / |A||B|$.

**Proof.** We first ignore the degree condition. By the theory of the linear Diophantine equation, given a particular solution $(u_0, v_0) \in \mathbb{F}_q [t]^2$, all other solutions in $\mathbb{F}_q [t]^2$ are of the form
\[
(u_0, v_0) + k(B, A),
\]
where $k \in \mathbb{F}_q [t]$ runs over all polynomials.

Given $u_0$, we may replace it by $u_1 = u_0 + kB$ where $\deg(u_1) < \deg(B)$ (or is zero), so that we may assume that the particular solution satisfies
\[
\deg(u_0) < \deg(B).
\]

In that case, if $k \neq 0$ then
\[
\deg(u_0 + kB) = \deg(kB)
\]
and \( u_0 + kB \) is monic if and only if \( k \) is monic. Hence if \( k \neq 0 \), then
\[
\deg(u_0 + kB) = n - \deg(A) \quad \Leftrightarrow \quad \deg(kB) = n - \deg(A) - \deg(B).
\]

Thus, the set of solutions of (7.2) is in one-to-one correspondence with the space \( M_{n-\deg(A)-\deg(B)} \) of monic \( k \) of degree \( n - \deg(A) - \deg(B) \). In particular, the number of solutions is \( q^n/|A||B| \).

Let
\[
S(\alpha, \beta; \gamma, \delta) := \# \{ x \in M_{\alpha}, y \in M_{\beta}, z \in M_{\gamma}, u \in M_{\delta} : xy - zu = 1 \}. \tag{7.8}
\]

Then we have the following lemma.

**Lemma 7.3.** For \( \alpha + \beta = n = \gamma + \delta \),
\[
S(\alpha, \beta; \gamma, \delta) = q^n \times \begin{cases} 
1, & \text{if min}(\alpha, \beta; \gamma, \delta) = 0, \\
1 - \frac{1}{q}, & \text{otherwise}.
\end{cases} \tag{7.9}
\]

**Proof.** We have some obvious symmetries from the definition
\[
S(\alpha, \beta; \gamma, \delta) = S(\beta, \alpha; \gamma, \delta) = S(\alpha, \beta; \delta, \gamma), \tag{7.10}
\]
and hence to evaluate \( S(\alpha, \beta; \gamma, \delta) \) it suffices to assume
\[
\alpha \leq \beta, \quad \gamma \leq \delta. \tag{7.11}
\]
Assuming (7.11), we write
\[
S(\alpha, \beta; \gamma, \delta) = \sum_{x \in M_{\alpha}, z \in M_{\gamma}} \sum_{y \in M_{\beta}, u \in M_{\delta}} \# \{ y \in M_{\beta}, u \in M_{\delta} : xy - zu = 1 \}. \tag{7.12}
\]
Note that \( \alpha, \gamma \leq n/2 \) (since \( \alpha + \beta = n \) and \( \alpha \leq \beta \)) and hence \( \alpha + \gamma \leq \frac{1}{2}(\alpha + \beta + \gamma + \delta) = n \). Thus, we may use lemma 7.2 to deduce that
\[
\# \{ y \in M_{\beta}, u \in M_{\delta} : xy - zu = 1 \} = q^{n-\alpha-\gamma} \tag{7.13}
\]
and therefore
\[
S(\alpha, \beta; \gamma, \delta) = q^{n-\alpha-\gamma} \sum_{x \in M_{\alpha}, z \in M_{\gamma}} \sum_{\gcd(x, z) = 1} 1. \tag{7.14}
\]
Recall the Möbius inversion formula, which says that, for monic \( f \), \( \sum_{d \mid f} \mu(d) \) equals 1 if \( f = 1 \), and 0 otherwise. Hence, we may write the coprimality condition \( \gcd(x, z) = 1 \) using the Möbius function as
\[
\sum_{d \mid x, d \mid z} \mu(d) \begin{cases} 
1, & \gcd(x, z) = 1, \\
0, & \text{otherwise},
\end{cases} \tag{7.15}
\]
and therefore
\[ S(\alpha, \beta; \gamma, \delta) = q^{n - \alpha - \gamma} \sum_{x \in \mathcal{M}_\alpha} \sum_{d \in \mathcal{M}_\delta} \mu(d) \]
\[ = q^{n - \alpha - \gamma} \sum_{\deg(d) \leq \min(\alpha, \gamma)} \mu(d) \cdot \frac{q^\alpha}{|d|} \cdot \frac{q^\gamma}{|d|} \]
\[ = q^n \sum_{\deg(d) \leq \min(\alpha, \gamma)} \frac{\mu(d)}{|d|^2} \]
\[ = q^n \sum_{\deg(d) \leq \min(\alpha, \beta, \gamma, \delta)} \frac{\mu(d)}{|d|^2} \]  \hspace{1cm} (7.16)

