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Hardy-Littlewood varieties and semisimple groups

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Summary. We are interested in counting integer and rational points in affine algebraic varieties, also under congruence conditions. We introduce the notions of a strongly Hardy-Littlewood variety and a relatively Hardy-Littlewood variety, in terms of counting rational points satisfying congruence conditions. The definition of a strongly Hardy-Littlewood variety is given in such a way that varieties for which the Hardy-Littlewood circle method is applicable are strongly Hardy-Littlewood.

We prove that certain affine homogeneous spaces of semisimple groups are strongly Hardy-Littlewood varieties. Moreover, we prove that many homogeneous spaces are relatively Hardy-Littlewood, but not strongly Hardy-Littlewood. This yields a new class of varieties for with the asymptotic density of integer points can be computed in terms of a product of local densities.

Introduction

Let X be an affine variety defined by polynomials with integer coefficients:

$$X = \{x \in \mathbf{C}^n : f_i(x) = 0, \quad i = 1, \dots, r\} \tag{0.0.1}$$

where $f_i \in \mathbf{Z}[X_1, \dots, X_n]$. For a euclidean norm $|\cdot|$ on $\mathbf{R}^n$, set

$$N(T,X) = |\{x \in X(\mathbf{Z}) : |x| \leqq T\}|$$

where $X(\mathbf{Z}) = X(\mathbf{C}) \cap \mathbf{Z}^n$. A basic problem of Diophantine analysis is to investigate the distribution of integer points in X, that is, the asymptotic behavior of $N(T,X)$ as $T \to \infty$.

In certain cases one can approximate $N(T,X)$ by a product of local densities. For simplicity, assume that

$$\operatorname{rank}\frac{\partial f_i}{\partial x_j} = r \tag{0.0.2}$$

everywhere on X (hence X is non-singular and $\dim X = n - r$). For a prime number p, set

$$\mu_p(X) = \lim_{k\to\infty} \frac{|\{x \in (\mathbf{Z}/p^k\mathbf{Z})^n : f_i(x) \equiv 0 \bmod p^k\}|}{p^{k \dim X}} . \tag{0.0.3}$$

We define the product $\mathfrak{S}(X) = \prod_p \mu_p(X)$, assuming that it is at least conditionally convergent; in the classical theory it is called the singular series. We also define the Hardy-Littlewood density at infinity *à la* Siegel, by

$$\mu_\infty(X, T) = \lim_{\varepsilon\to 0} \frac{\operatorname{vol}\{x \in \mathbf{R}^n : |x| \leqq T,\ |f_i(x)| < \varepsilon/2, \quad i = 1, \ldots, r\}}{\varepsilon^r} . \tag{0.0.4}$$

In the classical theory $\mu_\infty(X, T)$ is called the singular integral. In certain cases it is possible to prove that the counting function $N(T, X)$ behaves as the product of the singular series and the singular integral, i.e.

$$N(T, X) \sim \mathfrak{S}(X)\mu_\infty(X, T) \quad \text{as } T \to \infty , \tag{0.0.5}$$

where $\sim$ means that $N(T, X)$ is identically zero if the right hand side of (0.0.5) is identically zero, otherwise $\lim_{T\to\infty} N(T, X)/\mathfrak{S}(X)\mu_\infty(X, T) = 1$.

In the adelic setting, let X be a non-singular geometrically irreducible affine variety over $\mathbf{Q}$ embedded into a vector space W over $\mathbf{Q}$ with a norm $|\cdot|$. We assume that there exists a gauge form on X, i.e. a nowhere zero regular differential form ω of maximal degree. With ω one can associate p-adic measures m_p on $X(\mathbf{Q}_p)$ for any p, and a measure at infinity m_∞ on $X(\mathbf{R})$, cf. [We2]. Then one defines the Tamagawa measure $m = m_\infty \times \prod_p m_p$ on the set $X(\mathbf{A})$ of adelic points of X (convergence factors are necessary when the product is not absolutely convergent). We have $m = m_\infty \times m_f$, where m_f is a measure on the set of finite-adelic points $X(\mathbf{A}_f)$.

We wish to consider the density of integer points, or, more generally, rational points on X, subject to congruence conditions. We assume that all the connected components of $X(\mathbf{R})$ are non-compact. Choose an open compact subset $B_f \subset X(\mathbf{A}_f)$ and a connected component B_∞ of $X(\mathbf{R})$. Write $B_\infty^T = \{x \in B_\infty : |x| \leqq T\}$. We regard the product $B = B_\infty \times B_f$ as a kind of congruence condition on the $\mathbf{Q}$-points of X, and are interested in the counting function

$$N(X, T; B) = |X(\mathbf{Q}) \cap (B_\infty^T \times B_f)| .$$

In certain cases one can expect that this counting function grows approximately as the Tamagawa measure of $B_\infty^T \times B_f$. We will say that X is *a strongly Hardy-Littlewood variety* if for any $B = B_\infty \times B_f$ as above,

$$N(T, X; B) \sim m(B_\infty^T \times B_f) \quad \text{as } T \to \infty.$$

Thus we regard $m(B_\infty^T \times B_f)$ as the Hardy-Littlewood expectation of the number of rational points in $B_\infty^T \times B_f$. Note that $m(B_\infty^T \times B_f) = m_f(B_f)\, m_\infty(B_\infty^T)$, and that $m_f(B_f)$ does not depend on T.

The property of being a strongly Hardy-Littlewood variety is strong indeed. In particular it implies that X has strong approximation property, i.e $X(\mathbf{Q})$ is dense in $X(\mathbf{A}_f)$. It follows that for strongly Hardy-Littlewood varieties the following local-to-global principle holds: if $X(\mathbf{Z}_p) \neq \varnothing$ for all primes p, then $X(\mathbf{Z})$ is non-empty. The term "strongly Hardy-Littlewood variety" is chosen, because such results are usually proven by the Hardy-Littlewood circle method.

We say that X is *a (relatively) Hardy-Littlewood variety* if there exists a locally constant non-negative function $\delta : X(\mathbf{A}) \to \mathbf{R}$, constant on connected components B_∞ of $X(\mathbf{R})$ and not identically zero, such that for any $B = B_\infty \times B_f$ as above,

$$N(T,X;B) \sim \int_{B_\infty^T \times B_f} \delta(x)\,dm \quad \text{as } T \to \infty.$$

We call $\delta(x)$ the (relative) density function.

For a variety X as in (0.0.2), we have $m_f(X(\hat{\mathbf{Z}})) = \mathfrak{S}(X)$, see 1.8.3. We show in section 2 that if X is strongly Hardy-Littlewood, then (0.0.5) holds. If X is relatively Hardy Littlewood with relative density δ, then for any connected component B_∞ of $X(\mathbf{R})$ we have

$$|\{x \in X(\mathbf{Z}) \cap B_\infty : |x| \leqq T\}| \sim \int_{X(\hat{\mathbf{Z}})} \delta(B_\infty \times x)dx \cdot m_\infty(B_\infty^T)\,. \qquad (0.0.6)$$

Our definitions seem to depend on the choice of the gauge form ω. However in Section 2 we prove

Proposition 0.1. *If a variety X is Hardy-Littlewood with respect to a gauge form ω, then X is simply connected and any gauge form on X is of the form $\lambda\omega$ where $\lambda \in \mathbf{Q}^\times$.*

Now it follows from the product formula [We2] that the Hardy-Littlewood expectation (i.e. the Tamagawa measure) is defined uniquely, and so the property to be a strongly or relatively Hardy-Littlewood variety does not depend on the choice of a gauge form ω.

By the circle method, certain varieties as in (0.0.2) were proved to be strongly Hardy-Littlewood or at least to satisfy (0.0.5), see [Bi], [Sch], and also [Ig], [Pa], [FMT] and references therein. In most cases, the circle method can be applied when there are many variables relative to the number and degrees of the equations (the title of [Bi] is indicative). See also [HB] for a heuristic investigation of the density of integer points in a case that is beyond the range of the circle method.

Our goal in this paper is to investigate the distribution of integer points in affine homogeneous spaces of semisimple groups. For a large family of homogeneous varieties we prove that they are relatively Hardy-Littlewood and compute the relative density function. Some of these varieties are strongly Hardy-Littlewood (and therefore the density function is identically 1). The others are not strongly Hardy-Littlewood, and the density function turns out to take exactly two values, zero and a positive integer. Examples are the variety of all $n \times n$ matrices with given determinant, the variety of all $n \times n$ symmetric matrices with given determinant, or the variety of all $n \times n$ matrices with given irreducible characteristic polynomial (see Section 6 for details and other examples). It seems that Hardy-Littlewood varieties which are not *strongly* Hardy-Littlewood are beyond the range of the circle method.

We prove our results under following assumptions:

(0.2.1) Let G be a simply connected semisimple linear algebraic group, defined over $\mathbf{Q}$, without $\mathbf{Q}$-factors which are compact over $\mathbf{R}$. Assume that we are given a $\mathbf{Q}$-rational linear action (not necessarily effective) of G on a finite dimensional vector space W defined over $\mathbf{Q}$. Let X be a Zariski-closed orbit of G in W, defined over $\mathbf{Q}$. Assume that X has a $\mathbf{Q}$-rational point x_0, and that the stabilizer H of x_0 is connected and has no non-trivial $\mathbf{Q}$-characters.

(0.2.2). We assume that for any congruence subgroup $\Gamma \subset G(\mathbf{Q})$ and any rational point $x \in X(\mathbf{Q})$ the following asymptotic count holds:

$$|\{y \in x\Gamma : |y| \leqq T\}| \sim \frac{\mathrm{vol}(\Gamma \cap H_x \setminus H_x(\mathbf{R}))}{\mathrm{vol}(\Gamma \setminus G(\mathbf{R}))} m_\infty(B_\infty^T) \quad \text{as } T \to \infty ,$$

where H_x is the stabilizer of x in G, B_∞ is the $G(\mathbf{R})$-orbit of x, and the volumes are computed with respect to a compatible choice of measures in $G(\mathbf{R})$, $X(\mathbf{R})$ and $H_x(\mathbf{R})$.

Theorem 0.3. *Let X, G, H be as in* (0.2.1) *and* (0.2.2). *If H is simply connected, then X is strongly Hardy-Littlewood.*

Remark 0.3.1. Here we *assume* that X has a $\mathbf{Q}$-rational point x_0 (see [Bo4], [Bo5] for a Hasse principle for X which can be used to prove the existence of a rational point in X). We show in section 2 that if H is non-connected, then X cannot be Hardy-Littlewood, and therefore we assume in Theorem 0.3 that H is connected. If H is connected and has non-trivial $\mathbf{Q}$-characters, the "singular series" diverges, and so we assume that H has no non-trivial $\mathbf{Q}$-characters.

Remark 0.3.2. Assumption (0.2.2) was proved in [DRS] and [EM] for certain cases, in particular when X is a symmetric space, i.e. H is the fixed point set of an involution of G. In [EMS] significant progress is made in extending the range of cases where (0.2.2) is known to hold.

Corollory 0.3.3. *If X is a symmetric space of G and H is simply connected, then X is strongly Hardy-Littlewood.*

If H is connected but not necessarily simply connected, we prove that X is relatively Hardy–Littlewood. We will describe the density function below, after we introduce some notation and outline the proof.

The idea of the proof is the following. For a connected component B_∞ of X and an open subset B_f of $X(\mathbf{A})$, choose an open compact subgroup $K_f \subset G(\mathbf{A}_f)$ stabilizing B_f. Set $\Gamma = G(\mathbf{Q}) \cap (G(\mathbf{R}) \times K_f)$. Then Γ is a congruence subgroup of $G(\mathbf{Q})$, and by [B-HC], 6.9, the number of orbits of Γ

in $X(\mathbf{Q})\cap(B_\infty\times B_f)$ is finite. Let $x_1,\dots,x_h\in X(\mathbf{Q})\cap B$ be representatives of these orbits, and $H_1,\dots,H_h$ their stabilizers. Using Assumption (0.2.2), we can count separately the number of points in each Γ-orbit. After summing over the orbits, we obtain

$$N(T,X;B)\sim\sum_{i=1}^{h}\frac{\operatorname{vol}(\Gamma\cap H_i\setminus H_i(\mathbf{R}))}{\operatorname{vol}(\Gamma\setminus G(\mathbf{R}))}m_\infty(B_\infty^T)\quad\text{as } T\to\infty\,. \tag{0.3.4}$$

The sum in the right hand side above is a weighted sum of the same kind as that appearing in Siegel's weight formula [Si1], [Si2]. Our task is to compute this sum.

