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NODAL INTERSECTIONS AND L^p RESTRICTION THEOREMS ON THE TORUS*

BY

JEAN BOURGAIN

School of Mathematics
Institute for Advanced Study, Princeton, NJ 08540, USA
e-mail: bourgain@ias.edu

AND

ZEÉV RUDNICK

Raymond and Beverly Sackler School of Mathematical Sciences
Tel Aviv University, Tel Aviv 69978, Israel
e-mail: rudnick@post.tau.ac.il

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ABSTRACT

We study the number of intersections of the nodal lines of an eigenfunction of the Laplacian on the standard torus with a fixed reference curve, that is, the number of zeros of the eigenfunction restricted to the curve. An upper bound is the wave number k. When the curve has nowhere zero curvature, we conjecture that, up to a constant multiple, this should also be the correct lower bound. We give a lower bound which differs from this by an arithmetic quantity, given in terms of the maximal number of lattice points in arcs of size square root of the wave number k on a circle of radius k. According to a conjecture of Cilleruelo and Granville, this quantity is bounded, in which case we recover our conjecture. To get at the lower bound, we reduce the problem to giving a lower bound for the L^1 norm of the restriction of the eigenfunction to the curve, and then to an upper bound for the L^4 restriction norm.

1. Introduction

1.1. Nodal intersections. Let $\mathcal{C} \subset \mathbb{T}^2$ be a curve on the standard torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, which has nowhere-zero curvature. Let F be a real-valued eigenfunction of the Laplacian on $\mathbb{T}^2$ with eigenvalue λ^2: $-\Delta F = \lambda^2 F$. We want to estimate the number of nodal intersections

$$N_{F,\mathcal{C}} = \#\{x : F(x) = 0\} \cap \mathcal{C}, \tag{1.1}$$

that is, the number of zeros of F on $\mathcal{C}$.

If $\mathcal{C}$ is real analytic, then upper bounds of the form $N_{F,\mathcal{C}} \ll \lambda$ can be obtained from a result of Toth and Zelditch [13] (see also [4], [7]) once we have an exponential restriction lower bound $\int_{\mathcal{C}} |F|^2 \gg e^{-c\lambda} ||F||_2^2$ for the L^2-norm of F restricted to $\mathcal{C}$, in terms of the L^2-norm $||F||_2^2 = \int_{\mathbb{T}^2} |F(x)|^2 dx$. In the case of the torus, for any smooth $\mathcal{C}$ with non-vanishing curvature we have earlier obtained a uniform L^2-restriction bound [1]

$$\int_{\mathcal{C}} |F|^2 \gg ||F||_2^2 \tag{1.2}$$

(the implied constants depending only on the curve $\mathcal{C}$), and hence by [13] we get an upper bound for $\mathcal{C}$ analytic

$$N_{F,\mathcal{C}} \ll \lambda. \tag{1.3}$$

In our paper [4] we also obtained a lower bound for $N_{F,\mathcal{C}}$ when the curve $\mathcal{C}$ has non-vanishing curvature:

$$N_{F,\mathcal{C}} \gg \lambda^{1-o(1)}. \tag{1.4}$$

We conjecture that the correct lower bound is

$$N_{F,\mathcal{C}} \gg \lambda, \tag{1.5}$$

that is, the lower bound should be the same order of magnitude as the upper bound.

In this paper we approach conjecture (1.5) by giving a lower bound for $N_{F,\mathcal{C}}$ in terms of an arithmetic quantity, the maximal number B_λ of lattice points which lie on an arc of size $\sqrt{\lambda}$ on the circle $|x| = \lambda$:

$$B_\lambda = \max_{|x|=\lambda} \#\{\xi \in \mathcal{E} : |x-\xi| \le \sqrt{\lambda}\}, \tag{1.6}$$

where $\mathcal{E} = \mathcal{E}_\lambda$ is the set of all lattice points on the circle $|x| = \lambda$.

THEOREM 1.1: *If $\mathcal{C}$ is smooth with non-zero curvature, then*

$$N_{F,\mathcal{C}} \gg \lambda / B_\lambda^{5/2}. \tag{1.7}$$

According to the conjecture of Cilleruelo and Granville [6], $B_\lambda = O(1)$ is bounded, which, in view of Theorem 1.1, implies conjecture (1.5).

The conjecture of Cilleruelo and Granville is known for "almost all" λ [2], but individually we only know a bound of $B_\lambda \ll \log \lambda$; see §2.

To contrast with these results, we show in §8 that no lower bounds for $N_{F,\mathcal{C}}$ are possible when the curvature is zero, that is, for geodesic segments, in fact that $\liminf_\lambda N_{F,\mathcal{C}} = 0$. We also briefly discuss the situation on the sphere.

1.2. RELATION WITH L^p RESTRICTION THEOREMS. To prove Theorem 1.1 we start by giving a lower bound for $N_{F,\mathcal{C}}(\lambda)$ in terms of a lower bound for the restriction L^1-norm: In §5 we show

THEOREM 1.2: *If $\mathcal{C}$ is smooth with non-zero curvature, then*

$$N_{F,\mathcal{C}} \gg \lambda \cdot \left(\frac{1}{||F||_2} \int_{\mathcal{C}} |F| \right)^5. \tag{1.8}$$

We conjecture a uniform lower bound for the restriction L^1-norm, which will imply (1.5).

Next, in §6 we give a lower bound for $||F||_{L^1(\mathcal{C})} = \int_{\mathcal{C}} |F|$ in terms of the restriction L^4 norm:

$$||F||_{L^1(\mathcal{C})} \gg_{\mathcal{C}} \frac{||F||_2^3}{||F||_{L^4(\mathcal{C})}^2}. \tag{1.9}$$

Thus we find that we are reduced to giving an upper bound on the restriction L^4-norm. In §7 we show

Theorem 1.3: *If $\mathcal{C}$ is smooth with non-zero curvature, then*

$$||F||_{L^4(\mathcal{C})} \ll B_\lambda^{1/4} ||F||_2. \tag{1.10}$$

Inserting Theorem 1.3 into (1.9) we obtain

$$\frac{1}{||F||_2} \int_{\mathcal{C}} |F| \gg \frac{1}{\sqrt{B_\lambda}} \tag{1.11}$$

and using Theorem 1.2 we obtain Theorem 1.1.

1.3. Prior results. There are very few lower bounds on the number of nodal intersections available for other models. In the case of the modular domain $\mathbb{H}^2/SL_2(\mathbb{Z})$ and $\mathcal{C}$ being a closed horocycle, Ghosh, Reznikov and Sarnak [8] give a lower bound $N_{F,\mathcal{C}} \gg \lambda^{1/12-o(1)}$ for eigenfunctions F which are joint eigenfunctions of all Hecke operators, and assuming the Generalized Riemann Hypothesis they give a similar result when $\mathcal{C}$ is a sufficiently long segment of the infinite geodesic running between two cusps,

Concerning upper bounds, El-Hajj and Toth [7] show that for a bounded, piecewise-analytic convex domain with ergodic billiard flow and $\mathcal{C}$ an analytic interior curve with strictly positive geodesic curvature, the upper bound (1.3) holds for a density-one subsequence of eigenfunctions. For eigenfunctions on a compact hyperbolic surface, Jung [11] has recently obtained an upper bound analogous to (1.3) when $\mathcal{C}$ is a geodesic circle.

2. Lattice points and geometry

2.1. Lattice points in short arcs. We denote by $\mathcal{E} = \mathcal{E}_\lambda$ the set of lattice points on the circle $|x| = \lambda$. As is well known, $\#\mathcal{E} \ll \lambda^{o(1)}$ and can grow faster than any power of $\log \lambda$. Concerning lattice points in short arcs, Jarnik [10] showed that any arc of length $\lambda^{1/3}$ contains at most two lattice points. Cilleruelo and Córdoba [5] showed that for fixed $\delta > 0$, any arc of length $\lambda^{1/2-\delta}$

contains at most $M(\delta)$ lattice points. The natural conjecture here [6] is that the same statement holds for arcs of length $\lambda^{1-\delta}$. However, this is still open even for arcs of size $\sqrt{\lambda}$. That turns out to be a critical regime for us, and we set

$$B_\lambda = \max_{|x|=\lambda} \#\{\mu \in \mathcal{E} : |\mu - x| < \sqrt{\lambda}\} \tag{2.1}$$

to be the maximal number of lattice points in arcs of size $\sqrt{\lambda}$.