where we have used the fact that \( \alpha \leq \beta \) and \( \gamma \leq \delta \).

We next claim that
\[ \sum_{\deg(d) \leq \eta} \frac{\mu(d)}{|d|^2} = \begin{cases} 1, & \eta = 0, \\ 1 - \frac{1}{q}, & \eta \geq 1, \end{cases} \]  \hspace{1cm} (7.17)

which when we insert into (7.16) proves the lemma.

To prove (7.17), we sum over \( d \) of fixed degree
\[ \sum_{\deg(d) \leq \eta} \frac{\mu(d)}{|d|^2} = \sum_{0 \leq \xi \leq \eta} \frac{1}{q^{2\xi}} \sum_{d \in \mathcal{M}_\xi} \mu(d) \]  \hspace{1cm} (7.18)

and recall that [19, ch. 2, exercise 12]
\[ \sum_{d \in \mathcal{M}_\xi} \mu(d) = \begin{cases} 1, & \xi = 0, \\ -q, & \xi = 1, \\ 0, & \xi \geq 2, \end{cases} \]  \hspace{1cm} (7.19)

from which (7.17) follows.

\[ \square \]

**Proof of theorem 7.1.** We write
\[ \nu := \sum_{f \in \mathcal{M}_n} d_2(f) d_2(f + 1) \]
\[ = \#(x, y, z, u) \in \mathbb{F}_q[t] \text{ monic : } xy - zu = 1, \text{ deg}(xy) = n = \text{deg}(zu)). \]  \hspace{1cm} (7.20)

We partition this into a sum over variables with fixed degree, that is
\[ \nu = \sum_{\alpha + \beta = n, \gamma + \delta = n, \alpha, \beta, \gamma, \delta \geq 0} S(\alpha, \beta; \gamma, \delta). \]  \hspace{1cm} (7.21)

We now input the results of lemma 7.3 into (7.21) to deduce that
\[ \nu = \sum_{\alpha + \beta = n, \gamma + \delta = n, \alpha, \beta, \gamma, \delta \geq 0} q^n \times \begin{cases} 1, & \min(\alpha, \beta; \gamma, \delta) = 0, \\ 1 - \frac{1}{q}, & \text{otherwise.} \end{cases} \]  \hspace{1cm} (7.22)
Of the $(n + 1)^2$ quadruples of non-negative integers $(\alpha, \beta; \gamma, \delta)$ so that $\alpha + \beta = n = \gamma + \delta$, there are exactly $4n$ tuples $(\alpha, \beta; \gamma, \delta)$ for which $\min(\alpha, \beta) = \min(\gamma, \delta)$, namely they are

$$(n, 0; n, 0), \quad (n, 0; n, 0), \quad (0, n; n, 0) \quad \text{and} \quad (0, n; 0, n)$$

(7.23)

and the $4(n - 1)$ tuples of the form

$$(n, 0; i, n - i), \quad (0, n; i, n - i), \quad (i, n - i; n, 0) \quad \text{and} \quad (i, n - i; 0, n)$$

(7.24)

for $0 < i < n$.

Concluding, we have

$$v = (4 + 4(n - 1)) \cdot q^n + [(n + 1)^2 - (4 + 4(n - 1))] \cdot q^n \left(1 - \frac{1}{q}\right)$$

$$= q^n \left( (n + 1)^2 - \frac{1}{q} (n - 1)^2 \right),$$

(7.25)

proving the theorem. $\blacksquare$

It is easy to check that theorem 1.1 is compatible with the function field analogue of Estermann’s result. Taking $q \to \infty$ in (7.1), we recover the same results as presented in (1.17) with $k = 2$.