Here a finite group $C(H)$ comes into play. It is the the dual group to the Picard group $\operatorname{Pic}H$ of H. Kottwitz [Ko2] denotes it $A(H)$ and defines in terms of the Langlands dual group to H. We prefer to define it in terms of the algebraic fundamental group $\pi_1(H)$, which is a finitely generated abelian group with an action of the Galois group $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, see Section 3 for details. We set

$$C(H)=(\pi_1(H)_{\mathrm{Gal}})_{\mathrm{tors}},$$

the torsion subgroup of the group of coinvariants of $\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ in $\pi_1(H)$.

Assume that $B=B_\infty\times B_f$ is contained in one orbit $\mathcal{O}_A$ of $G(\mathbf{A})$ in $X(\mathbf{A})$. If $\mathcal{O}_A$ contains no rational points, then the sum in (0.3.4) is zero. So we need a criterion to determine whether $\mathcal{O}_A$ contains rational points. In section 3 we contruct *the Kottwitz invariant* $\kappa(\mathcal{O}_A)\in C(H)$ of an adelic orbit $\mathcal{O}_A=B_\infty\times\prod\mathcal{O}_p$ in terms of local invariants of orbits B_∞ and $\mathcal{O}_p$. It generalizes the product of the local Hasse-Minkowski invariants of an adelic quadratic form [Ca], and the Kottwitz invariant of an adelic conjugacy class in a reductive group [Ko2]. As in those cases, we have

Theorem 0.4. *$\mathcal{O}_A$ has a rational point if and only if $\kappa(\mathcal{O}_A)=0$.*

Now assume that $B=B_\infty\times B_f$ is contained in an orbit $\mathcal{O}_A$ containing rational points. In Section 4 we calculate the sum in (0.3.4), using the method of Tamagawa and Weil [We1] and calculations of the Tamagawa number of an algebraic group due to Ono, Sansuc and Kottwitz. We prove

Theorem 0.5. *If $B=B_\infty\times B_f$ is contained in an orbit $\mathcal{O}_A$ of $G(\mathbf{A})$ which contains rational points, then*

$$\sum_{i=1}^{h}\frac{\operatorname{vol}(\Gamma\cap H_i(\mathbf{R})\setminus H_i(\mathbf{R}))}{\operatorname{vol}(\Gamma\setminus G(\mathbf{R}))}=|C(H)|\cdot m_f(B_f)\,.$$

In section 5 we put all this together and prove

Theorem 0.6. *Let X,G, and H be as in* (0.2.1) *and* (0.2.2). *Then X is a Hardy-Littlewood variety with relative density function*

$$\delta(x)=\begin{cases}|C(H)|, & \kappa(x)=0\\ 0, & \kappa(x)\neq 0\end{cases}$$

where we regard the Kottwitz invariant $\kappa(\mathcal{O}_A)$ of an adelic orbit as a locally constant function on $X(\mathbf{A})$.

We discuss several examples in Section 6.

1. Gauge forms and Tamagawa measures

1.1. *Notation.* We will use the following notation throughout the paper.

If not otherwise stated, k is a field of characteristic zero. We write $\bar{k}$ for a fixed algebraic closure of k. When k is a number field, we write $\mathcal{V} = \mathcal{V}(k)$, $\mathcal{V}_f, \mathcal{V}_\infty$ for the sets of all the places of k, the finite (non-archimedean) places, the infinite (archimedean) places, respectively. For $v \in \mathcal{V}$, let k_v denote the completion of k at $v \in \mathcal{V}(k)$. Let $\mathfrak{o}_v$ denote the ring of integers of k_v, and $\mathfrak{p}_v$ denote the maximal ideal in $\mathfrak{o}_v$. We write $k(v)$ for the residue field $\mathfrak{o}_v/\mathfrak{p}_v$ and q_v for its order.

Let $\mathbf{A}, \mathbf{A}_f$ denote the ring of adeles of k and the ring of finite adeles (i.e. without archimedean components), respectively. Set $k_\infty = \prod_\infty k_v$ (here and in the sequel we write $\prod_\infty$ for $\prod_{v \in \mathcal{V}_\infty}$). We have $\mathbf{A} = k_\infty \times \mathbf{A}_f$. Note that if $k = \mathbf{Q}$ then $k_\infty = \mathbf{R}$.

By an algebraic variety we mean a geometrically irreducible variety. If not otherwise stated, all varieties are assumed to be non-singular.

For any k-group G we write $X^*(G) = \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_{m\bar{k}})$, where $\mathbb{G}_m$ is the multiplicative group. For a k-torus T we set $X_*(T) = \mathrm{Hom}(\mathbb{G}_{m_{\bar{k}}}, T_{\bar{k}})$.

1.2. Let X be a geometrically irreducible non-singular algebraic variety over a field k of characteristic 0. A gauge form on X is a nowhere zero regular differential form of degree $\dim X$.

In general, on an algebraic variety there may be no gauge forms. For example, there are no gauge forms on the projective line $\mathbb{P}^1$. Moreover, there exist *affine* varieties without gauge forms. For example, on a curve of genus $g \geq 2$ with a puncture at a non-Weierstrass point, there are no gauge forms, though such a curve can be embedded into an affine space $\mathbb{A}^n$ as a closed subvariety.

The following two subsections show that a gauge form exists when X is a homogeneous space or a generic level set of an algebra of polynomials on $\mathbb{A}^n$.

1.3. *Fibers and level sets.* Let X be a non-singular algebraic variety with a gauge form ω, all defined over k. Let $f : X \to S$ be a smooth k-morphism of varieties, where S is non-singular. We will define gauge forms on fibers X_s of f. This construction is well known; see [Se2] for a similar construction with measures.

For a k-point s_0 of S, there is a gauge form μ in a neighborhood U of s_0. One can define gauge forms ω_s on the fibers by "dividing" ω by μ. Namely, there exists a form η on X of degree $\dim X - \dim S$ such that $\eta \wedge f^*\mu = \omega$. Let ω_s be the restriction of η to X_s for $s \in U$. One can check that though η is not unique, the forms ω_s on X_s are defined uniquely by ω and μ, and are gauge forms. If μ' is another gauge form in another neighborhood U' of s_0, then on the intersection $U \cap U'$ we have $\mu' = \varphi\mu$ where φ is a regular function without zeros, so instead of ω_s we get another gauge form $\omega_s' = \varphi(s)^{-1}\omega_s$. We see that the gauge form ω_s on X_s is defined by ω uniquely up to a constant factor from $k^\times$.

The above construction may be applied to the family of affine varieties defined by an algebra of polynomials. Let $X = \mathbb{A}^n$ with the gauge form $\omega = dx_1 \wedge \cdots \wedge dx_n$. Let $\mathscr{A} \subset k[x_1,\ldots,x_n]$ be a finitely generated algebra of polynomials on $\mathbb{A}^n$. Set $S = \operatorname{Spec} \mathscr{A}$. The embedding $\mathscr{A} \hookrightarrow k[x_1,\ldots,x_n]$ defines a map $f : \mathscr{A}^n \to S$, and this map is dominant, i.e. the image is dense. It follows that S is irreducible. There exists a Zariski open subset $U \subset S$ such that f is smooth on $f^{-1}(U)$. Since S is irreducible, U is dense in S. The above construction defines gauge forms on fibers $X_s = f^{-1}(s)$ for k-points $s \in U$.

Let $f_1,\ldots,f_r$ be a system of generators of $\mathscr{A}$; it defines a map $f : \mathbb{A}^n \to \mathbb{A}^r$, and we can identify S with the image of f. If we write $s^1,\ldots,s^r$ for the coordinates of the point $s \in S(k)$ in $\mathbb{A}^r$, then X_s is defined by the equations

$$f_i(x) = s^i \quad (i = 1,\ldots r).$$

All the polynomials in $\mathscr{A}$ are constant on X_s. For a k-point $s \in U$ we say that X_s is a generic level set of the algebra $\mathscr{A}$. We have defined a gauge form ω_s on a generic level set X_s of $\mathscr{A}$.

1.4. *Homogeneous spaces.* Let G be a k-group. Then G admits a left-invariant gauge form. A group G is called unimodular if G admits an invariant (i.e. left- and right-invariant) gauge form. If G is unipotent, or connected reductive, or has no k-characters, then G is unimodular.

Let X be a homogeneous space of a connected reductive k-group G. We assume that X has a k-point x_0. Let H denote the stabilizer of x_0. Then X admits a G-invariant gauge form if and only if H is unimodular ([We2], Thm. 2.4.1). In particular, if H is a connected reductive group, then X admits an invariant gauge form.

We say that gauge forms ω on X, ω_G on G and ω_H on H match together algebraically if $\omega_G = \omega \cdot \omega_H$ in the sense of [We2], 2.4, p. 24.

1.5. When a gauge form ω on X exists, a natural question arises, whether ω is unique up to constant factor. In general ω may be not unique. For example, for any integer n the form $x^n dx$ is a gauge form on $\mathbb{G}_m$. We will show however that if X is a homogenous space of a semisimple algebraic group, or, more general, an algebraic variety with a finite fundamental group, then a gauge form on X is unique up to a constant factor.

Let ω be a gauge form on X, and let ω' be another gauge form. Then $\omega' = \varphi\omega$ where $\varphi \in k[X]^\times$, i.e. φ is a regular function without zeros on X. We show that if φ is non-constant, then X must have an infinite fundamental group.

Lemma 1.5.1. *Let X be a homogeneous space of a k-group G without non-trivial k-characters. Assume that X has a k-point x_0. Then $k[X]^\times = k^\times$.*

Proof. Consider the map $\nu : G \to X$ defined by $\nu(g) = x_0 g$. For any function φ without zeros on X, its pullback $\nu^*\varphi$ is a function without zeros on G. By Rosenlicht's theorem (cf. [Ro] Thm. 3, or [We2], Thm. 2.2.2, p. 15), any function without zeros on G is a product of a k-character of G and a constant. By hypothesis, G has no k-characters, hence $\nu^*\varphi$ is constant, and therefore φ is constant. □

The following lemma must be well-known. We include a proof for the reader's convenience.

Lemma 1.5.2. *Let X be an algebraic variety over an algebraically closed field k of characteristic zero. If $k[X]^\times \neq k^\times$, then for any $n \geqq 1$ the variety X admits an unramified Galois covering of degree n.*

Proof. Assume that $k[X]^\times \neq k^\times$. First we prove that there exists a function $f \in k[X]^\times$ such that f is not of the form f_1^m for any $f_1 \in k[X]^\times$ and any $m > 1$. Choose a non-constant function $f \in k[X]^\times$. It suffices to find a natural number N such that if f is an n-th power then $n|N$. We can assume that X is affine, and embed X as a Zariski-dense subset in the normalization $\bar{X}$ of the projective closure of X. We may write $\bar{X} - X = D_1 \cup \cdots \cup D_r$. We regard a function $f' \in k[X]^\times$ as a rational function on $\bar{X}$. With f' we associate its divisor $\operatorname{div}(f') = \sum_i n_i D_i$. We obtain a map $\operatorname{div} : k[X]^\times \to \mathbf{Z}^r$, $f' \mapsto (n_i)$, whose kernel is $k[\bar{X}]^\times = k^\times$. Since our f is non-constant, $\operatorname{div}(f) \neq 0$. For the non-zero element $\operatorname{div}(f) \in \mathbf{Z}^r$ there exists a natural number N such that if $\operatorname{div}(f)$ is divisible by a natural number n, then $n|N$. Hence if f is an n-th power then $n|N$, which was to be proved.