Lemma 2.1: *Let $B = B_\lambda(c)$ be the maximal number of lattice points of $\mathcal{E}$ in an arc of length $c\sqrt{\lambda}$, $c \geq 1/2$. Then*

$$B \ll c \log \lambda. \tag{2.2}$$

Proof. To see this, we recall that Cilleruelo and Córdoba [5] showed that if $P_1, \ldots, P_m \in \mathcal{E}$ are distinct lattice points on the circle of radius λ, then

$$\prod_{1 \leq i < j \leq m} |P_i - P_j| \geq \lambda^{e(m)}, \quad e(m) = \begin{cases} \frac{m}{2}(\frac{m}{2} - 1), & m \text{ even}, \\ \frac{1}{4}(m-1)^2, & m \text{ odd}. \end{cases} \tag{2.3}$$

Thus if $P_1, \ldots, P_m \in \mathcal{E}$ lie in an arc of diameter $D = \max |x - y| < \sqrt{\lambda}/2$, then by (2.3) we find

$$D^{m(m-1)/2} \geq \lambda^{m(m-2)/4} \tag{2.4}$$

and hence

$$m \leq \frac{\log \lambda}{2 \log 2} + 1 \ll \log \lambda. \tag{2.5}$$

Now for an arc of length $c\sqrt{\lambda}$, $c > 1/2$, divide it into $\approx 2c$ smaller arcs of length $\sqrt{\lambda}/2$ and use (2.5) to find that it contains $\ll c \log \lambda$ lattice points. ∎

2.2. Medians. Given a pair of points μ, ν on the circle $|x| = \lambda$, their median is $z = \frac{1}{2}(\mu + \nu)$. This gives a map from pairs of points on the circle λS^1 to points in the disc of radius λ:

$$z : \lambda S^1 \times \lambda S^1 \to \{|z| \leq \lambda\}.$$

By definition, if $\mu = \nu$ then $z = \mu$. Note that the origin is the median of all pairs of antipodal points $\{\mu, -\mu\}$.

Conversely, given a nonzero point in the interior of the punctured disk $\{0 < |x| < \lambda\}$, we can display it as the median of a unique (unordered) pair of points obtained as the intersection of the circle λS^1 with line through z perpendicular to the radial line between z and the origin; see Figure 1. In fact the formula for these points is

$$\mu_\pm(z) = z \pm \Delta(z)\frac{z^\perp}{|z^\perp|}, \tag{2.6}$$

where if $z=(x,y)$ then $z^\perp = (-y,x)$, and where we set (see Figure 1)

$$\Delta(z) = \sqrt{\lambda^2 - |z|^2} = \frac{1}{2}|\mu_+(z) - \mu_-(z)|. \tag{2.7}$$

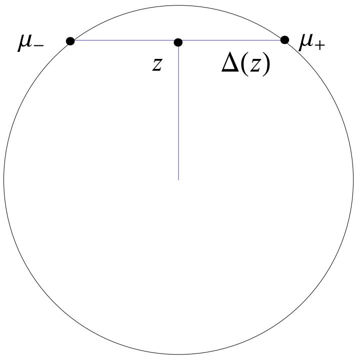

Figure 1. The median map and its inverse

Let $\mathcal{Z} = \mathcal{Z}_\lambda$ be the set of medians of integer points with $|\mu| = \lambda$. Note that $\#\mathcal{Z} \le (\#\mathcal{E})^2 \ll \lambda^{o(1)}$.

Lemma 2.2: *Given vectors $z, v \in \mathbb{R}^2$, the number of $w \in \mathcal{Z}_\lambda$ for which*

$$|\mu_+(w) - v| < \sqrt{\lambda} \tag{2.8}$$

and

$$|w - z| < \lambda^{1/3} \tag{2.9}$$

is at most $O(B_\lambda)$.

Proof. The medians w satisfying (2.8) have their corresponding lattice points $\mu_+(w)$ each lying in an arc of length about $\sqrt{\lambda}$, and hence there are at most B_λ

possibilities for $\mu_+(w)$. Given $\mu_+(w)$, we have at most 2 possibilities for $\mu_-(w)$: Indeed, since $w = (\mu_+(w) + \mu_-(w))/2$, we have

$$\mu_-(w) = 2w - \mu_+(w) = 2z - \mu_+(w) + 2(w - z). \tag{2.10}$$

Since $|w - z| < \lambda^{1/3}$, given z and $\mu_+(w)$ we know $\mu_-(w)$ up to an error of $O(\lambda^{1/3})$; by Jarnik's theorem, which states that an arc of size $\lambda^{1/3}$ contains at most two lattice points, this implies there are at most two possibilities for $\mu_-(w)$.

Since $w = (\mu_+(w) + \mu_-(w))/2$ is determined by knowing both $\mu_\pm(w)$, we see that there are at most $O(B_\lambda)$ possibilities for w. ∎

3. An oscillatory integral along the curve

3.1. Phase functions on the curve. Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ be the standard flat torus. An eigenfunction F of the Laplacian on $\mathbb{T}^2$ with eigenvalue λ^2 has a Fourier expansion

$$F(x) = \sum_{\mu\in\mathcal{E}} a_\mu e^{i\langle\mu,x\rangle}. \tag{3.1}$$

For F to be real valued forces $\overline{a_\mu} = a_{-\mu}$. The supremum of F is bounded by

$$||F||_\infty \le \sum_{\mu\in\mathcal{E}} |a_\mu| \le \frac{1}{2\pi}||F||_2\sqrt{\#\mathcal{E}}. \tag{3.2}$$

We normalize so that

$$4\pi^2||F||_2^2 = \sum_{\mu\in\mathcal{E}} |a_\mu|^2 = 1. \tag{3.3}$$

Let $\gamma : [0, L] \to \mathcal{C}$ be an arc-length parameterization of $\mathcal{C}$, so that $\gamma'(t)$ is the unit tangent vector to the curve at the point $\gamma(t)$. Denote by $n(t)$ the standard unit normal to the curve at the point $\gamma(t)$, so that $\gamma''(t) = \kappa(t)n(t)$ with $\kappa(t)$ the curvature. Let $K_{\min} > 0$ and $K_{\max}$ be the minimum and maximum values of the curvature, so that

$$0 < K_{\min} \le \kappa(t) \le K_{\max}. \tag{3.4}$$

By shrinking the curve $\mathcal{C}$, we may assume that its total curvature is $< \pi/2$.

We denote $f(t) = F(\gamma(t))$. Using the Fourier expansion of F, we write

$$f(t) = \sum_{\mu\in\mathcal{E}} a_\mu e^{i\langle\mu,\gamma(t)\rangle} = \sum_{\mu\in\mathcal{E}} a_\mu e^{i\lambda\phi_\mu(t)}, \tag{3.5}$$

where the phase function ϕ_μ is

$$\phi_\mu(t) = \left\langle \frac{\mu}{|\mu|}, \gamma(t) \right\rangle. \tag{3.6}$$

The derivative of ϕ_μ is

$$\phi'_\mu(t) = \left\langle \frac{\mu}{|\mu|}, \gamma'(t) \right\rangle = \sin\alpha_\mu(t), \tag{3.7}$$

where $\alpha_\mu(t)$ is the angle between the normal vector $n(t)$ and μ. Since we assume the total curvature of the curve is $< \pi/2$, the change in the angle α_μ is less than $\pi/2$. The second derivative is

$$\phi''_\mu(t) = \left\langle \frac{\mu}{|\mu|}, \gamma''(t) \right\rangle = \kappa(t)\cos\alpha_\mu(t). \tag{3.8}$$

By (3.4),

$$|\phi''_\mu| \le K_{\max}. \tag{3.9}$$

The third derivative is

$$\phi'''_\mu = \left\langle \frac{\mu}{|\mu|}, \gamma''' \right\rangle = \left\langle \frac{\mu}{|\mu|}, \kappa' n + \kappa n' \right\rangle = \left\langle \frac{\mu}{|\mu|}, \kappa' n - \kappa^2\gamma' \right\rangle \tag{3.10}$$

(since $n' = -\kappa\gamma'$) and hence, since $n \perp \gamma'$,

$$|\phi'''_\mu| \le (|\kappa'|_\infty^2 + K_{\max}^4)^{1/2} \tag{3.11}$$

is bounded independent of μ.

