THE AUTOCORRELATION OF THE MÖBIUS FUNCTION
AND CHOWLA’S CONJECTURE FOR THE RATIONAL
FUNCTION FIELD

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Abstract

We prove a function field version of Chowla’s conjecture on the autocorrelation of the Möbius function in the limit of a large finite field.

1. Introduction

There is a well-known equivalence between the Riemann hypothesis (RH) and square-root cancellation in sums of the Möbius function \( \mu(n) \), namely, RH is equivalent to

\[
\sum_{n \leq N} \mu(n) = O(N^{1/2+\epsilon}).
\]

This sum measures the correlation between \( \mu(n) \) and the constant function. Recent studies have explored the correlation between \( \mu(n) \) and other sequences; see [1, 2, 5]. Sarnak [8] showed that \( \mu(n) \) does not correlate with any ‘deterministic’ (i.e. zero entropy) sequence, assuming an old conjecture of Chowla [3] on the auto-correlation of the Möbius function, which asserts that given an \( r \)-tuple of distinct integers \( \alpha_1, \ldots, \alpha_r \) and \( \epsilon_i \in \{1, 2\} \), not all even, then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n + \alpha_1)\epsilon_1 \cdots \mu(n + \alpha_r)\epsilon_r = 0. \tag{1.1}
\]

Note that the number of non-zero summands here, that is, the number of \( n \leq N \) for which \( n + \alpha_1, \ldots, n + \alpha_r \) are all square-free, is asymptotically \( c(\alpha)N \), where \( c(\alpha) > 0 \) if the numbers \( \alpha_1, \ldots, \alpha_r \) do not contain a complete system of residues modulo \( p^2 \) for every prime \( p \) [6], so that (1.1) is about non-trivial cancellation in the sum.

Chowla’s conjecture (1.1) seems intractable at this time, the only known case being \( r = 1 \) where it is equivalent with the Prime Number Theorem. Our goal in this note is to prove a function field version of Chowla’s conjecture.

Let \( \mathbb{F}_q \) be a finite field of \( q \) elements and \( \mathbb{F}_q[x] \) be the polynomial ring over \( \mathbb{F}_q \). The Möbius function of a non-zero polynomial \( F \in \mathbb{F}_q[x] \) is defined to be \( \mu(F) = (-1)^r \) if \( F = cP_1 \cdots P_r \) with \( 0 \neq c \in \mathbb{F}_q \) and \( P_1, \ldots, P_r \) are distinct monic irreducible polynomials, and \( \mu(F) = 0 \) otherwise.

Let \( M_n \subseteq \mathbb{F}_q[x] \) be the set of monic polynomials of degree \( n \) over \( \mathbb{F}_q \), which is of size \( \#M_n = q^n \). The number of square-free polynomials in \( M_n \) is, for \( n > 1 \), equal to \( q^n - q^{n-1} \) [7, Chapter 2]. Hence, given \( r \) distinct polynomials \( \alpha_1, \ldots, \alpha_r \in \mathbb{F}_q[x] \), with \( \deg \alpha_j < n \), the number of \( F \in M_n \) for which all of \( F(x) + \alpha_j(x) \) are square-free is \( q^n + O(rq^{n-1}) \) as \( q \to \infty \).

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For $r > 0$, distinct polynomials $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q[x]$, with deg $\alpha_j < n$ and $\epsilon_i \in \{1, 2\}$, not all even, set

$$C(\alpha_1, \ldots, \alpha_r; n) := \sum_{F \in M_n} \mu(F + \alpha_1)^{\epsilon_1} \cdots \mu(F + \alpha_r)^{\epsilon_r}.$$  \hspace{1cm} (1.2)$$

For $r = 1$ and $n > 1$, we have $\sum_{F \in M_n} \mu(F) = 0$ [7, Chapter 2]. For $n = 1$, we have $\mu(F) = -1$ and the sum equals $(-1)^{\sum \epsilon_i} q^n$. For $n > 1, r > 1$, we show the following theorem.