For any natural n, the Kummer exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ induces the cohomology exact sequence

$$k[X]^\times \xrightarrow{n} k[X]^\times \to H^1_{\text{et}}(X, \mu_n).$$

A function $f \in k[X]^\times$ which is not an m-th power for any $m > 1$, defines an element of order n of $H^1_{\text{et}}(X, \mu_n)$, which corresponds to a Galois covering of X with Galois group $\mu_n \simeq \mathbf{Z}/n\mathbf{Z}$. □

Corollory 1.5.3. *Let X be a variety with a finite fundamental group over an algebraically closed field k of characteristic 0. Then a non-constant regular function f on X takes any value $a \in k$.* □

Corollory 1.5.4. *Let X be a variety over a field k of characteristic 0 with a finite geometric fundamental group. If X has a gauge form ω, then any other gauge form differs from ω by a constant factor.*

1.6. Let k be an algebraic number field. Let X be a non-singular k-variety, and ω a gauge form on X. With ω one can associate a Tamagawa measure m on the set $\boldsymbol{X(A)}$ of adelic points of X, see [We2], Ch. II. For any place $v \in \mathscr{V}(k)$ one associates a measure m_v on $X(k_v)$, cf. [We2], 2.2.

For a finite set $S \subset \mathcal{V}(k)$, let $\mathfrak{o}(S)$ denote the ring of S-integers in k (i.e. the elements of k, which are integer outside S). Fix a model of X over $\mathfrak{o}(S)$ for some finite S. For v outside S, set

$$\mu_v(X) = \int_{X(\mathfrak{o}_v)} m_v .$$

Then for almost all v we have $\mu_v(X) = q_v^{-\dim X}|X(k(v))|$, cf. [We2], Thm. 2.2.5.

If $\prod_v \mu_v(X)$ converges absolutely, one defines the Tamagawa measure m on $X(\mathbf{A})$ by

$$m = |\Delta_k|^{-\frac{1}{2}\dim X} \prod_{\mathcal{V}} m_v \tag{1.6.0}$$

where Δ_k is the discriminant of k (see [We2], 2.3 for details). By the product formula, the measure m does not change if ω is multiplied by a constant.

When $\prod_v \mu_v(X)$ does not converge absolutely, one needs convergence factors, cf. [We2], 2.3. A family (λ_v) of strictly positive numbers is called a family of convergence factors for X if $\prod_v \lambda_v^{-1} \mu_v(X)$ converges absolutely. Then one defines the Tamagawa measure by

$$m = |\Delta_k|^{-\frac{1}{2}\dim X} \prod_{\mathcal{V}} \lambda_v^{-1} m_v .$$

The Tamagawa measure m depends on the choice of convergence factors; a different choice will multiply the measure by a constant.

In case $\prod \mu_v(X)$ is conditionally convergent, we can normalize the Tamagawa measure by setting

$$m = |\Delta_k|^{-\frac{1}{2}\dim X} \Lambda \prod_v \lambda_v^{-1} m_v$$

where

$$\Lambda = \prod_{v|\infty} \lambda_v \lim_{x\to\infty} \prod_{p \leqq x} \prod_{v|p} \lambda_v . \tag{1.6.0.1}$$

Note that the convergence of (1.6.0.1) is equivalent to convergence of $\prod_v \mu_v$, and furthermore this normalization is independent of the choice of convergence factors.

We will recall the definition of the Tamagawa measure for connected unimodular groups and define the Tamagawa measure for homogeneous spaces.

1.6.1. Let G be a connected unimodular k-group G, and ω_G an invariant gauge form on G. Let ρ_G denote the representation of $\mathrm{Gal}(\bar{k}/k)$ in the space $X^*(G) \otimes \mathbf{Q}$, and let t_G be the the rank of the group of k-characters of G (i.e. the multiplicity of the trivial representation in ρ_G). The Tamagawa measure on $G(\mathbf{A})$ is defined by

$$m_f^G = r_G^{-1}|\Delta_k|^{-\frac{1}{2}\dim G} \prod_{\mathscr{V}_f}(\lambda_v^G)^{-1} m_v$$
$$m_\infty^G = \prod_{\mathscr{V}_\infty} m_v$$
$$m^G = m_\infty m_f \tag{1.6.1.1}$$

where

Δ_k is the discriminant of k;

$\lambda_v^G = L_v(1, \rho_G)^{-1}$ for $v \in \mathscr{V}$, where $L_v(s, \rho_G)$ is the local factor of the Artin L-function associated with ρ_G;

$r_G = \lim_{s \to 1}(s-1)^{t_G} L(s, \rho_G)$, where $L(s, \rho_G)$ is the corresponding Artin L-function, $L(s, \rho_G) = \prod_{\mathscr{V}_f} L_v(s, \rho_G)$.

If G is connected, the product in (1.6.1.1) converges absolutely, cf. [O], 1.1.1.

For an idele $a \in \mathbf{A}^\times$ its norm is defined by $|a| = \prod_v |a_v|$. Let $G(\mathbf{A})^1$ denote the set of all $g \in G(\mathbf{A})$ such that $|\chi(g)| = 1$ for any k-character $\chi : G \to \mathbb{G}_m$. The Tamagawa number of G is defined by $\tau(G) = m(G(\mathbf{A})^1/G(k))$.

1.6.2. Let G be a unimodular k-group, $H \subset G$ a unimodular k-subgroup, $X = H \backslash G$. By 1.4 there exists an invariant gauge form ω_X on X. We define the Tamagawa measure $m = m_X$ on $X(\mathbf{A})$ by

$$m_f = r_X^{-1}|\Delta_k|^{-\frac{1}{2}\dim X} \prod_{V_f}(\lambda_v^X)^{-1} m_v.$$
$$m_\infty = \prod_{\mathscr{V}_\infty} m_v$$
$$m = m_\infty m_f \tag{1.6.2.1}$$

where

$$\lambda_v^X = \frac{L_v(1, \rho_H)}{L_v(1, \rho_G)} = \frac{\lambda_v^G}{\lambda_v^H}$$
$$r_X = \lim_{s \to 1}(s-1)^{t_G - t_H} \frac{L(s, \rho_G)}{L(s, \rho_H)} = \frac{r_G}{r_H}$$

We will need two well-known lemmas.

Lemma 1.6.3. *Let k be any field, G a k-group and $H \subset G$ a k-subgroup. Set $X = H \backslash G$. Then there is a canonical exact sequence*

$$1 \to X(k)/G(k) \to H^1(k, H) \to H^1(k, G).$$

Proof. See [Se1], Ch. I, 5.4, Prop. 36. □

Lemma 1.6.4. *Let k be a number field, G a connected k-group, H a connected k-subgroup, and $X = H \backslash G$. For any point $x_\mathbf{A} \in X(\mathbf{A})$, the map $g \mapsto x_\mathbf{A} g : G(\mathbf{A}) \to X(\mathbf{A})$ is open. In particular, the orbits of $G(\mathbf{A})$ in $X(\mathbf{A})$ are open.*

Idea of proof. We use Lang's theorem [La] and Hensel's lemma. See [Se2], p. 654, for a more general statement. □

Now assume that G and H are connected unimodular and that gauge forms ω on X, ω_G on G and ω_H on H match together algebraically. One can check that the local measures m_v on $X(k_v), m_v^G$ on $G(k_v)$ and m_v^H on $H(k_v)$ match together topologically in the sense of [We2], 2.4, p. 25, i.e. $m_v^G = m_v \cdot m_v^H$.

Let m^G and m^H be the Tamagawa measures on $G(\mathbf{A})$ and $H(\mathbf{A})$, respectively.

Lemma 1.6.5. *When H is connected, the product in* (1.6.2.1) *converges absolutely, and any of the triples of measures (m^G, m^H, m) and (m_f^G, m_f^H, m_f) match together topologically.*

The lemma is a version of [We2], Thm. 2.4.2, without the hypothesis that the map $g \mapsto x_0 g : G \to X$ admits a local section.

Idea of proof. It suffices to prove that (1.6.2.1) converges absolutely. This means that $\prod_v (\lambda_v^X)^{-1} \mu_v(X)$ converges absolutely, cf. [We2], 2.3. The products $\prod_v (\lambda_v^G)^{-1} \mu_v(G)$ and $\prod_v (\lambda_v^H)^{-1} \mu_v(H)$ converge absolutely by [O], 1.1.1, and by definition $\lambda_v^X = (\lambda_v^H)^{-1} \lambda_v^G$. Since H is connected, by Lang's theorem $\mu_v^X = \mu_v^G (\mu_v^H)^{-1}$ for almost all v, and the lemma follows. □

Remark 1.6.6. The product in (1.6.2.1) converges absolutely even when H is non-connected (see 1.7 below). However in this case (1.6.1.1) does not converge for H. It can be shown that if G and H have no k-characters, the singular series is conditionally convergent and the normalization (1.6.0.1) coincides with (1.6.2.1).

1.7. *Non-connected stabilizer.* We show that the product (1.6.0) is absolutely convergent for $X = H \setminus G$ when G is connected semisimple and the stabilizer H is semisimple, even if we do not assume that H is connected. It suffices to prove that $\prod_v \mu_v(X)$ is absolutely convergent. Since for almost all primes we have $\mu_v(X) = q_v^{-\dim X} |X(k(v))|$, the convergence of (1.6.0) follows from

Proposition 1.7.1. *Let $X = H \setminus G$ be a homogenous space defined over a finite field $\mathbf{F}_q$, where G is connected semisimple, and H is semisimple, but not necessarily connected. Then*

$$\frac{|X(\mathbf{F}_q)|}{q^{\dim X}} = 1 + O(q^{-2}) .$$

We will need:

Lemma 1.7.2. *Let E be a finite group over the finite field $\mathbf{F}_q$. For a cohomology class $\xi \in H^1(\mathbf{F}_q, E)$ and a cocycle ψ representing ξ, set $e(\xi) = |{}_\psi E(\mathbf{F}_q)|$, where ${}_\psi E$ denotes the corresponding twisted group. Then*

$$\sum_{\xi \in H^1(\mathbf{F}_q, E)} \frac{1}{e(\xi)} = 1 . \qquad \square$$

1.7.3. *Idea of Proof of Proposition 1.7.1.* To count points in $X(\mathbf{F}_q)$, we decompose it into orbits under $G(\mathbf{F}_q)$. For one orbit $xG(\mathbf{F}_q)$ with stabilizer H_x, we have

$$|xG(\mathbf{F}_q)| = \frac{|G(\mathbf{F}_q)|}{|H_x(\mathbf{F}_q)|} = \frac{|G(\mathbf{F}_q)|}{|H_x^\circ(\mathbf{F}_q)|} \cdot \frac{1}{|\pi_0(H_x)(\mathbf{F}_q)|} . \qquad (1.7.3.1)$$

Set $E = \pi_0(H)$. Using Lemma 1.6.3 and Lang's theorem, we can show that the orbits of $G(\mathbf{F}_q)$ in $X(\mathbf{F}_q)$ are in one-to-one correspondence with the set $H^1(\mathbf{F}_q,E)$, and under this correspondence $|\pi_0(H_x)(\mathbf{F}_q)| = e(\xi)$.

For a connected semisimple group G over $\mathbf{F}_q$, it follows from [St, 11.16] that

$$(1 - q^{-2})^{\mathrm{rank}G} \leqq \frac{|G(\mathbf{F}_q)|}{q^{\dim G}} \leqq (1 + q^{-2})^{\mathrm{rank}G} .$$

Applying this inequality to G and H_x° in (1.7.3.1), we find

$$\frac{(1 - q^{-2})^{\mathrm{rank}G}}{(1 + q^{-2})^{\mathrm{rank}H}} \cdot \frac{1}{e(\xi)} \leqq \frac{|xG(\mathbf{F}_q)|}{q^{\dim X}} \leqq \frac{(1 + q^{-2})^{\mathrm{rank}G}}{(1 - q^{-2})^{\mathrm{rank}H}} \cdot \frac{1}{e(\xi)} , \qquad (1.7.3.2)$$

for the orbit corresponding to a cohomology class ξ. By summing (1.7.3.2) over $\xi \in H^1(\mathbf{F}_q,E)$ and applying Lemma 1.7.2, we obtain Proposition 1.7.1. □

We refer the reader to [Sp] for a treatment of the situation when G and H are not assumed to be semisimple.

1.8. We show that when X is as in (0.0.2), our Hardy-Littlewood expectation, i.e the Tamagawa measure of $B_\infty \times B_f$, coincides with the classical Hardy-Littlewood expectation, i.e. the product of the singular integral and the singular series.