Lemma 3.1: *For $0 < \sigma \ll 1$ sufficiently small, let B_μ be the set of points where $|\phi'_\mu(t)| < 2\sigma$. Then B_μ is an interval and*

$$\operatorname{length} B_\mu \ll \frac{\sigma}{K_{\min}}. \tag{3.12}$$

Proof. We have

$$|\phi'_\mu(t)| = \left|\left\langle \frac{\mu}{|\mu|}, \gamma'(t) \right\rangle\right| = |\sin\alpha_\mu(t)| < 2\sigma. \tag{3.13}$$

Since we assume the total curvature is $< \pi/2$, the change in the angle α_μ is less than $\pi/2$ and hence $B_\mu = (c_, c_+)$ consists of at most a single interval.

Since B_μ is in particular connected and

$$|\phi''_\mu(t)| = \kappa(t)|\cos\alpha_\mu| \ge K_{\min}\sqrt{1-4\sigma^2} \ge K_{\min}/2$$

on B_μ, we may assume that $\phi_\mu'' \geq K_{\min}/2 > 0$ on B_μ so that ϕ_μ' is monotonically increasing. Then $\phi_\mu'(c_-) \geq -2\sigma$, $\phi_\mu'(c_+) \leq +2\sigma$ and we have

$$4\sigma \geq \phi_\mu'(c_+) - \phi_\mu'(c_-) = (c_+ - c_-)\phi_\mu''(c) \tag{3.14}$$

for some $c \in (c_-, c_+)$ and hence

$$\operatorname{length} B_\mu = c_+ - c_- \leq \frac{4\sigma}{\phi_\mu''(c)} \leq \frac{8\sigma}{K_{\min}} \tag{3.15}$$

as claimed. ∎

3.2. Van der Corput's lemma. Let $[a,b]$ be a finite interval, $\phi \in C^\infty[a,b]$ a smooth and real valued phase function, and $A \in C^\infty[a,b]$ a smooth amplitude. For $\lambda > 0$ define the oscillatory integral

$$I(\lambda) := \int_a^b A(t) e^{i\lambda\phi(t)} dt. \tag{3.16}$$

We will need the following well-known result, due to van der Corput (see, e.g., [12])

Lemma 3.2: *Assume that $|\phi''| \geq 1$. Then*

$$|I(\lambda)| \ll \frac{1}{\lambda^{1/2}} \{||A||_\infty + ||A'||_1\}. \tag{3.17}$$

If $|\phi'| \geq 1$ and moreover ϕ' is monotonic, then

$$|I(\lambda)| \ll \frac{1}{\lambda} \{||A||_\infty + ||A'||_1\}, \tag{3.18}$$

the implied constants absolute.

3.3. An oscillatory integral along a curve. For each $0 \neq \xi \in \mathbb{R}^2$ define a phase function on the curve $\mathcal{C}$ by

$$\phi_\xi(t) = \left\langle \frac{\xi}{|\xi|}, \gamma(t) \right\rangle. \tag{3.19}$$

Let $A \in C^\infty[0,L]$ be a smooth amplitude, k real and

$$I(k) = \int A(t) e^{ik\phi_\xi(t)} dt. \tag{3.20}$$

Lemma 3.3: *For $|k| \geq 1$,*

$$|I(k)| \ll \frac{1}{|k|^{1/2}} \{||A||_\infty + ||A'||_1\}, \tag{3.21}$$

the implied constant depending only on the curve $\mathcal{C}$ (independent of ξ).

Proof. We wish to apply Lemma 3.2. Since the total curvature of $\mathcal{C}$ is $< \pi/2$, each of the phase functions ϕ_ξ has at most one stationary point (at a point where ξ is normal to the curve). Moreover, $\phi''_\xi(t) = \kappa(t)\cos\alpha_\xi(t)$ has at most one sign change since we restrict the total curvature to be $< \pi/2$.

Near a stationary point t_0, we have $|\phi'_\xi(t)| < 1/2$ if $|t-t_0| < 1/(2K_{\max})$, since

$$|\phi'_\xi(t)| = |\phi'_\xi(t) - \phi'_\xi(t_0)| = |t-t_0|\cdot|\phi''_\xi(t_1)| \le K_{\max}|t-t_0|. \tag{3.22}$$

If $|\phi'_\xi(t)| < 1/2$, then

$$|\phi''_\xi(t)| = \kappa(t)|\cos\alpha_\xi(t)| = \kappa(t)\sqrt{1-\phi'_\xi(t)^2} \ge K_{\min}\frac{\sqrt{3}}{2}. \tag{3.23}$$

Hence we may cut the curve, that is, the arc-length parameter interval $[0, L]$, into at most 4 segments on each of which either $|\phi''_\xi| \ge \frac{\sqrt{3}}{2}K_{\min} > 0$ or $|\phi'_\xi| \ge 1/2$ and ϕ''_ξ does not change sign, hence ϕ'_ξ is monotonic. Then we can invoke Lemma 3.2 to deduce that either (3.18) or (3.17) hold, and since $|k| \ge 1$ we have (3.17) valid in both cases. ∎

4. A bilinear inequality on the curve

As before let $\mathcal{E} = \{\mu \in \mathbb{Z}^2 : |\mu| = \lambda\}$. For each $\mu \in \mathcal{E}$ let $h_\mu(t) \in C^1_c(\mathbb{R})$ and $a_\mu \in \mathbb{C}$ with $\sum_{\mu\in\mathcal{E}} |a_\mu|^2 = 1$. Let

$$H(t) := \sum_{\mu\in\mathcal{E}} a_\mu h_\mu(t) e^{i\langle\mu,\gamma(t)\rangle}. \tag{4.1}$$

Lemma 4.1:

$$||H||_2^2 \le 2\max_{\mu\in\mathcal{E}} ||h_\mu||_2^2 + O\left(\frac{\#\mathcal{E}}{\lambda^{1/6}}\{\max_{\mu\in\mathcal{E}}||h_\mu||_\infty^2 + \max_{\mu\in\mathcal{E}}||h_\mu||_\infty \max_{\mu\in\mathcal{E}}||h'_\mu||_1\}\right). \tag{4.2}$$

Proof. Multiplying out gives

$$||H||_2^2 = \sum_{\mu,\nu} a_\mu \bar{a}_\nu \int h_\mu(t)\overline{h_\nu(t)} e^{i|\mu-\nu|\phi_{\mu-\nu}(t)}dt, \tag{4.3}$$

where if $\mu=\nu$ we set $\phi_0(t) \equiv 1$. We separate the double sum (4.3) to a sum over "close" pairs (μ,ν), that is, such that $|\mu-\nu| < \lambda^{1/3}$, and to a sum over the remaining "distant" pairs. We claim that the "close" pairs contribute

$$\text{close} \le 2\max_{\mu\in\mathcal{E}} ||h_\mu||_2^2 \tag{4.4}$$

while the "distant" pairs contribute at most

$$\text{distant} \ll \frac{\#\mathcal{E}}{\lambda^{1/6}}\{\max_{\mu\in\mathcal{E}}||h_\mu||_\infty^2 + \max_{\mu\in\mathcal{E}}||h_\mu||_\infty \max_{\mu\in\mathcal{E}}||h'_\mu||_1\}. \tag{4.5}$$

4.0.1. *Close pairs.* Given $\mu \in \mathcal{E}$, certainly we can take $\nu = \mu$ to get a "close" pair. By Jarnik's theorem [10], given $\mu \in \mathcal{E}$ there is at most one other element of $\mathcal{E}$ at distance $\le \lambda^{1/3}$ from μ, call it $\tilde{\mu}$ (if it exists). Estimating the integral trivially by

$$\left|\int h_\mu(t)\overline{h_\nu(t)}e^{i\langle \mu-\nu,\gamma(t)\rangle}dt\right| \le ||h_\mu||_2 \cdot ||h_\nu||_2 \tag{4.6}$$

we find that the contribution of "close" pairs is bounded by

$$\max_\mu ||h_\mu||_2^2 \cdot \sum_{\mu\in\mathcal{E}} |a_\mu|^2 + |a_\mu||a_{\tilde{\mu}}|. \tag{4.7}$$