(b) Higher divisor functions

Next, we want to check compatibility of our result in theorem 1.1 with what is conjectured over the integers. It is conjectured that

$$D_k(x; h) \sim x P_{2(k-1)}(\log x; h) \quad \text{as} \quad x \to \infty,$$

(7.26)

where $P_{2(k-1)}(u; h)$ is a polynomial in $u$ of degree $2(k - 1)$, whose coefficients depend on $h$ (and $k$). This conjecture appears in the work of Ivić [20] and Conrey & Gonek [5], and from their work, with some effort, we can explicitly write the conjectural leading coefficient for the desired polynomial. The conjecture over $\mathbb{Z}$ states that

$$P_{2(k-1)}(u; h) = \frac{1}{[(k - 1)!]^2} A_k(h) u^{2k-2} + \cdots,$$

(7.27)

where

$$A_k(h) = \sum_{m=1}^{\infty} \frac{c_m(h)}{m^2} C_{-k}(m)$$

(7.28)

with

$$C_{-k}(m) = m^{1-k} \prod_{d=1}^{m} \sum_{a_1=a_1}^{m} \sum_{a_k=a_k}^{m} e \left( \frac{h a_1 \cdots a_k}{m} \right),$$

(7.29)

where $e(x) = e^{2\pi ix}$ and $c_m(h)$ is the Ramanujan sum,

$$c_m(h) = \sum_{\substack{a=1 \\ (a, m) = 1}}^{m} e^{2\pi i a/m} h = \sum_{d|\gcd(m, h)} d \mu \left( \frac{m}{d} \right).$$

(7.30)

We now translate the conjecture above to the function field setting using the correspondence $x \leftrightarrow q^n$ and $\log x \leftrightarrow n$ and that summing over positive integers correspond to summing over monic polynomials in $\mathbb{F}_q[t]$. Under this correspondence, the function field analogue of the above polynomial is given in the following conjecture.
Conjecture 7.4. For $q$ fixed, let $0 \neq h \in \mathbb{F}_q[t]$. Then as $n \to \infty$,
\[ \sum_{f \in \mathcal{M}_n} d_k(f)d_k(f + h) \sim \frac{1}{[(k - 1)!]^2} A_{k,q}(h) q^m n^{2k-2}, \quad (7.31) \]
where
\[ A_{k,q}(h) = \sum_{m \in \mathbb{F}_q[t]} \frac{c_{m,q}(h)(\gcd(m,h))^{2(k-1)}}{|m|^{2(k-1)}} g_{k-1}(m/h), \quad (7.32) \]
where $|m| = q^{\deg(m)}$,
\[ g_{k-1}(f) = \#\{a_1, \ldots, a_{k-1} \text{ mod } f : a_1 \ldots a_{k-1} \equiv 0 \text{ mod } f\} \quad (7.33) \]
and
\[ c_{m,q}(h) = \sum_{d | \gcd(m,h)} |d| \mu \left( \frac{m}{d} \right) \quad (7.34) \]
is the Ramanujan sum over $\mathbb{F}_q[t]$. The sum above is over all monic polynomials $d \in \mathbb{F}_q[t]$, $\mu(f)$ is the Möbius function for $\mathbb{F}_q[t]$ and $\Phi(m)$ is the $\mathbb{F}_q[t]$ analogue for Euler’s totient function.

Remark 7.5. Note that
\[ C^2_{q,k}(m) = \frac{\gcd(m,h)^{2k-1}}{|m|^{2k-1}} g_{k-1} \left( \frac{m}{\gcd(m,h)} \right) \quad (7.35) \]
corresponds to $C_{k,q}(m)$ as given in (7.29).

Remark 7.6. Note that we establish this conjecture for $k = 2$ and $h = 1$ in theorem 7.1.

We now check that our theorem 1.1 is consistent with the conjecture (7.27) and (7.32) for the leading term of the polynomial $P_{2(k-1)}(u; h)$.