Let X be as in (0.0.2). Consider the map $f = (f_1,\dots,f_r) : \mathbb{A}^n \to \mathbb{A}^r$. Set $V_s = f^{-1}(s)$ for $s \in \mathbb{A}^r$. Then $X = V_0$.

Since $\mathrm{rank}(\partial f_i/\partial x_j) = r$ on X, the map f is smooth on $f^{-1}(U)$ for some Zariski-open set U in $\mathbb{A}^r$, and the construction of 1.3 defines gauge forms ω^s on the fibers V_s for $s \in U$. Set $\mathcal{U}_v = U(\mathbf{Q}_v) \subset \mathbf{Q}_v^r$. Let m_v^s denote the corresponding local measures on $V_s(\mathbf{Q}_v)$ for $s \in \mathcal{U}_v$.

From the construction of the forms ω^s it follows that for any compactly supported functions ϕ on $\mathcal{U}_v$ and ψ on $f^{-1}(\mathcal{U}_v) \subset \mathbf{Q}_v^n$, which are piecewise continuous when $v = \infty$ and locally constant when $v = p < \infty$, we have

$$\int \phi(f(x))\psi(x)\,dx_1 \dots dx_n = \int \phi(s) \left(\int_{V_s(\mathbf{Q}_v)} \psi\, dm_v^s \right) ds_1 \dots ds_r \qquad (1.8.0)$$

This equality defines the measures m_v^s uniquely for almost all s.

Lemma 1.8.1. *For any prime* p,

$$m_p(X(\mathbf{Z}_p)) = \lim_{l\to\infty} \frac{|\{x \in (\mathbf{Z}/p^l\mathbf{Z})^n : f_i(x) \equiv 0 \ mod\ p^l\}|}{p^{l \dim X}} .$$

Lemma 1.8.2.

$$m_\infty(\{x \in X(\mathbf{R}) : |x| \leqq T\}) = \lim_{\varepsilon \to 0} \frac{\mathrm{vol}\{x \in \mathbf{R}^n : |x| \leqq T,\ |f_i(x)| < \varepsilon/2,\ i = 1, \ldots, r\}}{\varepsilon^r} .$$

Proof of Lemmas 1.8.1 and 1.8.2. For $v = p < \infty$, set $\varepsilon = p^{-l}$,

$$\mathfrak{C}_\varepsilon = p^l \mathbf{Z}_p^r \subset \mathbf{Q}_p^r, \quad \mathfrak{B} = \mathbf{Z}_p^n \subset \mathbf{Q}_p^n .$$

Then $\mathrm{vol}(\mathfrak{C}_\varepsilon) = \varepsilon^r = p^{-lr}$. For l sufficiently big, $\mathfrak{C}_\varepsilon \subset U(\mathbf{Q}_p)$. Take for ψ the characteristic function of the compact open subset $\mathfrak{B} \cap f^{-1}(U) \subset \mathbf{Q}_p^n$, and for ϕ the characteristic function of $\mathfrak{C}_\varepsilon \cap U \subset \mathbf{Q}_p^r$. Then (1.8.0) yields

$$p^{-nl} |\{x \in (\mathbf{Z}/p^l\mathbf{Z})^n : f_i(x) \equiv 0 \bmod p^l\}| = \int_{\mathfrak{C}_\varepsilon} m_p^s(V_s(\mathbf{Z}_p))\, ds_1 \ldots ds_r$$

After dividing by $\mathrm{vol}(\mathfrak{C}_\varepsilon) = p^{-lr}$ and passing to limit, we get Lemma 1.8.1.

For $v = \infty$, for any $\varepsilon > 0$ set

$$\mathfrak{C}_\varepsilon = \{s \in \mathbf{R}^r : |s_i| \leqq \varepsilon/2,\ i = 1, \ldots, r\}, \quad \mathfrak{B} = \{x \in \mathbf{R}^n : |x| \leqq T\} .$$

For sufficiently small $\varepsilon, \mathfrak{C}_\varepsilon \in U(\mathbf{R})$. Take for ψ the characteristic function of the set $\mathfrak{B} \cap f^{-1}(U)$, and for ϕ the characteristic function of the cube $\mathfrak{C}_\varepsilon$. The equality (1.8.0) yields

$$\mathrm{vol}(f^{-1}(\mathfrak{C}_\varepsilon) \cap \mathfrak{B}) = \int_{\mathfrak{C}_\varepsilon} m_\infty^s(V_s(\mathbf{R}) \cap \mathfrak{B})\, ds_1 \ldots ds_r .$$

After dividing by $\mathrm{vol}(\mathfrak{C}_\varepsilon) = \varepsilon^r$ and passing to limit, we obtain Lemma 1.8.2. □

Corollary 1.8.3. *Let X be as in* (0.0.2), *and assume that the singular series $\mathfrak{S}(X)$ converges at least conditionally. Then*

$$m_f(X(\hat{\mathbf{Z}})) = \mathfrak{S}(X) \text{ and } \sum_{B_\infty} m_\infty(B_\infty^T) = \mu_\infty(X, T) . \quad \square$$

2. Hardy-Littlewood varieties

2.1. In this section $k = \mathbf{Q}$. Let X be an algebraic variety over $\mathbf{Q}$, and ω a gauge form on X. We assume that either the product (1.6.0) is absolutely convergent, or that we are given a canonical set λ_v of convergence factors. In any case we can define the Tamagawa measure m on $X(\mathbf{A})$ and measures m_f on $X(\mathbf{A}_f)$ and m_∞ on X_∞.

We assume that X is affine and is embedded into a vector space W as a closed subvariety. Suppose that all the connected components B_∞ of $X(\mathbf{R})$ are non-compact. Let B_∞ be a connected component of $X(\mathbf{R})$, and $B_f \subset G(\mathbf{A}_f)$

be any open compact subset. Set $B = B_\infty \times B_f$. For a positive number T, set $B_\infty^T = \{x \in B_\infty : |x| \leqq T\}$. We want to compare the counting function

$$N(T,X;B) = |\{x \in X(\mathbf{Q}) \cap B : |x| \leqq T\}|$$

with its Hardy-Littlewood expectation $m(B_\infty^T \times B_f) = m_f(B_f) m_\infty(B_\infty^T)$.

Definition 2.2. *A variety X is called strongly Hardy-Littlewood with respect to a gauge form ω, if for any B_∞ and any B_f as above,*

$$N(T,X;B) \sim m(B_\infty^T \times B_f) \quad \text{as } T \to \infty.$$

Definition 2.3. *Let δ be a locally constant non-negative function on $X(\mathbf{A})$, which is constant on connected components of $X(\mathbf{R})$ and not zero identically.*

A variety X is called (relatively) Hardy-Littlewood with density δ, with respect to a gauge form ω, if for any B_∞ and any B_f as above,

$$N(T,X;B_\infty \times B_f) \sim \int_{B_\infty^T \times B_f} \delta(x) dm \quad \text{as } T \to \infty.$$

Thus a strongly Hardy-Littlewood variety is a Hardy-Littlewood variety with constant density 1.

Proposition 2.4. *If X is strongly Hardy-Littlewood, then X has the strong approximation property: the image of $X(\mathbf{Q})$ in $X(\mathbf{A}_f)$ is dense.*

Proof. Let $B_f \subset X(\mathbf{A}_f)$ be an open subset. We must prove that $X(\mathbf{R}) \times B_f$ contains a rational point. We will prove a stronger assertion: for any connected component B_∞ of $X(\mathbf{R})$, the set $B_\infty \times B_f$ contains a rational point. We may assume that B_f is compact. Since X is strongly Hardy-Littlewood, the number of $\mathbf{Q}$-rational points in $B_\infty^T \times B_f$ grows asymptotically as $m_\infty(B_\infty^T) \cdot m_f(B_f)$ as $T \to \infty$, and in particular since $m_\infty(B_\infty^T) \cdot m_f(B_f) > 0$ (and is increasing), $X(\mathbf{R}) \times B_f$ contains a rational point. □

We may assume that X is defined by polynomials with integer coefficients. Then $X(\mathfrak{o}(S))$ makes sense for any finite set $S \subset \mathscr{V}(\mathbf{Q})$ containing ∞.

Proposition 2.5. *If X is Hardy-Littlewood, then there exists a finite set S containing ∞ such that the image of $X(\mathfrak{o}(S))$ in $\prod_{v \notin S} X(\mathfrak{o}_v)$ is dense.*

Proof. Choose a point $y = (y_\infty, y_f) \in X(\mathbf{A})$ such that $\delta(y) \neq 0$. Set $\Delta = \delta(y)$. Let U_f be an open compact neighborhood of y_f such that $\delta(x) = \Delta$ on $B_\infty \times U_f$, where B_∞ is the connected component of y_∞ in $X(\mathbf{R})$. We can choose U_f of the form $U_f = \prod U_p$. There exists a finite subset S of $\mathscr{V}(\mathbf{Q})$, containing ∞, such that $U_p = X(\mathbf{Z}_p)$ for $p \notin S$.

Now let B^S be any open subset of $\prod_{p \notin S} X(\mathbf{Z}_p)$. We must prove that the set $\prod_{v \in S} X(\mathbf{Q}_v) \times B^S$ contains rational points. Set $B_f = \prod_{p \in S \cap \mathscr{V}_f} U_p \times B^S$. The density δ is constant and positive on $B_\infty \times B_f$. An argument similar to that in the proof of Proposition 2.4 shows that $B_\infty \times B_f$ contains a rational point. We conclude that $\prod_{v\in S} X(\mathbf{Q}_v) \times B^S$ contains a rational point. □

We say that an algebraic variety X over a field k is geometrically simply connected if $X_{\bar{k}}$ has no non-trivial unramified coverings.

Proposition 2.6. *If a variety X over $\mathbf{Q}$ is Hardy-Littlewood with some density δ, then X is geometrically simply connected.*

Proof. Minchev [Min] proved that if X is a non-singular algebraic variety over a number field k, and for some $S \subset \mathscr{V}(k)$ the image of $X(\mathfrak{o}(S))$ in $\prod_{v\notin S} X(\mathfrak{o}_v)$ is dense, then X is geometrically simply connected. The proposition follows therefore from Proposition 2.5. □

Proposition 2.7. *If X is an affine homogeneous space of a connected group G, with non-connected stabilizer H, then X is not Hardy-Littlewood with respect to any gauge form.*

Proof. Indeed, then $H^\circ\backslash G$ is an unramified covering of $X = H\backslash G$, hence X is not simply connected. □

Remark 2.7.1. The fact that the image of $X(\mathfrak{o}(S))$ in $\prod_{v\notin S} X(\mathfrak{o}_v)$ is not dense for a homogeneous space X with non-connected stabilizer, was also proved in [Bo1]. It can also be easily proved by Kneser's method [Kn2], using a theorem on the finiteness of the number of orbits of an S-arithmetic group ([Brl], 8.10).

Corollary 2.8. *If X is a Hardy-Littlewood variety with respect to a gauge form ω, then the Tamagawa measure m' on $X(\mathbf{A})$ defined by any gauge form ω' on X, coincides with the Tamagawa measure m defined by ω.*

Proof. Indeed, by Proposition 2.6, X is geometrically simply connected. It follows from Corollary 1.5.4 that $\omega' = \lambda\omega$ for some $\lambda \in k^\times$. By the product formula, $m' = m$, cf. [We2], Thm. 2.3.1. □

We see that for a Hardy-Littlewood variety X, the Hardy-Littlewood expectation depends only on X, and not on the choice of a gauge form.

Proposition 2.9. *Let X be a strongly Hardy-Littlewood variety as in* (0.0.2). *Then*

$$N(T,X) \sim \mathfrak{S}(X)\,\mu_\infty(X,T)\,.$$

Proof. Since X is strongly Hardy-Littlewood, for any connected component B_∞ of $X(\mathbf{R})$ we have

$$N(T,X;B_\infty \times X(\hat{\mathbf{Z}})) \sim m(B_\infty^T \times X(\hat{\mathbf{Z}}))\,.$$

Summation over the connected components yields

$$N(T,X) \sim \sum_{B_\infty \subset X(\mathbf{R})} m_\infty(B_\infty^T)\, m_f(X(\hat{\mathbf{Z}}))\,.$$

By Corollary 1.8.3 the right hand side of the above equality equals $\mathfrak{S}(X)\mu_\infty(X,T)$, which proves the proposition. □

3. Rational points in adelic orbits

3.1. Let k be a number field, G a semisimple simply connected group over k, X a right homogeneous space of G defined over k. We assume that X has a k-point. Let H be the stabilizer of a k-point $x_0 \in X(k)$. Hereafter we assume that H is connected.