If μ does not have a close neighbor other than itself, the term $a_\mu a_{\tilde{\mu}}$ is zero. Otherwise, use $|a_\mu a_{\tilde{\mu}}| \le \frac{1}{2}(|a_\mu|^2 + |a_{\tilde{\mu}}|^2)$. Since each μ has at most one such close neighbor $\tilde{\mu}$, the sum over all $\mu \in \mathcal{E}$ is at most

$$\sum_{\mu\in\mathcal{E}} |a_\mu||a_{\tilde{\mu}}| \le \sum_{\mu\in\mathcal{E}} \frac{1}{2}(|a_\mu|^2 + |a_{\tilde{\mu}}|^2) \le \sum_{\mu\in\mathcal{E}} |a_\mu|^2 = 1 \tag{4.8}$$

and hence

$$\text{close} \le 2\max_\mu ||h_\mu||_2^2. \tag{4.9}$$

4.0.2. *Distant pairs.* We now bound the contribution of pairs μ, ν with $|\mu - \nu| > \lambda^{1/3}$ by

$$\text{distant} \le \sum_{|\mu-\nu|>\lambda^{1/3}} |a_\mu||a_\nu||I(\mu,\nu)|, \tag{4.10}$$

where

$$I(\mu,\nu) := \int h_\mu(t)\overline{h_\nu(t)}e^{i\langle \mu-\nu,\gamma(t)\rangle}dt = \int A_{\mu,\nu}(t)e^{i|\mu-\nu|\phi_{\mu-\nu}(t)}dt \tag{4.11}$$

with $A_{\mu,\nu} = h_\mu(t)\overline{h_\nu(t)}$.

By Lemma 3.3,

$$\begin{aligned} |I(\mu,\nu)| &\ll \frac{1}{|\mu-\nu|^{1/2}}(||A_{\mu,\nu}||_\infty + ||A'_{\mu,\nu}||_1) \\ &\ll \frac{1}{\lambda^{1/6}}\{\max_{\mu\in\mathcal{E}}||h_\mu||_\infty^2 + \max_{\mu\in\mathcal{E}}||h_\mu||_\infty \max_{\mu\in\mathcal{E}}||h'_\mu||_1\}. \end{aligned} \tag{4.12}$$

Using (4.12) and $\sum_{\mu,\nu\in\mathcal{E}}|a_\mu a_\nu| \le \#\mathcal{E}\sum_\mu |a_\mu|^2 = \#\mathcal{E}$ we find that

$$\begin{aligned}\text{distant } &\le \sum_{\mu,\nu} |a_\mu||a_\nu| \max_{|\mu-\nu|>\lambda^{1/3}} I(\mu,\nu)\\ &\ll \frac{\#\mathcal{E}}{\lambda^{1/6}}\{\max_{\mu\in\mathcal{E}}||h_\mu||_\infty^2 + \max_{\mu\in\mathcal{E}}||h_\mu||_\infty \max_{\mu\in\mathcal{E}}||h'_\mu||_1\}\end{aligned} \tag{4.13}$$

as claimed. ∎

5. Proof of Theorem 1.2

5.1. Overview. We denote $f(t) = F(\gamma(t))$, which is real valued, and want to count zeros of f on $[0, L]$. The idea is to detect sign changes of f by comparing $\int |f|$ and $|\int f|$.

Let C_1 be a parameter, which we will want to satsify $1 \ll C_1 = o(\lambda)$, and consider a partition of unity $\{\tau_j\}_{j\in J}$ of the interval $[0, L]$, where $\tau_j \ge 0$, $\sum_j \tau_j = \mathbf{1}_{[0,L]}$, so that

- (i) $\#J \approx \lambda/C_1$,
- (ii) τ_j supported in an interval of length $\approx C_1/\lambda$,
- (iii) $|\partial^r\tau_j/\partial t^r| \ll (\lambda/C_1)^r$,
- (iv) for each j, there is at most $O(1)$ values of k for which $\tau_j\tau_k \ne 0$ (independent of λ); in particular, for each point t there is at most $O(1)$ values of j so that $\tau_j(t) \ne 0$.

Let $J_0 \subseteq J$ be the set of indices j for which f has a sign change on $\operatorname{supp}\tau_j$. Since for each point t there is at most $O(1)$ values of j for which $t \in \operatorname{supp}\tau_j$, we have

$$\#\text{ sign changes of } f \gg \#J_0, \tag{5.1}$$

so that a lower bound for $\#J_0$ gives a lower bound for the number of sign changes of f.

If $j \notin J_0$, then $f\tau_j$ does not change sign and hence

$$\int |f|\tau_j = \left|\int f\tau_j\right|, \quad j \notin J_0. \tag{5.2}$$

Therefore

$$\int |f| = \sum_j \int |f|\tau_j = \sum_{j\notin J_0}\left|\int f\tau_j\right| + \int |f| \sum_{j\in J_0}\tau_j. \tag{5.3}$$

We will show that

$$\int |f| \sum_{j\in J_0} \tau_j \ll \left(\frac{\#J_0 C_1}{\lambda}\right)^{1/2} \tag{5.4}$$

and that

$$\sum_{j\notin J_0} \left|\int f\tau_j\right| \ll C_1^{-1/3}, \tag{5.5}$$

so that

$$\int |f| \ll \left(\frac{\#J_0 C_1}{\lambda}\right)^{1/2} + C_1^{-1/3}. \tag{5.6}$$

Taking $C_1^{-1/3} = \delta \int |f|$ with $\delta > 0$ sufficiently small gives

$$\#J_0 \gg \lambda\left(\int |f|\right)^5, \tag{5.7}$$

which proves Theorem 1.2. Note that our choice of C_1 indeed satisfies our requirements, indeed $1 \ll C_1 \ll \lambda^{o(1)}$ since $\int |f| \ll (\int |f|^2)^{1/2} \ll ||F||_2 \approx 1$, by the upper bound in the uniform L^2-restriction theorem [1], and $\int |f| \gg \lambda^{-o(1)}$ from the lower bound in the uniform L^2-restriction theorem, see (6.3).

5.2. Proof of (5.4). By Cauchy–Schwarz,

$$\int |f| \sum_{j\in J_0} \tau_j \le ||f||_2 \left\{\int \left(\sum_{j\in J_0} \tau_j\right)^2\right\}^{1/2} = ||f||_2 \left\{\sum_{j,k\in J_0} \int \tau_j\tau_k\right\}^{1/2}. \tag{5.8}$$

By the restriction upper bound of [1], $||f||_2 \ll 1$. Given j, we have $\int \tau_j\tau_k = 0$ except for $O(1)$ indices k (independent of j), including $k = j$, and for such k we have

$$\int \tau_j\tau_k \le \frac{1}{2}\left(\int \tau_j^2 + \int \tau_k^2\right) \le \max_k \int \tau_k^2. \tag{5.9}$$

Since

$$\int \tau_k^2 \le \int_{\operatorname{supp}\tau_k} 1 \ll \frac{C_1}{\lambda} \tag{5.10}$$

we obtain

$$\int \left(\sum_{j\in J_0} \tau_j\right)^2 \ll \#J_0 \max_k \int \tau_k^2 \ll \frac{\#J_0 C_1}{\lambda} \tag{5.11}$$

and hence

$$\sum_{j\in J_0}\int |f|\tau_j \ll \left(\frac{\#J_0 C_1}{\lambda}\right)^{1/2}. \tag{5.12}$$

5.3. Proof of (5.5). Our goal is to show that

$$\sum_{j\notin J_0}\left|\int f(t)\tau_j(t)dt\right| \tag{5.13}$$

is small.

Let $\sigma > 0$ be a (small) parameter, $\lambda^{-o(1)} < \sigma < 1$ and $0 \le \theta(x) \le 1$ a smooth, even function so that $\theta(x) = 1$ if $|x| < 1$, $\theta(x) = 0$ for $|x| > 2$ and set $\theta_\sigma(x) = \theta(\frac{x}{\sigma})$.

Write $f = f_0 + f_1$ where

$$f_0(t) = \sum_{\mu\in\mathcal{E}} a_\mu \theta_\sigma(\phi'_\mu(t))e^{i\lambda\phi_\mu(t)} \tag{5.14}$$

and

$$f_1(t) = \sum_{\mu\in\mathcal{E}} a_\mu (1-\theta_\sigma)(\phi'_\mu(t))e^{i\lambda\phi_\mu(t)}. \tag{5.15}$$

Thus in the Fourier expansion of f_1, none of the phase functions ϕ_μ has a critical point in the support of $(1-\theta_\sigma)\phi'_\mu$; in fact they satisfy $|\phi'_\mu(t)| \ge \sigma$.