The polynomial given by theorem 1.1 is
\[ \binom{n + k - 1}{k - 1^2} = \frac{1}{[(k - 1)!]^2} n^{2k-1} + \ldots. \quad (7.36) \]

We wish to show that, as $q \to \infty$, $A_{k,q}(h)/(k - 1)!^2$ matches the leading coefficient of $(\binom{n + k - 1}{k - 1})^2$, that is
\[ \lim_{q \to \infty} A_{k,q}(h) = 1. \quad (7.37) \]

Indeed, from (7.34) we note that $|c_{m,q}(h)| = O(h(1)$, and it is easy to see that
\[ g_{k-1}(n) \leq n^{k-1} d(n)^{k-1} \ll |n|^{k-2+\epsilon}, \quad \forall \epsilon > 0. \quad (7.38) \]

Thus, we find
\[ A_{k,q}(h) = 1 + O \left( \sum_{m \in \mathcal{M}_{\deg(m)>0}} \frac{1}{|m|^{2-\epsilon}} \right). \quad (7.39) \]
The series in the $O$ term is a geometric series:
\[ \sum_{m \in \mathcal{M}_{\deg(m)>0}} \frac{1}{|m|^{2-\epsilon}} = \sum_{n=1}^{\infty} \frac{1}{q^{2(1-\epsilon)}} \frac{1}{\# \mathcal{M}_n} = \sum_{n=1}^{\infty} \frac{1}{q^{1(1-\epsilon)}} = \frac{1/q^{1-\epsilon}}{1 - 1/q^{1-\epsilon}} \quad (7.40) \]
and hence tends to 0 as $q \to \infty$, giving (7.37).

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Appendix A. An explicit Chebotarev theorem

We prove an explicit Chebotarev theorem for function fields over finite fields. This theorem is known to experts, cf. [21, Theorem 4.1], [22, Proposition 6.4.8] or [23, Theorem 9.7.10]. However, there it is not given explicitly with the uniformity that we need to use. Therefore, we provide a complete proof.

(a) Frobenius elements

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements and algebraic closure \( \mathbb{F} \). We denote by \( \text{Fr}_q \) the Frobenius automorphism \( x \mapsto x^q \).

Let \( R \) be an integrally closed finitely generated \( \mathbb{F}_q \)-algebra with fraction field \( K \), and let \( \mathcal{F} \in R[T] \) be a monic separable polynomial of degree \( \deg \mathcal{F} = m \) such that

\[
\text{disc} \mathcal{F} \in R^*
\]  

is invertible. Let \( Y = (Y_1, \ldots, Y_m) \) be the roots of \( \mathcal{F} \), and put

\[
S = R[Y], \quad L = K(Y) \quad \text{and} \quad G = \text{Gal} \left( \frac{L}{K} \right).
\]

We identify \( G \) with a subgroup of \( S_m \) via the action on \( Y_1, \ldots, Y_m \):

\[
g(Y_i) = Y_{g(i)}, \quad g \in G \leq S_m.
\]

By (A 1) and Cramer’s rule, \( S \) is the integral closure of \( R \) in \( L \) and \( S/R \) is unramified. In particular, the relative algebraic closure \( \mathbb{F}_q^\mu \) of \( \mathbb{F}_q \) in \( L \) is contained in \( S \). For each \( \nu \geq 0 \) we let

\[
G_\nu = \{ g \in G : g(x) = x^{q^\nu}, \forall x \in \mathbb{F}_q^\mu \},
\]

the preimage of \( \text{Fr}_q^\nu \) in \( G \) under the restriction map. Since \( \text{Gal}(\mathbb{F}_q^\nu/\mathbb{F}_q) \) is commutative, \( G_\nu \) is stable under conjugation.