Let $\mathcal{O}_{\mathbf{A}}$ be an orbit of $G(\mathbf{A})$ in $X(\mathbf{A})$. In this section we are interested whether $\mathcal{O}_{\mathbf{A}}$ contains rational points. Our methods are those of [Ko2] and [Bo3]. We use cohomological techniques of [Ko1], [Ko2] in the form of [Bo2].

Recall that a connected algebraic group is simply connected if it is an extension of a simply connected semisimple group by a unipotent group. In the case when the stabilizer H is simply connected, we have

Theorem 3.2. *Let G,X and H be as in* 3.1. *Assume that H is simply connected. Then the embeddings $X(k) \hookrightarrow X(k_\infty) \hookrightarrow X(\mathbf{A})$ induce bijections of the orbit spaces*

$$X(k)/G(k) \xrightarrow{\sim} X(k_\infty)/G(k_\infty) \xrightarrow{\sim} X(\mathbf{A})/G(\mathbf{A}).$$

This result is stated in [Ig, p 138]. It follows from Corollary 3.7 below. We provide however an 'elementary' proof.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & X(k)/G(k) & \longrightarrow & H^1(k,H) & \longrightarrow & H^1(k,G) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & X(k_\infty)/G(k_\infty) & \longrightarrow & H^1(k_\infty,H) & \longrightarrow & H^1(k_\infty,G) \end{array}$$

where the rows are exact sequences of Lemma 1.6.3. By the Hasse principle for the simply connected groups H and G (Kneser–Harder–Chernousov, see [Ha], [PR, Ch. 6]) the right and middle vertical arrows are bijections, hence the left one is a bijection.

By Kneser's theorem [Kn1], $H^1(k_v,H) = 0$ for any finite v, hence $G(k_v)$ acts on $X(k_v)$ transitively for these v. Using Lemma 1.6.4 we obtain that $G(\mathbf{A}_f)$ acts on $X(\mathbf{A}_f)$ transitively. It follows that the map

$$X(k_\infty)/G(k_\infty) \to X(\mathbf{A})/G(\mathbf{A})$$

is a bijection. □

In order to describe the obstruction to the existence of a rational point in an adelic orbit in the case when H is not simply connected, we need the notion of the algebraic fundamental group.

3.3. *Algebraic fundamental group.* (cf. [Bo2], or the exposition in [Mi] App. B). Let $\bar{H}$ be a connected group over an algebraically closed field $\bar{F}$. First assume that $\bar{H}$ is reductive. Consider the homomorphism $\rho : \bar{H}^{\mathrm{sc}} \to \bar{H}^{\mathrm{ss}} \to \bar{H}$, where $\bar{H}^{\mathrm{sc}}$ is the universal covering of the derived group $\bar{H}^{\mathrm{ss}}$ of $\bar{H}$. Let $\bar{T}$ be a maximal torus of $\bar{H}$; set $\bar{T}^{(\mathrm{sc})} = \rho^{-1}(\bar{T})$. We define

$$\pi_1(\bar{H},\bar{T}) = X_*(\bar{T})/\rho_* X_*(\bar{T}^{(\mathrm{sc})})$$

where $X_*(\cdot)$ denotes the cocharacter group. If $\bar{T}' \subset \bar{H}$ is another torus, then there is an element $h \in \bar{H}(\bar{k})$ such that the inner automorphism int(h) of $\bar{H}$ takes $\bar{T}'$ to $\bar{T}$. One can easily check that int(h) induces a canonical isomorphism $\pi_1(\bar{H},\bar{T}') \to \pi_1(\bar{H},\bar{T})$. We can therefore write $\pi_1(\bar{H})$ for $\pi_1(\bar{H},\bar{T})$. Now if $\bar{H}$ is any connected group, not necessarily reductive, then we set $\pi_1(\bar{H}) = \pi_1(\bar{H}/\bar{H}^u)$ where $\bar{H}^u$ is the unipotent radical of $\bar{H}$.

The algebraic fundamental group $\pi_1(\bar{H})$ is a functor from the category of connected $\bar{F}$-groups to finitely generated abelian groups. Moreover, for a connected group H defined over any field F of characteristic 0, the Galois group Gal$(\bar{F}/F)$ acts on $\pi_1(H_{\bar{F}})$, so we get a functor $H \mapsto \pi_1(H_{\bar{F}})$ from the category of connected F-groups to Galois modules, finitely generated over $\mathbf{Z}$. We will write $\pi_1(H)$ for the corresponding Galois module. One can check that an inner twisting does not change the algebraic fundamental group.

Examples. If H is unipotent, then $\pi_1(H) = 0$. If H is a torus, then $\pi_1(H) = X_*(H)$. If H is a reductive group such that H^{ss} is simply connected, then $\pi_1(H) = X_*(H/H^{\mathrm{ss}})$. If H is semisimple, then $\pi_1(H)$ is a twisted form of the finite group $\ker[\rho : H^{\mathrm{sc}} \to H]$ (i.e. they are isomorphic as abelian groups but not as Galois modules); namely,

$$\pi_1(H) = \mathrm{Hom}(X^*(\ker\, \rho), \mathbf{Q}/\mathbf{Z}).$$

In particular, $\pi_1(\mathrm{PGL}_n) = \mathbf{Z}/n\mathbf{Z}$, while $\ker[\mathrm{SL}_n \to \mathrm{PGL}_n] = \mu_n$. It is worth mentioning that for any connected group H defined over $\mathbf{C}$, the algebraic fundamental group of H is just the usual topological fundamental group of $H(\mathbf{C})$.

3.4. *Coinvariants.* For a connected algebraic group H over a field F, we set, following Kottwitz ([Ko2], Introduction)

$$C(H) = (\pi_1(H)_{\mathrm{Gal}(\bar{F}/F)})_{\mathrm{tors}}\ ,$$

where $\pi_1(H)_{\mathrm{Gal}(\bar{F}/F)}$ denote the group of coinvariants of the Galois group, and $(\cdot)_{\mathrm{tors}}$ denotes the torsion subgroup. (Kottwitz writes $A(H)$ instead of $C(H)$.) For a connected group H over a number field k we set

$$C_v(H) := C(H_{k_v}) = (\pi_1(H)_{\mathrm{Gal}(\bar{k}_v/k_v)})_{\mathrm{tors}}$$

for any place v of k. We have a canonical map $i_v : C_v(H) \to C(H)$ induced by an inclusion $\mathrm{Gal}(\bar{k}_v/k_v) \to \mathrm{Gal}(\bar{k}/k)$.

Kottwitz relates the groups $C(H)$ and $C_v(H)$ to the first Galois cohomology of H. For a place v of k he defines a local map

$$\beta_v : H^1(k_v, H) \to C_v(H)$$

taking the neutral element of $H^1(k_v, H)$ to zero (cf. [Ko2], Thm. 1.2; our β_v is α_{H_v} in the notation of Kottwitz). For finite v, the map β_v is bijective.

3.5. *Kottwitz invariant.* We can now define the Kottwitz invariant of an adelic orbit $\mathcal{O}_{\mathbf{A}}$. Write $\mathcal{O}_{\mathbf{A}} = \prod' \mathcal{O}_v$, where $\mathcal{O}_v$ is an orbit of $G(k_v)$ in $X(k_v)$ for a place v of k, and $\prod'$ denotes the restricted product. By Lemma 1.6.3 an orbit $\mathcal{O}_v$ defines a cohomology class $\xi_v \in H^1(k_v, H)$. We define local invariants by

$$\kappa_v(\mathcal{O}_v) = \beta_v(\xi_v) \quad \in C_v(H).$$

For a point $x_v \in X(k_v)$ we set $\kappa(x_v) = \kappa(x_v G(k_v))$. Then $\kappa(x_v)$ is a locally constant map $X(k_v) \to C_v(H)$.

Applying Lang's theorem and Hensel's lemma, we see that for almost all v we have $\mathcal{O}_v = x_0 G(k_v)$ and therefore $\kappa_v(\mathcal{O}_v) = 0$. We now define the Kottwitz invariant of $\mathcal{O}_{\mathbf{A}}$ by

$$\kappa(\mathcal{O}_{\mathbf{A}}) = \sum_v i_v(\kappa_v(\mathcal{O}_v)) \quad \in C(H)\ .$$

For a point $x_{\mathbf{A}} \in X(\mathbf{A})$ we set $\kappa(x_{\mathbf{A}}) = \kappa(x_{\mathbf{A}} G(\mathbf{A}))$; it is a locally constant map $X(\mathbf{A}) \to C(H)$.

Theorem 3.6. *Let G, X and H be as in 3.1. An orbit $\mathcal{O}_{\mathbf{A}}$ of $G(\mathbf{A})$ in $X(\mathbf{A})$ contains a k-rational point if and only if $\kappa(\mathcal{O}_{\mathbf{A}}) = 0$.*

In the case when the group H is either semisimple or a torus, the result is known to experts: it is a standard application of the Tate duality for finite groups and tori. Kottwitz ([Ko2], Lemma 6.3) proved Theorem 3.6 in the case when X is a conjugacy class in G.

Proof. By construction if $\mathcal{O}_{\mathbf{A}}$ contains a k-point then $\kappa(\mathcal{O}_{\mathbf{A}}) = 0$. We must prove that if $\kappa(\mathcal{O}_{\mathbf{A}}) = 0$ then $\mathcal{O}_{\mathbf{A}}$ contains a k-point.

For $v \in \mathcal{V}$ consider the localization maps $\mathrm{loc}_v : H^1(k, H) \to H^1(k_v, H)$. The family (loc_v) defines a map

$$\mathrm{loc} : H^1(k, H) \to \bigoplus_v H^1(k_v, H)$$

where $\oplus_v$ denotes the subset in the direct product consisting of (ξ_v) such that $\xi_v = 1$ for almost all v. Consider the map

$$\beta : \bigoplus_v H^1(k_v, H) \xrightarrow{\oplus \beta_v} \bigoplus_v C_v(H) \xrightarrow{\sum i_v} C(H)\ .$$

Kottwitz proved that im loc = kerβ ([Ko2], 2.5, 2.6).

An adelic orbit $\mathcal{O}_{\mathbf{A}}$ defines a class $\xi_{\mathbf{A}} := (\xi_v) \in \oplus_v H^1(k_v, H)$, see 3.5. By definition, $\kappa(\mathcal{O}_{\mathbf{A}}) = \beta(\xi_{\mathbf{A}})$. Since $\kappa(\mathcal{O}_{\mathbf{A}}) = 0$, we have $\beta(\xi_{\mathbf{A}}) = 0$. Hence $\xi_{\mathbf{A}} = \mathrm{loc}(\xi)$ for some $\xi \in H^1(k, H)$.

Let η denote the image of ξ in $H^1(k,G)$. Then $\mathrm{loc}_v(\eta)=1$ for any place v of k. By the Hasse principle for the simply connected group G, $\eta=1$. By Lemma 1.6.3, ξ defines a $G(k)$-orbit $\mathcal{O}_k$ in $X(k)$, and we see that $\mathcal{O}_k\subset\mathcal{O}_{\mathbf{A}}$. □

Corollary 3.7. *If $C(H)=0$, then any adelic orbit $\mathcal{O}_{\mathbf{A}}$ contains a rational point.*

Proof. Indeed, then $\kappa(\mathcal{O}_{\mathbf{A}})=0$ for any adelic orbit $\mathcal{O}_{\mathbf{A}}$. □

3.8. *Remarks.* (i) Conversely, if H has no k_v-characters for some place v of k (e.g. if H is semisimple) and $C(H)\neq 0$, then there exists an adelic orbit $\mathcal{O}_{\mathbf{A}}$ without rational points.

(ii) Theorem 3.2 follows from Corollary 3.7. Indeed, if H is simply connected, then $\pi_1(H)=0$, hence $C(H)=0$.