We have

$$\sum_{j\notin J_0}\left|\int f(t)\tau_j(t)dt\right| \le \sum_{j\notin J_0}\left|\int f_0(t)\tau_j(t)dt\right| + \sum_{j\notin J_0}\left|\int f_1(t)\tau_j(t)dt\right|. \tag{5.16}$$

We will show that

$$\sum_{j\notin J_0}\left|\int f_0(t)\tau_j(t)dt\right| \ll \sigma^{1/2} \tag{5.17}$$

and

$$\sum_{j\notin J_0}\left|\int f_1(t)\tau_j(t)dt\right| \ll \frac{1}{C_1\sigma}, \tag{5.18}$$

which gives

$$\sum_{j\notin J_0}\left|\int f\tau_j\right| \ll \sigma^{1/2} + \frac{1}{C_1\sigma}. \tag{5.19}$$

Choosing $C_1 = \sigma^{-3/2}$ gives

$$\sum_{j\notin J_0} \left| \int f\tau_j \right| \ll C_1^{-1/3} \tag{5.20}$$

proving (5.5).

5.4. Proof of (5.17). We have

$$\sum_{j\notin J_0} \left| \int f_0(t)\tau_j(t)dt \right| \leq \int |f_0(t)|dt \ll ||f_0||_2 \tag{5.21}$$

and hence (5.17) follows from:

Lemma 5.1:

$$||f_0||_2 \ll \sigma^{1/2} \tag{5.22}$$

Proof. We wish to apply Lemma 4.1 with $h_\mu = \theta(\frac{\phi'_\mu}{\sigma})$. We clearly have $||h_\mu||_\infty \leq 1$. Moreover,

$$||h_\mu||_2^2 \leq \int \theta\left(\frac{\phi'}{\sigma}\right) \leq \text{length}\{t : |\phi'_\mu(t)| < 2\sigma\} \tag{5.23}$$

and hence $||h_\mu||_2^2 \ll \sigma$ by Lemma 3.1. Likewise

$$||h'_\mu||_1 = \int \left|\theta'\left(\frac{\phi'_\mu}{\sigma}\right)\right| \frac{|\phi''_\mu|}{\sigma} \leq \frac{K_{\max}|\theta'|_\infty}{\sigma} \text{length}\{t : |\phi'_\mu(t)| < 2\sigma\} \tag{5.24}$$

and hence $||h'_\mu||_1 \ll 1$. Inserting into Lemma 4.1 gives

$$||f_0||_2^2 \ll \sigma + \frac{\#\mathcal{E}}{\lambda^{1/6}}, \tag{5.25}$$

which gives our claim provided $\sigma \gg \lambda^{-o(1)}$. ∎

5.5. Proof of (5.18). We expand and integrate by parts

$$\int f_1\tau_j = \frac{1}{i\lambda}\int f_2\tau_j + \frac{1}{i\lambda}\int f_3\tau'_j \tag{5.26}$$

where

$$f_2 = \sum_{\mu\in\mathcal{E}} a_\mu \left(\frac{1-\theta_\sigma(\phi'_\mu)}{\phi'_\mu}\right)' e^{i\lambda\phi_\mu} \tag{5.27}$$

and

$$f_3 = \sum_{\mu\in\mathcal{E}} a_\mu \frac{1-\theta_\sigma(\phi'_\mu)}{\phi'_\mu} e^{i\lambda\phi_\mu}. \tag{5.28}$$

Hence

$$\sum_{j\notin J_0}\left|\int f_1\tau_j\right| \le \frac{1}{\lambda}\Big(\int |f_2|\sum_{j\notin J_0}\tau_j + \int |f_3|\sum_{j\notin J_0}|\tau_j'|\Big)$$
$$\le \frac{1}{\lambda}||f_2||_2 + \frac{1}{\lambda}||f_3||_2\Big\{\int\Big(\sum_{j\notin J_0}|\tau_j'|\Big)^2\Big\}^{1/2}. \tag{5.29}$$

We have

$$\int\Big(\sum_{j\notin J_0}|\tau_j'|\Big)^2 = \sum_{j\notin J_0}\sum_{k\notin J_0}\int|\tau_j'\tau_k'| \ll \sum_{j\notin J_0}\int(\tau_j')^2, \tag{5.30}$$

since for each j there are only $O(1)$ values of k for which $\tau_j'\tau_k' \neq 0$. Hence

$$\int\Big(\sum_{j\notin J_0}|\tau_j'|\Big)^2 \ll \sum_{j\notin J_0}\int(\tau_j')^2 \ll \Big(\frac{\lambda}{C_1}\Big)^2. \tag{5.31}$$

Therefore

$$\sum_{j\notin J_0}\left|\int f_1\tau_j\right| \ll \frac{1}{\lambda}||f_2||_2 + \frac{1}{C_1}||f_3||_2. \tag{5.32}$$

Using Lemma 4.1 we find

$$||f_2||_2 \ll \frac{1}{\sigma^2}, \quad ||f_3||_2 \ll \frac{1}{\sigma} \tag{5.33}$$

once we note that

$$\left|\frac{1-\theta_\sigma(\phi_\mu')}{\phi_\mu'}\right| \ll \frac{1}{\sigma}, \tag{5.34}$$

$$\left|\Big\{\frac{1-\theta_\sigma(\phi_\mu')}{\phi_\mu'}\Big\}'\right| \ll \frac{1}{\sigma^2} \tag{5.35}$$

and, using (3.11), that

$$\left|\Big\{\frac{1-\theta_\sigma(\phi_\mu')}{\phi_\mu'}\Big\}''\right| \ll \frac{1}{\sigma^3} \tag{5.36}$$

(we assume throughout that $\sigma \gg \lambda^{-o(1)}$). This gives

$$\sum_{j\notin J_0}\left|\int f_1\tau_j\right| \ll \frac{1}{\lambda}\frac{1}{\sigma^2} + \frac{1}{C_1}\frac{1}{\sigma} \ll \frac{1}{C_1\sigma} \tag{5.37}$$

proving (5.18).

6. Relating L^1 and L^4 restriction theorems

We briefly explain the relation between L^1 and L^4 restriction theorems given in (1.9), namely

$$||F||_{L^1(\mathcal{C})} \gg_{\mathcal{C}} \frac{||F||_2^3}{||F||_{L^4(\mathcal{C})}^2}. \tag{6.1}$$

By Cauchy–Schwarz $\int_{\mathcal{C}} |F| \ll ||F||_{L^2(\mathcal{C})}$, and by the upper bound in the L^2-restriction theorem [1] we have $||F||_{L^2(\mathcal{C})} \ll ||F||_2$ so that

$$\int_{\mathcal{C}} |F| \ll ||F||_2. \tag{6.2}$$

As for lower bounds, we certainly have $\int_{\mathcal{C}} |F|^2 \leq ||F||_\infty \int_{\mathcal{C}} |F|$ and, combining the lower bound in the L^2-restriction theorem [1], $\int_{\mathcal{C}} |F|^2 \gg ||F||_2^2$ with the upper bound on the L^∞ norm $||F||_\infty \leq \sqrt{\#\mathcal{E}}||F||_2/2\pi$ (see (3.2)) we obtain

$$\frac{1}{||F||_2} \int_{\mathcal{C}} |F| \gg \frac{1}{\sqrt{\#\mathcal{E}}}. \tag{6.3}$$

We want to improve the bound (6.3) for $\int_{\mathcal{C}} |F|$. To start with, we use interpolation (log-convexity of the L^p norm) to give a lower bound for $||F||_{L^1(\mathcal{C})} = \int_{\mathcal{C}} |F|$ in terms of the L^2 and L^4 norms on the curve:

$$||F||_{L^2(\mathcal{C})} \leq ||F||_{L^1(\mathcal{C})}^{1/3} \cdot ||F||_{L^4(\mathcal{C})}^{2/3}, \tag{6.4}$$

which improves on (6.3) as it does not contain any component which is a-priori unbounded in λ.

Inserting the uniform L^2 restriction lower bound $||F|||_{L^2(\mathcal{C})} \gg ||F||_2$ of [1] into (6.4) gives (6.1) as claimed.