For every \( \Phi \in \text{Hom}_{\mathbb{F}_q}(S, \mathbb{F}) \) with \( \Phi(R) = \mathbb{F}_q^\nu \) there exists a unique element in \( G \), which we call the Frobenius element and denote by

\[
\left[ \frac{S/R}{\Phi} \right] \in G,
\]

such that

\[
\Phi \left( \left[ \frac{S/R}{\Phi} \right] x \right) = \Phi(x)^{q^\nu}, \quad \forall x \in S.
\]

Since \( S \) is generated by \( Y \) over \( R \), it suffices to consider \( x \in \{Y_1, \ldots, Y_k\} \) in (A 5). If we further assume that \( \Phi \in \text{Hom}_{\mathbb{F}_q^\nu}(S, \mathbb{F}) \), then (A 5) gives that \( [S/R/\Phi]x = x^{q^\nu} \) for all \( x \in \mathbb{F}_q^\nu \), hence

\[
\Phi(R) = \mathbb{F}_q^\nu \quad \implies \quad \left[ \frac{S/R}{\Phi} \right] \in G_\nu.
\]

Lemma A.1. For every \( g \in S_m \) and \( \nu \geq 1 \) there exists \( V_{g,\nu} = (v_{ij}^\nu) \in \text{GL}_m(\mathbb{F}) \) such that \( \text{Fr}_q^\nu \) acts on the rows of \( V_{g,\nu} \) as \( g \) acts on \( Y \):

\[
v_{ij}^\nu = v_{g(i)j}.
\]

Proof. By replacing \( q \) by \( q^\nu \), we may assume without loss of generality that \( \nu = 1 \). By relabelling, we may assume without loss of generality that

\[
g = (s_1 \cdots e_1)(s_2 \cdots e_2) \cdots (s_k \cdots e_k),
\]

where \( s_1 = 1, s_{i+1} = e_i + 1 \) and \( e_k = m \).
Let $V$ be the block diagonal matrix

$$V = \begin{pmatrix} V_1 & & \\ & V_2 & \\ & & \ddots \\ & & & V_k \end{pmatrix},$$

where

$$V_i = \begin{pmatrix} 1 & \zeta_i & \cdots & \zeta_i^{\lambda_i - 1} \\ 1 & \zeta_i^q & \cdots & \zeta_i^{q(\lambda_i - 1)} \\ & \vdots & \ddots & \vdots \\ 1 & \zeta_i^{q^{\lambda_i - 1}} & \cdots & \zeta_i^{q^{\lambda_i - 1}(\lambda_i - 1)} \end{pmatrix}$$

is the Vandermonde matrix corresponding to an element $\zeta_i \in F$ of degree $\lambda_i = e_i - s_i$ over $F_q$. So $\det V_i = \prod_{1 \leq j < j' \leq \lambda_i} (\zeta_i^{q^{j'} - 1} - \zeta_i^{q^{j} - 1}) \neq 0$, hence $V$ is invertible, and by definition $\text{Fr}_q$ acts on the rows of $V$ as the permutation $g$. 

**Lemma A.2.** Let $\Phi : S \to F$ with $\Phi(R) = F_{q^r}$ and let $g \in G_v$. Then

$$\left[ \frac{S/R}{\Phi} \right] = g \iff V^{-1} \begin{pmatrix} \Phi(Y_1) \\ \vdots \\ \Phi(Y_m) \end{pmatrix} \in F_{q^r}^m,$$

where $V = V_{g,v}$ is the matrix from lemma A.1.

**Proof.** Let $z_1, \ldots, z_m \in F$ be the unique solution of the linear system

$$\Phi(Y_i) = \sum_{j=1}^m v_{ij} z_j, \quad i = 1, \ldots, m,$$

i.e.

$$\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = V^{-1} \begin{pmatrix} \Phi(Y_1) \\ \vdots \\ \Phi(Y_m) \end{pmatrix}.$$

If $z_i \in F_{q^r}$, i.e. $z_i^{q^r} = z_i$, we get by applying $\text{Fr}_{q^r}$ on (A 10) that

$$\Phi(Y_i)^{q^r} = \sum_{j=1}^m v_{ij}^{q^r} z_i = \sum_{j=1}^m v_{g(i)j} z_i = \Phi(Y_{g(i)}).$$

Hence $\left[ (S/R)/\Phi \right] = g$ by (A 5).