3.9. *Remark.* If we do not know in advance whether X has a rational point, let $\bar{x}\in X(\bar{k})$ be a $\bar{k}$-point, and $\bar{H}$ its stabilizer which we assume to be connected. The Galois group acts on $\pi_1(\bar{H})$, and one can define $C(\bar{H})$. Then one can define the Kottwitz invariant $\kappa(\mathcal{O}_{\mathbf{A}})\in C(\bar{H})$ of an adelic orbit and prove that $\mathcal{O}_{\mathbf{A}}$ contains a rational point if and only if $\kappa(\mathcal{O}_{\mathbf{A}})=0$. This generalizes [Ko2], 6.3.

4. Weight formula

4.1. Let G be a simply connected semisimple group over a number field k, X a right homogeneous space of G. We assume that X has a k-point x_0 and that the stabilizer H of x_0 is connected, unimodular and has no non-trivial k-characters.

Let $\mathcal{O}_{\mathbf{A}}$ be an orbit of $G(\mathbf{A})$ in $X(\mathbf{A})$ containing a rational point x_0. Let B be an open subset in $\mathcal{O}_{\mathbf{A}}$ of the form $B=\mathcal{O}_\infty\times B_f$, where $\mathcal{O}_\infty$ is an orbit of $G(k_\infty)$ in $X(k_\infty)$ and B_f is an open compact subset in $X(\mathbf{A}_f)$. There exists an open compact subgroup $K_f\subset G(\mathbf{A}_f)$ of the form $K_f=\prod K_v$ such that $B_fK_f=B_f$. We fix K_f and set $K=G(k_\infty)\times K_f$. Set $\Gamma=G(k)\cap K$; it is an arithmetic subgroup of $G(k)$. For $x\in X(k)$ we write H_x for the stabilizer of x in G, and set $\Gamma_x=\Gamma\cap H_x(k), K_x=K\cap H_x(\mathbf{A})$. We are interested in the orbits of the arithmetic group Γ in the 'arithmetic set' $X(k)\cap B$.

Fix a G-invariant gauge form ω_X on X. Choose an invariant gauge form ω_G on G. For any orbit $\mathcal{O}$ of Γ in $X(k)\cap B$ we define its *weight* $w(\mathcal{O})$ as follows. Choose $x\in\mathcal{O}$. We can normalize a gauge form ω_H on H_x so that the gauge forms ω_H,ω_G and ω_X match together algebraically. We obtain measures m_H on $H_x(\mathbf{A})$, $m_{H,\infty}$ on $H_x(k_\infty)$ etc. We define the weight $w(\mathcal{O})$ by

$$w(\mathcal{O})=\frac{m_{H,\infty}(H(k_\infty)/\Gamma_x)}{m_{G,\infty}(G(k_\infty)/\Gamma)}.$$

It is clear that $w(\mathcal{O})$ does not depend on the choice of $x \in \mathcal{O}$ and ω_G on G.

Theorem 4.2. *Let $G, X, \mathcal{O}_{\mathbf{A}}, B, K, \Gamma$ be as in* 4.1. *Assume that G has no k-factors G' such that $G'(k_\infty)$ is compact. Let H be the stabilizer of a point $x_0 \in X(k)$. Then*

$$\sum_{\mathcal{O} \subset X(k) \cap B} w(\mathcal{O}) = |C(H)| m_{X,f}(B_f).$$

Theorem 4.2 is inspired by [We1]. Our methods are those of [We1] combined with the calculation of the Tamagawa number of a connected group due to Ono, Sansuc and Kottwitz.

4.3. The weight $w(\mathcal{O})$ depends on the choice of the gauge form ω_X. In order to prove Theorem 4.2 we define another, canonical weight $w_{\text{can}}(\mathcal{O})$. Since H_x has no non-trivial k-characters, we have $H_x(\mathbf{A})^1 = H_x(\mathbf{A})$, hence $\tau(H_x) = m_H(H_x(\mathbf{A})/H_x(k))$. We set

$$w_{\text{can}}(\mathcal{O}) = \frac{m_H(K_x H_x(k)/H_x(k))}{\tau(H_x)}$$

Clearly $w_{\text{can}}(\mathcal{O}) \leqq 1$.

We want to compute $\sum w_{\text{can}}(\mathcal{O})$ where $\mathcal{O}$ runs over the orbits of Γ in $X(k) \cap B$.

Proposition 4.4. *If B_f is an orbit of K_f in $X(\mathbf{A}_f)$ and $B = B_\infty \times B_f$, then*

$$\sum_{\mathcal{O} \subset X(k) \cap B} w_{\text{can}}(\mathcal{O}) = |(X(k) \cap \mathcal{O}_{\mathbf{A}})/G(k)|$$

Proof. It suffices to prove that $\sum w_{\text{can}}(\mathcal{O})$, $\mathcal{O}$ running over the Γ-orbits in $xG(k) \cap B$, equals 1 for any $x \in X(k) \cap \mathcal{O}_{\mathbf{A}}$.

Let $x_0 G(k)$ be any orbit of $G(k)$ in $X(k) \cap \mathcal{O}_{\mathbf{A}}$. First we wish to prove that $x_0 G(k) \cap B \neq \emptyset$. Since G is semisimple simply connected and has no k-factors G' such that $G'(k_\infty)$ is compact, by the strong approximation theorem ([Kn2], [Pl]) $G(k)K = G(\mathbf{A})$. Hence $x_0 G(k) K = x_0 G(\mathbf{A}) = \mathcal{O}_{\mathbf{A}}$. Choose $x_{\mathbf{A}} \in B$ and write $x_{\mathbf{A}} = x_0 g_{\mathbf{A}}$ where $g_{\mathbf{A}} \in G(\mathbf{A})$. We can write $g_{\mathbf{A}} = g_k g_K$ where $g_k \in G(k)$, $g_K \in K$. We have $x_0 g_k = x_{\mathbf{A}} g_K^{-1}$, where $x_0 g_k \in X(k)$ and $x_{\mathbf{A}} g_K^{-1} \in B$. Hence $x_0 G(k) \cap B$ contains the rational point $x_0 g_k$ and thus is non-empty.

Now let $x_0 \in X(k) \cap B$. To prove the Proposition it suffices to prove that $\sum w_{\text{can}}(\mathcal{O})$ over the Γ-orbits $\mathcal{O}$ in $x_0 G(k) \cap x_0 K$ equals 1.

Write $K_H = K \cap H(\mathbf{A})$, where $H = \text{Stab}(x_0)$ (then $K_H = K_{x_0}$ in the notation of 4.1). With any Γ-orbit $\mathcal{O} \subset x_0 G(k) \cap x_0 K$ we associate a double coset $D(\mathcal{O}) \in K_H \backslash H(\mathbf{A})/H(k)$ as follows. Choose $x \in x_0 G(k) \cap x_0 K$ and write $x = x_0 g_k = x_0 g_K$ where $g_k \in G(k)$, $g_K \in K$. Set $h = g_k g_K^{-1}$. Then $h \in H(\mathbf{A})$. We write $D(\mathcal{O})$ for the double coset $K_H h^{-1} H(k)$ of h in $K_H \backslash H(\mathbf{A})/H(k)$; it does not depend on the choice of $x \in \mathcal{O}$.

Lemma 4.4.1. [We1]. *(i) The map D is a bijection of the set of orbits of Γ in $x_0 G(k) \cap x_0 K$ onto the set of double cosets $K_H \backslash H(\mathbf{A})/H(k)$*

(ii) $w_{\text{can}}(\mathcal{O}) = \tau(H)^{-1}\ m_H(D(\mathcal{O})/H(k))$.

Proof. (i) is straightforward. To prove (ii) we consider the isomorphism $h' \mapsto g_k h' g_k^{-1} : H_x \to H$. This isomorphism takes the double coset $K_x H_x(k)$ to the set $(g_k K g_k^{-1} \cap H)H(k) = hK_H h^{-1}H(k)$ (because $g_k = hg_K$). Hence

$$m_H(K_x H_x(k)/H_x(k)) = m_H(hK_H h^{-1}H(k)/H(k)).$$

Since the Tamagawa measure m_H on $H(\mathbf{A})$ is invariant,

$$m_H(hK_H h^{-1}H(k)/H(k)) = m_H(K_H h^{-1}H(k)/H(k)).$$

Thus

$$\begin{aligned} w_{\text{can}}(\mathcal{O}) &= \tau(H_x)^{-1}\ m_H(K_x H_x(k)/H_x(k)) = \tau(H)^{-1}\ m_H(K_H h^{-1}H(k)/H(k)) \\ &= \tau(H)^{-1}\ m_H(D(\mathcal{O})/H(k)) \end{aligned}$$

which proves the lemma. □

To complete the proof of Proposition 4.4 we note that by Lemma 4.4.1(i) the double cosets $D(\mathcal{O})$ for $\mathcal{O} \subset x_0G(k) \cap x_0K$ are pairwise distinct and together constitute all of the group $H(\mathbf{A})$, so

$$\sum_{\mathcal{O} \subset x_0 G(k) \cap x_0 K} w_{\text{can}}(\mathcal{O}) = \sum \frac{m_H(D(\mathcal{O})/H(k))}{\tau(H)} = \frac{m_H(H(\mathbf{A})/H(k))}{\tau(H)} = 1$$

which proves the proposition. □

4.5. *Proof of Theorem* 4.2. We wish to compare the weights $w(\mathcal{O})$ and $w_{\text{can}}(\mathcal{O})$. Write $K_x = K \cap H_x(\mathbf{A}), \Gamma_x = H_x(k) \cap K = \Gamma \cap H_x(k)$. We have $K_x = H_x(k_\infty) \times K_{x,f}$, where $K_{x,f} \subset H_x(\mathbf{A}_f)$. There is an evident map $K_x/\Gamma_x \to H_x(k_\infty)/\Gamma_x$ with fiber $K_{x,f}$, whence

$$m_H(K_x/\Gamma_x) = m_{H,\infty}(H_x(k_\infty)/\Gamma_x)\ m_{H,f}(K_{x,f})\,.$$

Thus

$$w_{\text{can}}(\mathcal{O}) = \tau(H_x)^{-1}\ m_{H,\infty}(H_x(k_\infty)/\Gamma_x)\ m_{H,f}(K_{x,f})\,.$$

On the other hand, by the strong approximation theorem $KG(k) = G(\mathbf{A})$, so we find as above that

$$\tau(G) = m_G(KG(k)/G(k)) = m_{G,\infty}(G(k_\infty)/\Gamma)\ m_{G,f}(K_f)\,.$$

Since the gauge forms on G, X and H_x match together algebraically, we have

$$m_{G,f}(K_f) = m_{H,f}(K_{x,f})\ m_{X,f}(xK_f)\,.$$

We see that

$$w(\mathcal{O}) := \frac{m_{H,\infty}(H_x(k_\infty)/\Gamma_x)}{m_{G,\infty}(G(k_\infty)/\Gamma)} = \frac{\tau(H_x)}{\tau(G)} m_{X,f}(xK_f)\ w_{\text{can}}(\mathcal{O})\,.$$

By results of [O], [Sa], [Ko1] (5.1.1), [Ko3],

$$\tau(G) = 1, \qquad \tau(H_x) = |I(H_x)|^{-1}\,|C(H_x)|$$

where for any connected k-group H', $I(H')$ denotes the Tate-Shafarevich group,

$$I(H') = \ker[H^1(k, H') \to \prod_v H^1(k_v, H')]\,.$$

Kottwitz has shown ([Ko1], (4.2.2)) that $I(H')$ can be computed in terms of the Galois module $\pi_1(H)$, hence it does not change under inner twisting. Since the group H_x is an inner form of H, we obtain

$$w(\mathcal{O}) = |I(H)|^{-1}|C(H)|m_{X,f}(x_f K_f)\; w_{\mathrm{can}}(\mathcal{O})\,. \tag{4.5.1}$$

Note that

$$|(X(k) \cap \mathcal{O}_{\mathbf{A}})/G(k)| = |I(H)|\,. \tag{4.5.2}$$

Now assuming that B_f is an orbit of K_f in $X(\mathbf{A}_f)$, we find from Proposition 4.4 and formulas (4.5.1), (4.5.2) that

$$\begin{aligned}\sum_{\mathcal{O}\subset X(k)\cap B} w(\mathcal{O}) &= \sum |I(H)|^{-1}|C(H)|m_{X,f}(x_f K_f)\; w_{\mathrm{can}}(\mathcal{O}) \\ &= |I(H)|^{-1}|C(H)|m_{X,f}(x_f K_f)|I(H)| \\ &= |C(H)|m_{X,f}(x_f K_f)\,.\end{aligned}$$