**Theorem 1.1.** Fix $r > 1$ and assume that $n > 1$ and $q$ is odd. Then for any choice of distinct polynomials $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q[x]$, with max deg $\alpha_j < n$, and $\epsilon_i \in \{1, 2\}$, not all even

$$|C(\alpha_1, \ldots, \alpha_r; n)| \leq 2rnq^{n-1/2} + 3rn^2q^{n-1}.$$  \hspace{1cm} (1.3)$$

Thus, for fixed $n > 1$,

$$\lim_{q \to \infty} \frac{1}{\#M_n} \sum_{F \in M_n} \mu(F + \alpha_1)^{\epsilon_1} \cdots \mu(F + \alpha_r)^{\epsilon_r} = 0,$$  \hspace{1cm} (1.4)$$

under the assumption of Theorem 1.1, giving an analogue of Chowla’s conjecture (1.1).

Our starting point is Pellet’s formula, see, for example, [4, Lemma 4.1], which asserts that for the polynomial ring $\mathbb{F}_q[x]$ with $q$ odd (hence the restriction on the parity of $q$ in Theorem 1.1), the Möbius function $\mu(F)$ can be computed in terms of the discriminant $\text{disc}(F)$ of $F(x)$ as

$$\mu(F) = (-1)^{\text{deg} F} \chi_2(\text{disc}(F)),$$  \hspace{1cm} (1.5)$$

where $\chi_2$ is the quadratic character of $\mathbb{F}_q$. That will allow us to express $C(\alpha_1, \ldots, \alpha_r; n)$ as a character sum and to estimate it.

2. **Reduction to a counting problem**

2.1. **Character sums**

We use Pellet’s formula (1.5) to write

$$C(\alpha_1, \ldots, \alpha_r; n) = (-1)^{rn} \sum_{F \in M_n} \chi_2(\text{disc}(F + \alpha_1)^{\epsilon_1} \cdots \text{disc}(F + \alpha_r)^{\epsilon_r}).$$  \hspace{1cm} (2.1)$$

Since $\text{disc}(F)$ is polynomial in the coefficients of $F$, (2.1) is an $n$-dimensional character sum; we will estimate it by trivially bounding all but one variable. We single out the constant term $t := F(0)$ of $F \in M_n$ and write $F(x) = f(x) + t$, with

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x,$$  \hspace{1cm} (2.2)$$

Thus, for fixed $n > 1$,
and set
\[ D_f(t) := \text{disc}(f(x) + t), \]  
which is a polynomial of degree \( n - 1 \) in \( t \). Therefore, we have
\[ |C(\alpha_1, \ldots, \alpha_r; n)| \leq \sum_{a \in \mathbb{F}_q^{n-1}} \left| \sum_{t \in \mathbb{F}_q} \chi_2(D_{f+a_1}(t)^{\epsilon_1} \cdots D_{f+a_r}(t)^{\epsilon_r}) \right|. \]  
(2.4)

We use Weil's theorem (the RH for curves over a finite field), which implies that for a polynomial \( P(t) \in \mathbb{F}_q[t] \) of positive degree, which is not proportional to a square of another polynomial, we have
\[ \left| \sum_{t \in \mathbb{F}_q} \chi_2(P(t)) \right| \leq (\deg P - 1)q^{1/2}, \quad P(t) \neq cH^2(t). \]  
(2.5)