Conversely, if $\left[ (S/R)/\Phi \right] = g$, then $\Phi(Y_i)^{q^r} = \Phi(Y_{g(i)})$ by (A 2) and (A 5). We thus get that $\text{Fr}_{q^r}$ permutes the equations in (A 10), hence $\text{Fr}_{q^r}$ fixes the unique solution of (A 10). That is to say, $z_i^{q^r} = z_i$, as needed.
Next, we describe the dependence of the Frobenius element when varying the homomorphisms. For $\phi \in \text{Hom}_{\mathbb{F}_q}(R, \mathbb{F})$ we define

$$
\left( \frac{S/R}{\phi} \right) = \left\{ \left[ \frac{S/R}{\Phi} \right] : \phi \in \text{Hom}_{\mathbb{F}_q}(S, \mathbb{F}) \text{ prolongs } \phi \right\}.
$$

(A 11)

Unlike the case when working with ideals, this set is not a conjugacy class in $G$, as we fix the action on $\mathbb{F}_{q^v}$. However, as we will prove below, the group $G_0$ acts regularly on $(S/R)/\phi$ by conjugation. In particular, if $G_0 = G$, or equivalently if $L \cap \mathbb{F} = \mathbb{F}_q$ (with $\mathbb{F}$ denoting an algebraic closure of $\mathbb{F}_q$), then $((S/R)/\phi)$ is a conjugacy class.

To state the result formally, we recall that a group $\Gamma$ acts regularly on a set $\Omega$ if the action is free and transitive, i.e. for every $\omega_1, \omega_2 \in \Omega$ there exists a unique $\gamma \in \Gamma$ with $\gamma \omega_1 = \omega_2$.

**Lemma A.3.** Let $\phi \in \text{Hom}_{\mathbb{F}_q}(R, \mathbb{F})$ and let $H$ be the subset of $\text{Hom}_{\mathbb{F}_q}(S, \mathbb{F})$ consisting of all homomorphisms prolonging $\phi$. Assume that $\phi(R) = \mathbb{F}_{q^v}$.

1. The group $G_0$ defined in (A 3) acts regularly on $H$ by $g : \Phi \mapsto \Phi \circ g$.
2. For every $g \in G_0$ and $\Phi \in H$, we have

$$
\left[ \frac{S/R}{\phi \circ g} \right] = g^{-1} \left[ \frac{S/R}{\phi} \right] g.
$$

3. Let $\Phi \in H$, let $g = [S/R/\Phi]$, let $H_g = \{ \Psi \in H : [S/R/\Psi] = g \}$ and let $C_{G_0}(g)$ be the centralizer of $g$ in $G_0$. Then $C_{G_0}(g)$ acts regularly on $H_g$.
4. $\#H_g = \#G_0/\#C = \#G/\mu \cdot \#C$, where $C$ is the conjugacy class of $g$ in $G_0$.

**Proof.** We consider $G_0 \leq G$ as subgroups of $S_m$ via the action on $Y_1, \ldots, Y_m$. Let $g \in G_0$ and $\Phi \in H$. Then $g(x) = x$ and $\Phi(x) = x$, thus $\Phi \circ g(x) = x$, for all $x \in \mathbb{F}_{q^v}$. Thus, $\Phi \circ g \in H$. If $\Phi \circ g = \Phi$, then $\Phi(Y_{g(i)}) = \Phi(Y_i)$ for all $i$. Since $\text{disc} \mathbb{F} \in \mathbb{F}^*$ it follows that $\Phi(\text{disc} \mathbb{F}) \neq 0$, thus $\Phi$ maps $\{Y_1, \ldots, Y_m\}$ injectively onto $\{\Phi(Y_1), \ldots, \Phi(Y_m)\}$. We thus get that $Y_{g(i)} = Y_i$, hence $g$ is trivial. This proves that the action is free.