We have proved Theorem 4.2 under the assumption that B consists of only one orbit of K. In the general case we obtain the assertion of the theorem by summation over the orbits of K in B. □

Corollary 4.6. *If in addition H is semisimple simply connected and has no k-factors H' such that $H'(k_\infty)$ is compact, then any orbit of K in B contains exactly one orbit of Γ in $X(k) \cap B$.*

Proof. Assume that B consists of one orbit of K. We have already proven that there is a point x in $X(k) \cap B$. By the Hasse principle for the simply connected group H, we have $I(H) = 1$, and therefore $X(k) \cap \mathcal{O}_{\mathbf{A}} = xG(k)$. By Weil's Lemma 4.4.1(i), the set of orbits of Γ in $X(k) \cap B = xG(k) \cap xK$ is in a one-to-one correspondence with the set of double cosets $(K \cap H_x) \setminus H_x(\mathbf{A})/H_x(k)$. Since H is simply connected and has no direct factors H' defined over k such that $H'(k_\infty)$ is compact, by the strong approximation theorem the set $(K \cap H_x) \setminus H_x(\mathbf{A})/H_x(k)$ consists of one element. Hence the set $X(k) \cap B$ contains exactly one orbit of Γ. □

Corollary 4.7. *With the assumptions of Corollary* 4.6, *if $K = \prod K_v$ and $B = \prod B_v$, then $|(X(k) \cap B)/\Gamma| = \prod |B_v/K_v|$* □

5 Counting integer points in homogeneous spaces

5.1. Let G be a semisimple group defined over **Q**, acting on a **Q**-vector space W. Consider a Zariski-closed G-orbit X of G in W, defined over **Q**. We assume that $X(\mathbf{Q}) \neq \emptyset$. Let H be the stabilizer of some rational point in $X(\mathbf{Q})$. We suppose that G is connected, semisimple and simply connected, without compact factors defined over **Q**, and that H is connected and has no non-trivial characters defined over **Q**. Note that H is reductive (cf. [B-HC], 3.5). We fix a G-invariant gauge form ω on X; it defines a measure m_∞ on $X(\mathbf{R})$. Choose a euclidean norm in $W_{\mathbf{R}}$.

5.2. We assume throughout this section that the following asymptotic count holds: For any arithmetic group $\Gamma \subset G$, and point $x \in X(\mathbf{Q})$ with stabilizer H_x,

$$|\{y \in x\Gamma : |y| \leqq T\}| \sim \frac{\mathrm{vol}(\Gamma \cap H_x \setminus H_x(\boldsymbol{R}))}{\mathrm{vol}(\Gamma \setminus G(\mathbf{R}))} \times m_\infty(xG(\mathbf{R}) \cap \{|y| \leqq T\}) \quad \text{as} \quad T \to \infty\,,$$

where the invariant measures $m_{G,\infty}$ on $G(\mathbf{R}), m_{H,\infty}$ on $H_x(\mathbf{R})$, and m_∞ on $X(\mathbf{R})$ are compatible.

This assumption is proved (for certain norms) in [DRS] and [EM] when $H \setminus G$ is *symmetric*, i.e. H is the group of fixed points of an involution of G, and in [EMS] in a more general setting. However, results of [Esk], [EMS] indicate that (5.2.1) is probably not valid in general.

Theorem 5.3. *For G, X, H as above, X is Hardy-Littlewood, with density function*

$$\delta(x) = \begin{cases} |C(H)|, & \kappa(x) = 0 \\ 0, & \kappa(x) \neq 0 \end{cases}.$$

Proof. Let $B_f \subset X(\mathbf{A}_f)$ be a non-empty compact open subset. Let $B_\infty \subset X(\mathbf{R})$ be an orbit of $G(\mathbf{R})$. Set $B = B_f \times B_\infty, B_\infty^T = \{x \in B_\infty : |x| \leqq T\}$.

Recall that

$$N(T, X; B) := |X(\mathbf{Q}) \cap (B_\infty^T \times B_f)|.$$

By Lemma 1.6.4 the orbits of $G(\mathbf{A}_f)$ in $X(\mathbf{A}_f)$ are open. We may therefore assume that $B \subset \mathcal{O}_{\mathbf{A}}$ for some orbit $\mathcal{O}_{\mathbf{A}}$ of $G(\mathbf{A})$. If $\kappa(\mathcal{O}_{\mathbf{A}}) \neq 0$, then by Theorem 3.6 there are no **Q**-points in $\mathcal{O}_{\mathbf{A}}$, and hence in B, which proves that $\delta(x) = 0$ when $\kappa(x) \neq 0$.

Now assume that $\kappa(\mathcal{O}_{\mathbf{A}}) = 0$. By Theorem 3.6 there are **Q**-points in $\mathcal{O}_{\mathbf{A}}$. We must show that as $T \to \infty$,

$$N(T, X; B) \sim |C(H)| m(B_\infty^T \times B_f)\,. \tag{5.3.1}$$

Pick a compact open subgroup $K_f \subset G(\mathbf{A}_f)$ such that $B_f K_f = B_f$. Set $K = K_f \times G(\mathbf{R})$. Set $\Gamma = G(\mathbf{Q}) \cap K$; this is an arithmetic subgroup of $G(\mathbf{Q})$. We will use Γ to count points in $X(\mathbf{Q}) \cap B$.

It is clear that the set $X(\mathbf{Q}) \cap B$ is Γ-invariant. Let $\mathcal{O}$ be an orbit of Γ in $X(\mathbf{Q})$. By (5.2.1),

$$|\mathcal{O} \cap (B_\infty^T \times B_f)| \sim m_\infty(B_\infty^T)\, w(\mathcal{O})$$

where

$$w(\mathcal{O}) = \frac{m_{H,\infty}(\Gamma \cap H_x \setminus H_x(\mathbf{R}))}{m_{G,\infty}(\Gamma \setminus G(\mathbf{R}))} .$$

By [B-HC], 6.9, the number of orbits $\mathcal{O} \subset X(\mathbf{Q}) \cap B$ is finite. Summing over all the orbits $\mathcal{O} \subset X(\mathbf{Q}) \cap B$, we see that

$$N(T,X;B) \sim m_\infty(B_\infty^T) \sum_{\mathcal{O} \subset X(\mathbf{Q}) \cap B} w(\mathcal{O}).$$

By Theorem 4.2 the sum in the right hand side equals $|C(H)| \cdot m_f(B_f)$. Thus

$$N(T,X;B) \sim |C(H)| m_\infty(B_\infty^T) m_f(B_f) = |C(H)| m(B_\infty^T \times B_f) . \quad \square$$

Theorem 5.4. *Let G, X, H be as in 5.1 and 5.2. If $C(H) = 0$ then X is strongly Hardy-Littlewood.*

Proof. Indeed, then by Theorem 5.3 X is Hardy-Littlewood with constant density 1, hence strongly Hardy-Littlewood. $\square$

Remark 5.4.1. If X, G, H are as in 5.1 and 5.2, and $C(H) \neq 0$, then X is not strongly Hardy-Littlewood. Indeed, by Theorem 5.3 X is Hardy-Littlewood with density δ taking values 0 and $|C(H)|$, and since we assume that X has a rational point, there exists $B = B_\infty \times B_f$ such that the density δ on B equals $|C(H)| \neq 1$.

Corollary 5.5. *Let G, X, H be as in 5.1. Assume that X is a symmetric space of G and H is semisimple and simply connected. Then X is strongly Hardy-Littlewood.*

Proof. Since H is simply connected, $C(H) = 0$. Since X is symmetric, the asymptotics (5.2.1) hold for X, cf. [DRS], [EM]. By Theorem 5.4, X is strongly Hardy-Littlewood. $\square$

6. Examples

6.1. $W = M_n$ $(n \geqq 2)$, the space of $n \times n$ matrices with the Hilbert-Schmidt norm

$$\|X\|^2 = \operatorname{tr}(X^t X) = \sum_{i,j} x_{ij}^2 .$$

$G = SL_n \times SL_n$ acting on W by left and right multiplication: $X \mapsto g_1^{-1} X g_2$, where $X \in M_n, (g_1, g_2) \in G \times G$.

For an integer $q \neq 0$, take $V_q = \{X \in M_n : \det X = q\}$. Then V_q is a closed orbit of G, with stabilizer H isomorphic to SL_n. Both G and H are semisimple simply connected, and the homogeneous space V_q is symmetric. Clearly V_q has a **Z**-point. By Corollary 5.5, V_q is a strongly Hardy–Littlewood variety.

This example is discussed in detail in [DRS], where it is shown that for all $\varepsilon > 0$,

$$N(T, V_q) \sim E_{HL}(V_q, T) + O_\varepsilon(T^{n^2-n-1/(n+1)+\varepsilon})$$

where $E_{HL}(V_q, T) = \mathfrak{S}(V_q)\mu_\infty(V_q, T)$ is the Hardy-Littlewood expectation, and that

$$E_{HL}(V_k, T) \sim c_{n,k} T^{n^2-n} ,$$

where

$$c_{n,k} = \zeta(2)^{-1} \cdot \ldots \cdot \zeta(n)^{-1} \sum_{d_1 \cdot \ldots \cdot d_n = k} d_2^{-1} d_3^{-2} \ldots d_n^{1-n}, \frac{\pi^{n^2/2}}{\Gamma\left(\frac{n^2-n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)} .$$

6.2. $W = \{X \in M_{2n} : X^t = -X\} (n \geqq 2)$, the space of skew-symmetric matrices, with the Hilbert-Schmidt norm. Let $G = SL_{2n}$, with the action $X \mapsto g^t X g$.

For $q \neq 0$, set $V = \{X \in W : \mathrm{Pff}(X) = q\}$ where $\mathrm{Pff}(X)$ is the *Pfaffian* of a skew-symmetric matrix X, so that $\mathrm{Pff}(X)^2 = \det(X)$. The variety V is a symmetric homogeneous space of G, with stabilizer $H_X \simeq \mathrm{Sp}_{2n}$. Both G and H are connected, semisimple and simply connected. Clearly V has a **Z**-point. By Corollary 5.5, the variety V is strongly Hardy-Littlewood.

6.3. $W = \{X \in M_n : X^t = X\} (n \geqq 3)$, the space of symmetric matrices, with the Hilbert-Schmidt norm. Let $G = SL_n$, with the action $X \mapsto g^t X g$.

For $q \neq 0$, set $V_q = \{X \in W : \det(X) = q\}$. It is a symmetric homogeneous space of G with stabilizer $H_X = \mathrm{SO}(W, X)$, the special orthogonal group of the quadratic form defined by a symmetric matrix X. The variety V_q has a **Z**-point $X_0 = \mathrm{Diag}(q, 1, \ldots, 1)$.

Since $n \geqq 3, H$ is connected semisimple, but not simply connected. We have $\pi_1(H) = \mathbf{Z}/2\mathbf{Z}$, whence $C(H) = \mathbf{Z}/2\mathbf{Z} \simeq \{-1, 1\}$. By Theorem 5.3, V is Hardy–Littlewood with density function taking values 0 and 2.

In this case the local invariants $\kappa_v(\mathcal{O}_v)$ of an adelic orbit $\mathcal{O}_\mathbf{A} = \prod_v \mathcal{O}_v$ can be related to the classic Hasse-Minkowski invariants $c_v(X_v)$ (cf. [Ca]) where $(X_v) \in \mathcal{O}_\mathbf{A}$, $X_v \in \mathcal{O}_v$. Namely,

$$\kappa_v(\mathcal{O}_v) = c_v(X_v) c_v(X_0)$$

We have

$$\kappa(\mathcal{O}_\mathbf{A}) := \prod_v \kappa_v(\mathcal{O}_v) = \prod_v c_v(X_v) ,$$

because $\prod_v c_v(X_0) = 1$ by the product formula.