For us, the relevant polynomial is \( P(t) = D_{f+a_1}(t)^{\epsilon_1} \cdots D_{f+a_r}(t)^{\epsilon_r} \), which has degree \( \leq 2r(n - 1) \). Instead of requiring that it not be proportional to a square, we impose the stronger requirement that for some \( i \) with \( \epsilon_i \) odd, \( D_{f+a_i}(t) \) has positive degree and is square-free and that for all \( j \) such that \( j \neq i \), \( D_{f+a_i}(t) \) and \( D_{f+a_j}(t) \) are coprime. We denote the set of coefficients \( a \) satisfying the stronger condition by \( G_n \) (the ‘good’ as, where we can apply (2.5)), and let \( G_n^c = \mathbb{F}_q^{n-1} \setminus G_n \) be the complement of \( G_n \), where we use the trivial bound \( q \) on the character sum. Thus, we deduce that we can bound
\[ |C(\alpha_1, \ldots, \alpha_r; n)| \leq \sum_{a \in G_n} (2r(n - 1) - 1)q^{1/2} + \sum_{a \not\in G_n} q \]
\[ \leq (2r(n - 1) - 1)q^{n-1/2} + q\#G_n^c, \]  
(2.6)

where we have used the trivial bound \( \#G_n \leq q^{n-1} \) for the first part of the sum. Theorem 1.1 will follow from the following proposition.

**Proposition 2.1.** Assume that \( n > 1 \) and \( \max \deg \alpha_j < n \). Then
\[ \#G_n^c \leq 3rn^2q^{n-2}. \]

**2.2. Bounding \( \#G_n^c \)**

We can write \( G_n^c \subset A_n \cup B_n \) where:

1. \( A_n = A_{n,i} \) is the set of those \( a \in \mathbb{F}_q^{n-1} \) for which \( D_{f+a_i}(t) \) is either a constant or is not square-free, that is,
\[ A_n = \{ a \in \mathbb{F}_q^{n-1} : D_{f+a_i}(t) \text{ is constant or disc}(D_{f+a_i}) = 0 \}. \]  
(2.7)

2. \( B_n = \bigcup_{j \neq i} B(j) \), where \( B(j) \) are those \( a \) as for which \( D_{f+a_i}(t) \) and \( D_{f+a_j}(t) \) have a common zero, which can be written as the vanishing of their resultant
\[ B(j) = \{ a \in \mathbb{F}_q^{n-1} : \text{Res}(D_{f+a_i}(t), D_{f+a_j}(t)) = 0 \}. \]  
(2.8)
What is crucial is that $A_n$ and each $B(j)$ are the zero sets of a polynomial equation in the coefficients $a_i$; this is a key property of the discriminant and the resultant.

We will need the following elementary but useful uniform upper bound on the number of zeros of polynomials (cf. [9, Section 4, Lemma 3.1]).

**Lemma 2.2.** Let $h(X_1, \ldots, X_m) \in \mathbb{F}_q[X_1, \ldots, X_m]$ be a non-zero polynomial of total degree at most $d$. Then the number of zeros of $h(X_1, \ldots, X_m)$ in $\mathbb{F}_q^m$ is at most

$$\#\{x \in \mathbb{F}_q^m : h(x) = 0\} \leq dq^{m-1}.$$  

(2.9)

As we will see below (see Section 2.3), the equation defining $A_n$ has total degree $3(n-1)(n-2)$ in the coefficients $a_1, \ldots, a_{n-1}$, and the equation defining $B(j)$ has total degree $\leq 3(n-1)^2$. Therefore, by Lemma 2.2, if we show that the equations defining $A_n, B(j)$ are not identically zero, then we will have proved

$$\#A_n \leq 3n^2q^{n-2}$$  

(2.10)

and

$$\#B_n \leq 3(r-1)n^2q^{n-2}.$$  

(2.11)

This immediately gives Proposition 2.1.

In order to show that a polynomial $h \in \mathbb{F}_q[X_1, \ldots, X_m]$ is not identically zero, we may instead consider it as a polynomial defined over $\bar{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$. In this context, we can investigate the zero set $Z_h = \{a \in \bar{\mathbb{F}}_q^m : h(a) = 0\}$, which is a subvariety of the affine space $\mathbb{A}^m$. The polynomial $h$ is not identically zero if and only if $Z \neq \mathbb{A}^m$. This shall be our main tool in the following sections.