Next, we prove that the action is transitive. Let $\Phi, \Psi \in H$. Then $\ker \Phi$ and $\ker \Psi$ are prime ideals of $S$ that lie over the prime ideal $\ker \phi$ of $R$, hence over the prime $\ker \mathbb{F}_{q^v}$ of $R \mathbb{F}_{q^v}$. By [24, VII, 2.1], there exists $g_1 \in \text{Gal}(L/K\mathbb{F}_{q^v}) = G_0$ such that $\ker(\Phi \circ g_1^{-1}) = g_1 \ker \Phi = \ker \Psi$. Replace $\Phi$ by $\Phi \circ g_1^{-1}$ to assume without loss of generality that $\ker \Phi = \ker \Psi$. Hence $\Phi = \alpha \circ \Psi$, where $\alpha$ is an automorphism of the image $\Phi(S) = \Psi(S)$ that fixes both $R \mathbb{F}_{q^v}$ and $\Phi(R) = \mathbb{F}_{q^v}$. That is to say, $\alpha = \mathbb{F}_{q^v}^\rho$, where $\rho$ is a common multiple of $\nu$ and $\mu$. By (A 5)

$$
\Phi(x) = \Psi(x)^{\rho/\nu} = \Psi \left( \frac{S/R}{\Psi} \right)^{\rho/\nu} = \ldots = \Psi \left( \frac{S/R}{\Psi} \right)^{\rho/\nu} x,
$$

so $\Phi = \Psi \circ g$, where $g = [(S/R)/\Psi]^{\rho/\nu}$. Since, for $x \in \mathbb{F}_{q^v}$ we have $g(x) = x^{\rho/\nu}$ and $\mu \mid \rho$, we have $g(x) = x$, so $g \in G_0$. This finishes the proof of (1).

To see (2) note that

$$
\Phi \left( g \left[ \frac{S/R}{\phi \circ g} \right] x \right) = \Phi \circ g \left( \left[ \frac{S/R}{\phi \circ g} \right] x \right) = \Phi(g(x))^{\rho/\nu} = \Phi(\Phi(x))^{\rho/\nu} = \Phi \left( \left[ \frac{S/R}{\phi} \right] g x \right), \quad \text{for all } x \in S,
$$

so $g[(S/R)/\Phi \circ g] = [(S/R)/\Phi]|g$ (since $\Phi$ is unramified), as claimed.

The rest of the proof is immediate, as (3) follows immediately from (1) and (2), and (4) follows from (3).

By (A 6) and lemma A.3, it follows that if $\Phi(R) = \mathbb{F}_{q^v}$, then $((S/R)/\phi) \subseteq G_v$ is an orbit of the action of conjugation from $G_0$. 


Let $C \subseteq G$ be such an orbit, i.e. $C = C_g = (hgh^{-1} : h \in G_0)$, $g \in G_v$. Then $C \subseteq G_v$, since the latter is stable under conjugation (see after (A 3)). The explicit Chebotarev theorem gives the asymptotic value of the denominator of $(S/R)/\phi = C$:

$$P_{v,C} = \frac{\#\{\phi \in \Hom_{\mathbb{F}_q}(R, \mathbb{F}) : \phi(R) = \mathbb{F}_{q'}\text{ and } ((S/R)/\phi) = C\}}{\#\{\phi \in \Hom_{\mathbb{F}_q}(R, \mathbb{F}) : \phi(R) = \mathbb{F}_{q'}\}}.$$

**Theorem A.4.** Let $v \geq 1$, let $C \subseteq G_v$ be an orbit of the action of conjugation from $G_0$. Then

$$P_{v,C} = \frac{\#C}{\#G_v} + O_{\deg F, \text{cmp}(R)}(q^{-1/2}),$$

as $q \to \infty$.

We define cmp$(R)$ below.

Before proving this theorem, we need to recall the Lang–Weil estimates, which play a crucial role in the proof of the theorem and in particular give the asymptotic value of the denominator of $P_{v,C}$.