6.4. Let $F = (f_{ij})$ be an indefinite integral quadratic form in n variables, $n \geqq 3$. We take $W = \mathbf{Q}^n, G = \mathrm{Spin}(W, F)$; it is a simply connected semisimple group.

For $q \neq 0$ set $V = \{x \in W : F(x) = q\}$, it is a symmetric homogeneous space of G. Assume that V has a $\mathbf{Q}$-point. The stabilizer H is isomorphic to Spin_{n-1}, hence connected.

If $n \geqq 4$, then H is semisimple and simply connected. By Corollary 5.5, V is strongly Hardy-Littlewood. This was earlier proved by the circle method, cf. [Da]. For $n = 4$ the proof requires Kloosterman's method of "levelling" [Est].

For $n = 3$, the stabilizer H is a 1-dimensional torus. If H is split over $\mathbf{Q}$, which happens when F is isotropic over $\mathbf{Q}$ and $-q\det(F)$ is a square, then the singular series diverges, and $N(T, V) \sim cT\log T$, while $\mu_\infty(V_q, T) \sim cT$. Thus $N(T, V)$ differs from the Hardy-Littlewood expectation by a factor of order $\log T$ [DRS]. We always assume that H has no $\mathbf{Q}$-characters, in this case it means that H is anisotropic, i.e. $-q\det(F)$ is not a square. The asymptotics (5.2.1) hold, cf. [DRS], and $C(H) = \mathbf{Z}/2\mathbf{Z}$. By Theorem 5.3 V_q is Hardy-Littlewood with density taking values 2 and 0.

Note that by Proposition 2.4, for a strongly Hardy–Littlewood variety V strong approximation holds. In particular, if $V(\mathbf{R}) \neq \emptyset$ and for all $p, V(\mathbf{Z}_p) \neq \emptyset$, then $V(\mathbf{Z}) \neq \emptyset$. However for a non-strongly Hardy-Littlewood variety, strong approximation may not hold. We provide two specific examples.

6.4.1 Take $F(x_1, x_2, x_3) = -9x_1^2 + 2x_1x_2 + 7x_2^2 + 2x_3^2$, and consider the quadric $V = \{x \in \mathbf{Q}^3 : F(x) = 1\}$. It turns out that $V(\mathbf{Z})$ has $\mathbf{Z}_p$-points for any prime p, but has no integer points. Indeed, $F(-\frac{1}{2}, \frac{1}{2}, 1) = 1$, hence F represents 1 over $\mathbf{Z}_p$ for $p > 2$. In addition, $F(4, 1, 1) = -127$, and using Hensel's lemma, one can easily check that F represents 1 over $\mathbf{Z}_2$. We are grateful to J.H. Conway, R. Schultze-Pillot and D. Zagier (personal communications) for different proofs of the following

Claim 6.4.1.1. *F does not represent 1 over $\mathbf{Z}$.*

'Elementary' proof (after D. Zagier). Assume that there exist integer numbers x_1, x_2, x_3 such that $F(x_1, x_2, x_3) = 1$. We may write the equation as

$$2x_3^2 - 1 = (x_1 - x_2)^2 + 8(x_1 - x_2)(x_1 + x_2) .$$

Easy calculations modulo 16 show that $x_1 - x_2 \equiv \pm 3 \bmod 8$. It follows that $x_1 - x_2$ and also $2x_3^2 - 1$ must have a prime factor p congruent to $\pm 3 \bmod 8$. On the other hand, if a prime p divides $2x_3^2 - 1$, then $2x_3^2 \equiv 1 \bmod p$, and 2 is a square mod p. Then by the quadratic reciprocity law $p \equiv \pm 1 \bmod 8$. Contradiction. □

Note that this phenomenon (non-representability of an integer by an integral form F when there is no congruence obstruction) cannot occur when the genus of F contains only one class (see [Ca, Ch. 9 Thm. 1.3]). The example above is a "minimal" one: here the discriminant $D(F) = -128$, and any indefinite integral ternary quadratic form of discriminant $|D| < 128$ has only one class in its genus, cf. [CS, Ch. 15, 9.7].

6.4.2. Take $F(x) = x_1^2 + x_2^2 - x_3^2$, and consider the quadric $V = \{x : F(x) = -1\}$. There is an integer point $x_0 = (0, 0, 1) \in V(\mathbf{Z})$. We try to find integer points $x = (x_1, x_2, x_3)$ in V such that $x \equiv x_0 \bmod 8$ and $x_3 < 0$. Set:

$$B_2 = \{x \in V(\mathbf{Z}_2) : x \equiv x_0 \bmod 8\}, \quad B_f = B_2 \times \prod_{p>2} X(\mathbf{Z}_p),$$

$$B_\infty^+ = \{x \in V(\mathbf{R}) : x_3 > 0\}, \quad B_\infty^- = \{x \in V(\mathbf{R}) : x_3 < 0\}.$$

Claim 6.4.2.1.

$$V(\mathbf{Q}) \cap (B_\infty^- \times B_f) = \varnothing$$
$$N(T, V; B_\infty^+ \times B_f) \sim 2E_{HL}(T, V; B_\infty^+),$$

where $E_{HL}(T, V; B_\infty^+) = 2m((B_\infty^+)^T \times B_f)$ *is the Hardy-Littlewood expectation.*

Idea of proof. We compute local invariants κ_v with respect to the base point x_0.

By Lang's theorem and Hensel's lemma, $B_p \subset x_0 G(\mathbf{Z}_p)$ for any finite prime p, hence, in multiplicative notation, κ_p equals 1 on B_p. At infinity, $B_\infty^+ = x_0 G(\mathbf{R})$ and $B_\infty^- \cap x_0 G(\mathbf{R}) = \varnothing$, hence κ_∞ equals $+1$ on B_∞^+ and equals -1 on B_∞^- (again in multiplicative notation). We see that the product over all the places, $\kappa(x)$, equals $+1$ on $B_\infty^+ \times B_f$ and equals -1 on $B_\infty^- \times B_f$. Now the Claim follow from Theorem 5.3. □

In other words, Claim 6.4.2.1 means that

$$\{x \in \mathbf{Z}^3 : |x| \leqq T,\ F(x) = -1, x \equiv (0,0,1) \bmod 8,\ x_3 < 0\} = \varnothing$$
$$|\{x \in \mathbf{Z}^3 : |x| \leqq T,\ F(x) = -1, x \equiv (0,0,1) \bmod 8,\ x_3 > 0\}|$$
$$\sim 2\mathrm{E}_{HL}(T, V; B_\infty^+).$$

One can also prove the first assertion of Claim 6.4.2.1 by an 'elementary' argument similar to that of 6.4.1.1.

6.5. $G = SL_2$, and W is the space

$$W = \{f(x,y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n\}$$

of binary forms of degree n ("binary n-ics", see [Di]), on which $G = SL_2$ acts by linear substitutions. We assume that $n \geqq 3$. As a norm we take

$$\|f\|^2 = \sum_{i=0}^{n} \binom{n}{i}^{-1} |a_i|^2.$$

Let $\mathscr{A} = \mathbf{Q}[W]^{SL_2}$ be the algebra of invariants of binary n-ics. For $\alpha \in \operatorname{Spec} \mathscr{A}$, we denote by V_α the corresponding level set. For generic α, the level set V_α is a single G-orbit with finite stabilizer H.

If $n \geqq 4$ is even, then the generic stabilizer is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ for $n = 4$, and $\{\pm 1\}$ for $n \geqq 6$, and is thus disconnected. By Proposition 2.7, V_α is not Hardy-Littlewood. For $n = 3$, the generic stabilizer H is isomorphic to $\mathbf{Z}/3\mathbf{Z}$; again V_α is not Hardy-Littlewood.

If n is odd, $n \geqq 5$, then the generic stabilizer is trivial.The asymptotic count (5.2.1) is proved in [DRS (for all $n \geqq 3$). By Theorem 5.4, a generic level set V_α is strongly Hardy-Littlewood.

6.6. $W = M_n$ ($n \geqq 2$), the space of $n \times n$ matrices, with the Hilbert-Schmidt norm. Take $G = SL_n$, with the action $(X, g) \mapsto X^g = g^{-1} X g$ for $X \in M_n$, $g \in G$.

For a given monic polynomial with integer coefficients $f(t) = t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbf{Z}[t]$, we consider the variety of $n \times n$ matrices having $f(t)$ as characteristic polynomial:

$$V_f = \{X \in M_n : \det(tI - X) = f(t)\} .$$

We assume that f is irreducible over $\mathbf{Q}$. Then f has no multiple roots, and therefore V_f is a homogeneous space of G. The variety V_f has an integer point

$$X_0 = \begin{pmatrix} 0 & 0 & \dots & -a_n \\ 1 & 0 & \dots & -a_{n-1} \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & -a_1 \end{pmatrix}$$

The stabilizer H of X_0 is an $(n-1)$-dimensional torus, isomorphic to $\ker[\mathrm{Nm}: K^\times \to \mathbf{Q}^\times]$ where $K = \mathbf{Q}(\alpha), \alpha$ is a root of f. The group H is connected and has no non-trivial $\mathbf{Q}$-characters. The asymptotic count (5.2.1) is proved in [EMS]. By Theorem 5.3, V_f is a Hardy-Littlewood variety with density function taking values 0 and $|C(H)|$.

Let L be a normal closure of K. To describe $C(H)$, let us fix an ordering of the roots of $f(t)$. We can now identify $\mathrm{Gal}(L/\mathbf{Q})$ with a subgroup $\Gamma \subset S_n$, and its subgroup $\mathrm{Gal}(L/K)$ with the stabilizer Γ_1 of 1 in Γ. The Galois group Γ acts on $\pi_1(H) = X_*(H) \simeq \{a \in \mathbf{Z}^n : \sum_i a_i = 0\}$ by permuting the coordinates. The group $C(H)$ can be computed as follows:

Claim 6.6.1. *$C(H) \simeq$ coker $\left[\mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{Gal}(L/\mathbf{Q})^{\mathrm{ab}}\right]$.*

Proof. The short exact sequence of $\mathrm{Gal}(L/\mathbf{Q})$-modules

$$0 \to X_*(H) \to \mathrm{Ind}_{\Gamma_1}^{\Gamma} \mathbf{Z} \to \mathbf{Z} \to 0$$

gives rise to a cohomology exact sequence

$$\cdots \to H^{-2}(\Gamma_1, \mathbf{Z}) \to H^{-2}(\Gamma, \mathbf{Z}) \to H^{-1}(\Gamma, X_*(H)) \to 0 .$$

We can identify $H^{-1}(\Gamma, X_*(H))$ with $C(H)$. Further, $H^{-2}(\Gamma, \mathbf{Z}) = H_1(\Gamma, \mathbf{Z}) = \Gamma^{\mathrm{ab}} = \mathrm{Gal}(L/\mathbf{Q})^{\mathrm{ab}}$, and similarly, $H^{-2}(\Gamma_1, \mathbf{Z}) = \Gamma_1^{\mathrm{ab}} = \mathrm{Gal}(L/K)^{\mathrm{ab}}$. Thus $C(H) \simeq$ coker $\left[\mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{Gal}(L/\mathbf{Q})^{\mathrm{ab}}\right]$. □

Specific examples

(1) $\mathrm{Gal}(L/\mathbf{Q}) = S_n$: We have $C(H) \simeq$ coker $\left[S_{n-1}^{\mathrm{ab}} \to S_n^{\mathrm{ab}}\right] = 0$, and so V_f is strongly Hardy-Littlewood for $n \geqq 3$.

(2) $\mathrm{Gal}(L/\mathbf{Q}) = A_n$: In this case $C(H) \simeq$ coker $\left[A_{n-1}^{\mathrm{ab}} \to A_n^{\mathrm{ab}}\right]$. One can check that for $n \geqq 4$, $C(H) = 0$, while for $n = 2$, $C(H) \simeq \mathbf{Z}/2\mathbf{Z}$, and for $n = 3$, $C(H) \simeq \mathbf{Z}/3\mathbf{Z}$. We find that in this case, V_f is strongly Hardy-Littlewood if and only if $n \geqq 4$.

(3) $K/\mathbf{Q}$ is a Galois extension: Then $L = K$, and $C(H) = \mathrm{Gal}(K/\mathbf{Q})^{\mathrm{ab}}$.

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