### 2.3. Resultant and discriminant formulas

The discriminant $\text{disc}(F)$ of a polynomial $F(x) = a_n x^n + \cdots + a_0$, $a_n \neq 0$, is given in terms of its roots $r_1, \ldots, r_n$ in the algebraic closure $\bar{\mathbb{F}}_q$ as $\text{disc} F = a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2$, and is a homogeneous polynomial with integer coefficients in $a_0, \ldots, a_n$, with degree of homogeneity $2n-2$, has total degree $2n-2$, and has degree $n-1$ as a polynomial in $a_0$. Moreover, if $a_i$ is regarded as having degree $i$, then $\text{disc}(F)$ is homogeneous of degree $n(n-1)$, that is, for every monomial $c_r \prod_i a_i^{r_i}$ in $\text{disc}(F)$,

$$\sum_i ir_i = n(n-1).$$  

(2.12)

The resultant of two polynomials $F(x) = a_n x^n + \cdots$, $G = b_m x^m + \cdots$, of degrees $n$ and $m$, is

$$\text{Res}(F, G) = a_n^m b_m^n \prod_{F(\rho) = 0} \prod_{G(\eta) = 0} (\rho - \eta).$$  

(2.13)

It is a homogeneous polynomial of degree $m+n$ in the coefficients of $F$ and $G$, in fact it is homogeneous of degree $m$ in $a_0, \ldots, a_n$ and of degree $n$ in $b_0, \ldots, b_m$. Moreover, if $a_i, b_i$ are regarded as having degree $i$, then $\text{Res}(F, G)$ is homogeneous of degree $mn$. We have

$$\text{Res}(F, G) = a_n^m \prod_{F(\rho) = 0} G(\rho) = (-1)^{mn} b_m^n \prod_{G(\eta) = 0} F(\eta).$$  

(2.14)
Furthermore, the discriminant of a polynomial $F(x) = a_nx^n + \cdots + a_0$ of degree $n$ may be computed in terms of the resultant as

$$\text{disc } F = (-1)^{n(n-1)/2} a_n^{n-\deg(F)-2} \text{Res}(F, F').$$

(2.15)

We apply this to compute the discriminant of $D_f(t) = \text{disc}(f(x) + t)$, $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x$. The discriminant $\text{disc}(D_f(t))$ is a polynomial in the coefficients $a_1, \ldots, a_{n-1}$ of $f(x)$. We claim that the total degree of $\text{disc } D_f(t)$ is $3(n-1)(n-2)$. Indeed, $D_f(t) = \sum_{j=0}^{n-1} b_j t^j$ is a polynomial of degree $n-1$ in $t$, and since it is homogeneous of degree $2(n-1)$ in $t, a_1, \ldots, a_{n-1}$ we find that $b_j$ are polynomials of total degree $2(n-1) - j$ in the $a_j$s. Now $\text{disc } D_f(t) = \sum c_r \prod_j b_j^r$ has total degree $2(n-1) - 2 = 2(n-2)$ in the $b_j$s, that is, $\sum r_j = 2(n-2)$, and by (2.12), $\sum_j r_j = (n-1)(n-2)$. Thus, the total degree of $\text{disc } D_f(t)$ in $a_1, \ldots, a_{n-1}$ is

$$\sum_j r_j \deg b_j = \sum r_j (2(n-1) - j) = 2(n-1) \sum r_j - \sum j r_j$$

$$= 2(n-1) \cdot 2(n-2) - (n-1)(n-2) = 3(n-1)(n-2),$$

as claimed.

Arguing similarly, one sees that the resultant $\text{Res}(D_f(t), D_{f+\alpha}(t))$ has total degree $3(n-1)^2$ in the coefficients $a_1, \ldots, a_{n-1}$.