7. An upper bound on the restriction L^4 norm: Proof of Theorem 1.3

The aim of this section is to reduce getting a uniform upper bound for the 4-th moment $\int_{\mathcal{C}} |F|^4$, to counting lattice points in arcs of length $\sqrt{\lambda}$ by showing that

$$\int_{\mathcal{C}} |F|^4 = \int |f(t)|^4 dt \ll B_\lambda, \tag{7.1}$$

where, as in (1.6),

$$B_\lambda = \max_{|x|=\lambda} \#\{\xi \in \mathcal{E} : |x - \xi| \leq \sqrt{\lambda}\}. \tag{7.2}$$

7.1. Computing $\int|f|^4$. Recall

$$f(t)=\sum_\mu a_\mu e^{i\langle\mu,\gamma(t)\rangle}. \tag{7.3}$$

We may break up f into $O(1)$ terms, each the sum over frequencies μ lying in an arc of size $\lambda/100$. By the triangle inequality, it suffices to prove the restriction L^4 bound for such f, and from now on we assume that f is of this form.

In order to compute the 4-th moment $\int|f|^4$, write

$$f(t)^2=\sum_{\mu,\nu} a_\mu a_\nu e^{i\langle\mu+\nu,\gamma(t)\rangle}=\sum_\mu a_\mu a_{-\mu}+\sum_{0\neq z\in\mathcal{Z}} b_z e^{2i\langle z,\gamma(t)\rangle}, \tag{7.4}$$

where for a median $z=(\mu+\nu)/2\in\mathcal{Z}$ (see §2.2), we set $b_z=2a_\mu a_\nu$. The assumption that all the frequencies μ lie in an arc of size $\lambda/100$ implies that the medians $z\in\mathcal{Z}$ appearing in (7.4) satisfy $|z|>\lambda/2$, and that $a_\mu a_{-\mu}=0$ for all μ. Observe that

$$\sum_{0\neq z\in\mathcal{Z}}|b_z|^2\ll\left(\sum_\mu|a_\mu|^2\right)^2=\frac{1}{(2\pi)^4}||F||_2^4. \tag{7.5}$$

Hence we can we write

$$f(t)^2=g_0(t)+g(t) \tag{7.6}$$

with

$$g_0(t)=\sum_{0<\Delta(z)\leq\sqrt{\lambda}} b_z e^{2i\langle z,\gamma(t)\rangle} \tag{7.7}$$

and

$$g(t)=\sum_z{}^{*}\, b_z e^{2i\langle z,\gamma(t)\rangle}, \tag{7.8}$$

where we denote

$$\sum_z{}^{*}:=\sum_{\substack{z\in\mathcal{Z}\\|z|\geq\lambda/2\\|\Delta(z)|>\sqrt{\lambda}}}. \tag{7.9}$$

Therefore

$$||f||_4=||f^2||_2^{1/2}\leq(||g_0||_2+||g||_2)^{1/2}, \tag{7.10}$$

so that it suffices to show

$$||g_0||_2^2\ll B_\lambda||b||^2,\quad ||g||_2^2\ll B_\lambda||b||^2 \tag{7.11}$$

where $b = (b_z) \in \mathbb{C}^{\mathcal{Z}}$.

By Lemma 3.3, if $z \neq w$ then

$$\int e^{2i\langle z-w,\gamma(t)\rangle}dt \ll \frac{1}{|z-w|^{1/2}}, \tag{7.12}$$

and since the integral is trivially bounded by $\ll 1$, we can write this for any pair $z, w \in \mathcal{Z}$ as

$$\int e^{2i\langle z-w,\gamma(t)\rangle}dt \ll \frac{1}{|z-w|_+^{1/2}} \tag{7.13}$$

where

$$|z|_+ = \max(1, |z|). \tag{7.14}$$

Therefore

$$\int |g|^2 \ll \sum_z{}^* \sum_w{}^* \frac{|b_z||b_w|}{|z-w|_+^{1/2}}. \tag{7.15}$$

Moreover, we may restrict the sum to $|w-z| < \lambda^\epsilon$ at a cost of $O(\lambda^{-\epsilon/2}||b||^2\#\mathcal{Z}) = o(||F||_2^4)$ since $\#\mathcal{Z} \ll \lambda^{o(1)}$. Denoting

$$\sum_{z,w}{}^* := \sum_z{}^* \sum_{w:\ |z-w|<\lambda^\epsilon}{}^* \tag{7.16}$$

we have found that

$$||g||^2 \ll \sum_{z,w}{}^* \frac{|b_z b_w|}{|z-w|_+^{1/2}} \tag{7.17}$$

and likewise

$$||g_0||^2 \ll \sum_{\substack{0<\Delta(z),\Delta(w)\leq\sqrt{\lambda} \\ |z-w|<\lambda^\epsilon}} \frac{|b_z b_w|}{|z-w|_+^{1/2}}. \tag{7.18}$$

Thus we see that it suffices to show:

PROPOSITION 7.1: *Let $b = (b_z) \in \mathbb{C}^{\mathcal{Z}}$. Then*

(i)
$$\sum_{\substack{0<\Delta(z),\Delta(w)<\sqrt{\lambda} \\ |z-w|<\lambda^\epsilon}} \frac{|b_z b_w|}{|z-w|_+^{1/2}} \ll B_\lambda ||b||^2$$

and

(ii)
$$\sum_{z,w}{}^* \frac{|b_z b_w|}{|z-w|_+^{1/2}} \ll B_\lambda ||b||^2.$$

7.2. Proof of Proposition 7.1 (i). By Schur's test,

$$\sum_{\substack{0<\Delta(z),\Delta(w)\leq\sqrt{\lambda}\\ |z-w|<\lambda^\epsilon}} \frac{|b_z b_w|}{|z-w|_+^{1/2}} \leq \max_{0<\Delta(z)\leq\sqrt{\lambda}} \sum_{\substack{0<\Delta(w)\leq\sqrt{\lambda}\\ |z-w|<\lambda^\epsilon}} \frac{1}{|z-w|_+^{1/2}} ||b||^2 \tag{7.19}$$

and so it suffices to show that

$$\sum_{\substack{0<\Delta(w)\leq\sqrt{\lambda}\\ |z-w|<\lambda^\epsilon}} \frac{1}{|z-w|_+^{1/2}} \ll B_\lambda. \tag{7.20}$$

Replacing $|z-w|_+$ by 1 we are reduced to showing that

$$\#\{0<\Delta(w)\leq\sqrt{\lambda}, \quad |z-w|<\lambda^\epsilon\} \ll B_\lambda. \tag{7.21}$$

We have

$$\mu_+(w)-\mu_+(z) = (w-z) + (\Delta(w)-\Delta(z))\frac{z^\perp}{|z|} + \Delta(w)\left(\frac{w^\perp}{|w|}-\frac{z^\perp}{|z|}\right). \tag{7.22}$$

Since $\Delta(z),\Delta(w)<\sqrt{\lambda}$ we have $|z|,|w|\sim\lambda$, and hence

$$|\frac{w^\perp}{|w|}-\frac{z^\perp}{|z|}| \ll \frac{|z-w|}{\lambda} \ll \lambda^{-1+\epsilon}. \tag{7.23}$$

Thus

$$|\mu_+(w)-\mu_+(z)| \ll \sqrt{\lambda}. \tag{7.24}$$

By Lemma 2.2 we see that there are at most $O(B_\lambda)$ possibilities for w. This proves (7.21). ∎

7.3. A dyadic subdivision. We turn to the proof of part (ii) of Proposition 7.1. For $K\geq 1$, let

$$S_K = \{z\in\mathcal{Z}, \quad K\sqrt{\lambda}\leq\Delta(z)<2K\sqrt{\lambda}\}. \tag{7.25}$$

We write

$$\sum_{z,w}{}^{*} \frac{|b_z b_w|}{|z-w|_+^{1/2}} = \sum_{K,L \text{ dyadic}} \langle A_{K,L} b^{(K)}, b^{(L)}\rangle, \tag{7.26}$$

the sum over $K=2^k$, $L=2^\ell$, with

$$\langle A_{K,L} b^{(K)}, b^{(L)}\rangle = \sum_{\substack{w\in S_L\\ z\in S_K}}{}^{*} \frac{|b_z b_w|}{|z-w|_+^{1/2}}, \tag{7.27}$$

where $b^{(K)} = (|b_z|)_{z\in S_K}$, $b^{(L)} = (|b_w|)_{w\in S_L}$, and $A_{K,L} : \mathbb{C}^{S_K} \to \mathbb{C}^{S_L}$ is the matrix

$$A_{K,L} = \left(\frac{1}{|z-w|_+^{1/2}} \right)_{z\in S_K, w\in S_L} \tag{7.28}$$

with zeros whenever one of the conditions $\Delta(z), \Delta(w) > \sqrt{\lambda}$, $|z|, |w| > \lambda/2$ or $|z-w| < \lambda^\epsilon$ is violated.