Let $U$ be a closed subvariety of $\mathbb{A}^n_{\mathbb{F}_q}$ that is geometrically irreducible. Lang–Weil estimates give that

$$\#U(\mathbb{F}_q) = q^{\dim U} + O_{n, \text{deg } U}(q^{\dim U - 1/2}).$$

(A 12)

Note that both $n$ and $\deg U$ are stable under base change. This may be reformulated in terms of $\mathbb{F}_q$-algebras, to say that if

$$R \cong \mathbb{F}_q[X_1, \ldots, X_n, f_0^{-1}]/(f_1, \ldots, f_k)$$

(A 13)

then

$$\#\{\phi \in \Hom_{\mathbb{F}_q}(R, \mathbb{F}) : \phi(R) = \mathbb{F}_{q'}\} = q^{\dim R} + O_{\text{cmp}(R)}(q^{\dim R - 1/2}),$$

(A 14)

provided $R \otimes \mathbb{F}$ is a domain, where cmp$(R)$ is a function of $\sum \deg f_i$ and $n$, taking minimum over all presentations (A 13). By the remark following (A 12), it follows that if two $\mathbb{F}_q$-algebras $S$ and $S'$ become isomorphic over $\mathbb{F}$, then cmp$(S')$ is bounded in terms of cmp$(S)$. A final property needed is that if $R \to S$ is a finite map of degree $d$, then cmp$(S)$ is bounded in terms of cmp$(R)$ and $d$.

**Proof.** Let $g \in C$, let $V = V_{g,v}$ be as in (A 7) and let $S' = R[Z]$, where $Z = V^{-1}Y$. Note that $Z$ is the unique solution of the linear system

$$Y_i = \sum_{j=1}^n w_{ij}Z_j, \quad i = 1, \ldots, n.$$  

(A 15)

Let $N = \#\Hom_{\mathbb{F}_q}(S', \mathbb{F}_{q'})$. By (A 9), the number of $\Phi \in \Hom_{\mathbb{F}_q}(S, \mathbb{F})$ with $[(S/R)/\Phi] = g$ equals $N$. By lemma A.3, for each $\phi$ there exist exactly $\#G_0/\#C$ homomorphisms $\Phi \in \Hom_{\mathbb{F}_q}(S, \mathbb{F})$ with $[(S/R)/\Phi] = g$ prolonging $\phi$. Hence,

$$\#\left\{ \phi \in \Hom_{\mathbb{F}_q}(R, \mathbb{F}) : \phi(R) = \mathbb{F}_{q'} \text{ and } \left( \frac{S/R}{\phi} \right) = C \right\} = \#C/\#G_0 \cdot N.$$  

Since $G_v$ is a coset of $G_0$, $\#G_0 = \#G_v$. Hence, it suffices to prove that $N = q^{\dim R} + O_{\deg F, \text{deg } F}(q^{\dim R - 1/2})$. As $R \to S'$ is a finite map of degree $\deg F$, we get that $\dim R = \dim S'$ and cmp$(S')$ is bounded in terms of cmp$(R)$ and $\deg F$. It suffices to show that $S' \cap \mathbb{F} \subseteq \mathbb{F}_{q'}$ since then by (A 14) we have

$$N = q^{\dim S'} + O_{\deg F}(q^{\dim S' - 1/2}) = q^{\dim R} + O_{\deg F}(q^{\dim R - 1/2}),$$

and the proof is done.

Let $L$ be the fraction field of $S$ and $K$ of $R$. Since $L/K$ is Galois and $L \cap \mathbb{F} = \mathbb{F}_{q'}$, it follows that there exists an automorphism $\tau$ of $L\mathbb{F}$ such that $\tau|_L = g$ and $\tau|_{\mathbb{F}} = \text{Fr}_{q'}$. By (A 7) $\tau$ permutes the equations (A 15), hence fixes $Z$ and thus $S'$. In particular, if $x \in S' \cap \mathbb{F}$, then $x^{\mathbb{F}} = \tau(x) = x$, so $x \in \mathbb{F}_{q'}$, as was needed to complete the proof. ■
References


Correction to ‘Shifted convolution and the Titchmarsh divisor problem over \( \mathbb{F}_q[t] \)’

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Two of the equations in the above article contained a typographical error.

Equation (1.17) should read as follows:

\[
\frac{1}{q^n} \sum_{f \in M_n} d_k(f)d_k(f + h) = \binom{n+k-1}{k-1}^2 + O(q^{-1/2}). \quad (1.17)
\]

Equation (7.36) should read as follows:

\[
\binom{n+k-1}{k-1}^2 = \frac{1}{[(k-1)!]^2} q^{2(k-1)} + \cdots. \quad (7.36)
\]