We use Schur's test for the operator norm:

$$||A_{K,L}||_{2\to 2} \le ||A_{K,L}||_{1\to 1}^{1/2} \cdot ||A_{K,L}^*||_{1\to 1}^{1/2} \tag{7.29}$$

to bound

$$|\langle A_{K,L} b^{(K)}, b^{(L)}\rangle| \le ||A_{K,L}||_{1\to 1}^{1/2} \cdot ||A_{K,L}^*||_{1\to 1}^{1/2} \cdot ||b^{(K)}|| \cdot ||b^{(L)}||, \tag{7.30}$$

where $||b^{(K)}||$ is the ℓ^2-norm. We will show

Proposition 7.2: *For $K \le L$,*

$$||A_{K,L}||_{1\to 1} \ll B_\lambda \tag{7.31}$$

and

$$||A_{K,L}^*||_{1\to 1} \ll \frac{K}{L} B_\lambda. \tag{7.32}$$

Therefore

$$\begin{aligned} \sum_{z,w}{}^{*} \frac{|b_z b_w|}{|z-w|_+^{1/2}} &\ll B_\lambda \sum \sum_{K,L \text{ dyadic}} \left(\frac{\min(K,L)}{\max(K,L)} \right)^{1/2} ||b^{(K)}|| \cdot ||b^{(L)}|| \\ &\ll B_\lambda \left\{ \max_K \sum_{\substack{L=2^\ell \text{ dyadic} \\ L\ge K}} \left(\frac{K}{L}\right)^{1/2} \right\} \cdot \sum_K ||b^{(K)}||^2. \end{aligned} \tag{7.33}$$

Since $\sum_K ||b^{(K)}||^2 = ||b||^2$ and

$$\sum_{\substack{L=2^\ell \text{ dyadic} \\ L\ge K}} \left(\frac{K}{L}\right)^{1/2} \ll 1 \tag{7.34}$$

we will have proved part (ii) in Proposition 7.1. ∎

7.4. Proof of Proposition 7.2. By Schur's test,

$$||A_{K,L}||_{1\to 1} \le \max_{z\in S_K} \sideset{}{^*}\sum_{\substack{w\in S_L\\ |z-w|<\lambda^\epsilon}} \frac{1}{|z-w|_+^{1/2}} \tag{7.35}$$

and

$$||A^*_{K,L}||_{1\to 1} \le \max_{w\in S_L} \sideset{}{^*}\sum_{\substack{z\in S_K\\ |z-w|<\lambda^\epsilon}} \frac{1}{|z-w|_+^{1/2}}. \tag{7.36}$$

Lemma 7.3: *Let $z \in S_K$. Then*

$$\#\{w \in S_L : |w-z| < \lambda^\epsilon\} \ll LB_\lambda. \tag{7.37}$$

Proof. For $0 \le l \le L-1$, set

$$S_{L,l} = \{w \in S_L : (L+l)\sqrt{\lambda} \le \Delta(w) < (L+l+1)\sqrt{\lambda}\}. \tag{7.38}$$

We show that for $z \in S_K$, $w \in S_{L,l}$ and $|z-w| < \lambda^\epsilon$, we have

$$|\mu_+(w) - v| \ll \sqrt{\lambda} \tag{7.39}$$

where

$$v = \mu_+(z) + ((L+l)\sqrt{\lambda} - \Delta(z))\frac{z^\perp}{|z|}. \tag{7.40}$$

By Lemma 2.2 we see that there are at most $O(B_\lambda)$ possibilities for w in $S_{L,l}$ subject to $|w-z| < \lambda^\epsilon$:

$$\#\{w \in S_{L,l} : |w-z| < \lambda^\epsilon\} \ll B_\lambda. \tag{7.41}$$

Since

$$S_L = \bigcup_{l=0}^{L-1} S_{L,l} \tag{7.42}$$

we find

$$\#\{w \in S_L : |w-z| < \lambda^\epsilon\} \ll LB_\lambda \tag{7.43}$$

as claimed.

To prove (7.39), we use (2.6) to get

$$\mu_+(w) - \mu_+(z) = w - z + \Delta(w)\frac{w^\perp}{|w|} - \Delta(z)\frac{z^\perp}{|z|}. \tag{7.44}$$

Note that $|w^\perp| = |w|$, and since $|z| \geq \lambda/2$ and $|z-w| < \lambda^\epsilon$ then $|w| = |z| + O(\lambda^\epsilon) \gg \lambda$ and so

$$\frac{w^\perp}{|w|} = \frac{z^\perp}{|z^\perp|} + O(\lambda^{-1+\epsilon}). \tag{7.45}$$

Hence

$$\begin{aligned} \mu_+(w) - \mu_+(z) &= O(\lambda^\epsilon) + \Delta(w)\left(\frac{z^\perp}{|z^\perp|} + O(\lambda^{-1+\epsilon})\right) - \Delta(z)\frac{z^\perp}{|z^\perp|} \\ &= (\Delta(w) - \Delta(z))\frac{z^\perp}{|z^\perp|} + O(\lambda^\epsilon). \end{aligned} \tag{7.46}$$

Writing $\Delta(w) = (L + l + \theta)\sqrt{\lambda}$ with $0 \leq \theta < 1$ we find

$$\mu_+(w) - \mu_+(z) = ((L+l)\sqrt{\lambda} - \Delta(z))\frac{z^\perp}{|z^\perp|} + \theta\sqrt{\lambda}\frac{z^\perp}{|z^\perp|} + O(\lambda^\epsilon) \tag{7.47}$$

proving (7.39). ∎

We now give a lower bound for the difference of medians $|z-w|$:

Lemma 7.4: *Let $K \leq L$, $w \in S_{L,l}$, $z \in S_{K,k}$, $0 < |w-z| < \lambda^\epsilon$. Then*

(i) *if $2K < L$ then $|z-w| \gg L^2$;*
(ii) *if $K = L/2$ and $l \neq 0$ then $|z-w| \gg Ll$;*
(iii) *if $K = L$ and $l \neq k, k \pm 1$ then $|z-w| \gg L|l-k|$.*

Proof. To bound $|z-w|$, the condition $K \leq L$ allows us to assume $|z| \geq |w|$. Then

$$\begin{aligned} |z-w| \geq |z| - |w| &= \frac{\lambda^2 - |w|^2}{\lambda + |w|} - \frac{\lambda^2 - |z|^2}{\lambda + |z|} \\ &\geq \frac{\Delta(w)^2}{\lambda + |z|} - \frac{\Delta(z)^2}{\lambda + |z|} \\ &\geq \frac{\Delta(w)^2 - \Delta(z)^2}{2\lambda} \\ &\gg \frac{L}{\sqrt{\lambda}}|\Delta(w) - \Delta(z)| \gg L(L + l - K - k - 1). \end{aligned} \tag{7.48}$$

If $K \leq L/4$ then $L(L+l-K-k-1) \geq L(L-2K) \geq L^2/2$. If $K = L/2$ then $L|L+l-K-k-1| \geq L(L+l-2K) = Ll$, useful if $l \neq 0$. Finally, if $K = L$ then $L(L+l-K-k-1| = L|l-k-1| \geq \frac{1}{2}L|l-k|$ if $|l-k| \geq 2$, the exceptional cases being $l = k, k \pm 1$. ∎

7.5. Bounding $||A_{K,L}||_{2\to 2}$. We want to show that, if $K \le L$, then

$$||A_{K,L}||_{1\to 1} \le \max_{z\in S_K} \sum_{\substack{w\in S_L \\ |z-w|<\lambda^\epsilon}} \frac{1}{|z-w|_+^{1/2}} \ll B_\lambda \tag{7.49}$$

and

$$||A^*_{K,L}||_{1\to 1} \le \max_{w\in S_L} \sum_{\substack{z\in S_K \\ |z-w|<\lambda^\epsilon}} \frac{1}{|z-w|_+^{1/2}} \le \frac{K}{L} B_\lambda, \tag{7.50}$$

We assume first that $2K < L$, that is, $K = 2^k$, $L = 2^\ell$ with $\ell \ge k+2$. Let

$$S_L(z,\lambda^\epsilon) = \{w \in S_L : |w-z| < \lambda^\epsilon\}. \tag{7.51}$$

We have

$$\begin{aligned} ||A_{K,L}||_{1\to 1} &\le \max_{z\in S_K} \sum_{w\in S_L(z,\lambda^\epsilon)} \frac{1}{|z-w|_+^{1/2}} \\ &\le \max_{z\in S_K} \frac{\#S_L(z,\lambda^\epsilon)}{\min_{w\in S_L(z,\lambda^\epsilon)} |z-w|_+^{1/2}}. \end{aligned} \tag{7.52}$$

According to Lemma 7.3,

$$\#S_L(z,\lambda^\epsilon) \ll LB_\lambda, \tag{7.53}$$

and by Lemma 7.4, if $2K < L$ then

$$\min_{w\in S_L(z,\lambda^\epsilon)} |z-w| \gg L^2. \tag{7.54}$$

Hence we find (for $2K < L$)

$$||A_{K,L}||_{1\to 1} \ll B_\lambda. \tag{7.55}$$

Arguing in the same way with the roles of L and K reversed gives, for $2K < L$, that

$$||A^*_{K,L}||_{1\to 1} \le \max_{z\in S_K} \frac{\#S_K(z,\lambda^\epsilon)}{\min_{w\in S_K(z,\lambda^\epsilon)} |z-w|_+^{1/2}} \le \frac{KB_\lambda}{L}. \tag{7.56}$$

7.6. The cases $K = L/2, L$. It remains to deal with the case $K = L/2$ and $K = L$. We use the decomposition $S_L = \bigcup_{l=0}^{L-1} S_{L,l}$ in (7.38) to write

$$\begin{aligned}\sum_{w\in S_L(z,\lambda^\epsilon)} \frac{1}{|z-w|_+^{1/2}} &\ll \sum_{l=0}^{L-1} \sum_{w\in S_{L,l}(z,\lambda^\epsilon)} \frac{1}{|z-w|_+^{1/2}} \\ &\ll \sum_{l=0}^{L-1} \frac{\#S_{L,l}(z,\lambda^\epsilon)}{\min_{w\in S_{L,l}(z,\lambda^\epsilon)} |z-w|_+^{1/2}} \\ &\ll B_\lambda \sum_{l=0}^{L-1} (\min_{w\in S_{L,l}(z,\lambda^\epsilon)} |z-w|_+^{1/2})^{-1},\end{aligned} \tag{7.57}$$

where we have used (7.41). Applying Lemma 7.4 gives for $K = L/2$

$$\sum_{w\in S_L(z,\lambda^\epsilon)} \frac{1}{|z-w|_+^{1/2}} \ll B_\lambda \left(1 + \sum_{l=1}^{L-1} \frac{1}{(Ll)^{1/2}}\right) \ll B_\lambda, \tag{7.58}$$

and if $K = L$ and $z \in S_{L,k}$ we get

$$\sum_{w\in S_L(z,\lambda^\epsilon)} \frac{1}{|z-w|_+^{1/2}} \ll B_\lambda \Bigg(\sum_{\substack{0\le l\le L-1 \\ |l-k|\ge 2}} \frac{1}{L^{1/2}|l-k|^{1/2}} + O(1)\Bigg) \ll B_\lambda. \tag{7.59}$$

Thus we find that $||A^*_{K,L}||_{1\to 1}, ||A^*_{K,L}||_{1\to 1} \ll B_\lambda$ for $K = L/2, L$, concluding the proof of Proposition 7.2. ∎

8. Exceptions on the sphere and the torus

8.1. Nodal intersections with geodesics on the torus. We conclude by pointing out that no lower bound on $N_{F,\mathcal{C}}$ is possible without the assumption on non-vanishing of the curvature of $\mathcal{C}$, that is, when $\mathcal{C}$ is flat.

When $\mathcal{C}$ is a segment of a closed geodesic on the torus, there are arbitrarily large eigenvalues λ for which there are eigenfunctions F_λ vanishing identically on $\mathcal{C}$, that is, for which $\mathcal{C} \subset \mathcal{N}_{F_\lambda}$. Indeed, if the curve is a segment of the rational line $px + qy = c$, with $(0,0) \neq (p,q) \in \mathbb{Z}^2$, then taking $F_n(x,y) = \sin n(qx - py - c)$, $n = 1, 2, \dots$, gives an eigenfunction which has eigenvalue $n^2(p^2+q^2)$ and which vanishes on the entire closed geodesic. See [3] for further discussion of such "persistent components."

For the case when $\mathcal{C}$ is a segment of an unbounded geodesic, we claim that there are always arbitrarily large eigenvalues λ_k^2 for which there is an eigenfunction F_k with $N_{F_k,\mathcal{C}} = 0$. To see this, take an irrational $\beta \notin \mathbb{Q}$, and $\vec{v}_0 \in \mathbb{R}^2$ and let $\mathcal{C}$ be the irrational line segment $\{\vec{v}_0 + t(1,-\beta) : |t| < 1\}$. Let $\vec{n}_k = (p_k, q_k)$ be a sequence of good rational approximations of β:

$$\left|\beta - \frac{p_k}{q_k}\right| < \frac{1}{q_k^2} \tag{8.1}$$

with $q_k \to +\infty$. Let

$$F_k(\vec{x}) = \cos(\vec{n}_k \cdot (\vec{x} - \vec{v}_0)), \tag{8.2}$$

which is an eigenfunction with eigenvalue $\lambda_k^2 = p_k^2 + q_k^2$. Then on $\mathcal{C}$ we have

$$F_k(\vec{v}_0 + t(1,-\beta)) = \cos(t(p_k - q_k\beta)). \tag{8.3}$$

Since

$$|t(p_k - q_k\beta)| \leq |p_k - q_k\beta| < \frac{1}{q_k} \tag{8.4}$$

we see that

$$F_k(\vec{v}_0 + t(1,-\beta)) = 1 + O\left(\frac{1}{q_k^2}\right) \tag{8.5}$$

and so for $k \gg 1$, $F_k|_{\mathcal{C}}$ has no zeros.

8.2. The sphere. On the sphere, a basis of eigenfunctions is provided by the spherical harmonics. Let us restrict attention to zonal spherical harmonics. They are of the form $Y_\ell^0 = P_\ell(\cos\theta)$, where θ is the colattitude, and $P_\ell(x)$ are the Legendre polynomials

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{j=0}^{\lfloor \ell/2 \rfloor} (-1)^j \binom{\ell}{j} \binom{2\ell - 2j}{\ell - 2j} x^{\ell - 2j} \tag{8.6}$$

which are orthogonal polynomials on the interval $[-1,1]$. The nodal set of the zonal spherical harmonic Y_ℓ^0 is the union of the parallels $\theta = \theta_{\ell,j}$, $j = 1,\dots,\ell$ where $x_{\ell,j} = \cos\theta_{\ell,j}$ are the zeros of the Legendre polynomial $P_\ell(x)$.

Since $P_\ell(-x) = (-1)^\ell P_\ell(x)$, for odd ℓ we have $P_\ell(0) = 0$, and so we find that the zonal spherical harmonics vanish on the equator $\mathcal{C}(\pi/2) = \{\theta = \pi/2\}$ for odd ℓ, that is, $N_{\mathcal{C}(\pi/2), Y_\ell^0} = \infty$.

For other parallels $\mathcal{C}(\theta_0) = \{\theta = \theta_0\}$, $0 < \theta_0 < \pi/2$, we claim that there are infinitely many ℓ with $N_{\mathcal{C}(\theta_0), Y_\ell^0} = 0$. Thus even though the parallels have nonzero curvature, no analogue for the lower bound of Theorem 1.1 can hold on the sphere. To see this, note that if $\cos\theta_0$ is not one of the countably many zeros of the $P_\ell(x)$, then all the Y_ℓ^0 never vanish there. If $P_L(\cos\theta_0) = 0$, then we claim that $P_p(\cos\theta_0) \neq 0$ for all prime $p > L+1$. Indeed, when $p > 2$ is prime, Holt [9] showed in 1912 that $P_p(x)/x$ are irreducible over the rationals, and since $\deg P_\ell = \ell$, we must have $\gcd(P_p(x)/x, P_L(x)) = 1$ and, in particular, they have no common zeros.

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