Assume gcd$(q, n) = 1$. Then $f'(t) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots$ has degree $n-1$ and by (2.14) and (2.15) we find

$$D_f(t) = \text{disc}_{x}(f(x) + t) = (-1)^{n(n-1)/2} a_n^n \prod_{f'\rho=0} (t + f(\rho))$$

(2.16)

has degree $n-1$, with roots $-f(\rho)$ as $\rho$ runs over the $n-1$ roots of $f'(x)$.

In the case where gcd$(q, n) > 1$, $f'(t) = -a_{n-1}x^{n-2} + \cdots$ has degree $n-2$ provided that $a_{n-1} \neq 0$, in which case by (2.14) and (2.15) we have

$$D_f(t) = \text{disc}_{x}(f(x) + t) = (-1)^{n(n-1)/2} a_{n-1}^n \prod_{f'\rho=0} (t + f(\rho)),$$

(2.17)

which has degree $n-2$ and again has roots $-f(\rho)$ as $\rho$ runs over the $n-2$ roots of $f'(x)$.

3. Non-vanishing of the resultant

**Proposition 3.1.** Given a non-zero polynomial $\alpha \in \mathbb{F}_q[x]$, with $\deg \alpha < n$, then $\alpha \mapsto \text{Res}(D_f(t), D_{f+\alpha}(t))$ is not the zero polynomial, that is, the polynomial function

$$R(\alpha) := \text{Res}_{\alpha}(D_f(t), D_{f+\alpha}(t)) \in \mathbb{Z}[\tilde{a}]$$

(3.1)

is not identically zero.

Applying this to $\alpha = \alpha_j - \alpha_i$ for each $j \neq i$ will show that (2.11) holds.
Proof. Write \( \alpha(x) = A_{n-1}x^{n-1} + \cdots + A_0 \in \mathbb{F}_q[x] \) with \( \deg \alpha < n \).

Let \( p \) be the characteristic of \( \mathbb{F}_q \). Assume first that \( p \nmid n \). Then, by (2.14) and (2.16), we find

\[
\text{Res}(D_f, D_{f+\alpha}) = \frac{n^2(n-1)}{(n-1)!} \prod_{f'(\rho_1) = 0} (f(\rho_2) + \alpha(\rho_2) - f(\rho_1)).
\]

If \( p \mid n \), but \( a_{n-1} \neq 0 \) and \( a_{n-1} + A_{n-1} \neq 0 \), then by (2.14) and (2.17), we find

\[
\text{Res}(D_f, D_{f+\alpha}) = \frac{a_n(n-2)}{(n-2)!} \prod_{f'(\rho_1) = 0} (f(\rho_2) + \alpha(\rho_2) - f(\rho_1)).
\]

Note that when \( a_{n-1} = 0 \) or \( a_{n-1} + A_{n-1} = 0 \), the resultant \( \text{Res}(D_f, D_{f+\alpha}) \) is given by different polynomials than in the above case. However, this might affect at most \( 2q^{n-2} \) ‘bad’ \( \vec{a} \)s, which is a negligible amount, and the conclusion of (2.11) remains valid.

In both cases above, the ‘bad’ \( \vec{a} \)s are those for which there are \( \rho_1, \rho_2 \in \overline{\mathbb{F}_q} \) such that

\[
f'(\rho_1) = 0, \quad f'(\rho_2) = -\alpha'(\rho_2), \quad f(\rho_2) - f(\rho_1) = -\alpha(\rho_2).
\]

This is a linear system of equations for \( \vec{a} \in \mathbb{A}^{n-1} \), which has the form

\[
M(\rho)a = b(\rho), \quad \rho = (\rho_1, \rho_2),
\]

for a suitable \( 3 \times (n-1) \) matrix \( M(\rho) \) and vector \( b(\rho) \in \mathbb{A}^3 \). Thus, over \( \overline{\mathbb{F}_q} \), the solutions of \( R(\vec{a}) = 0 \) are precisely those \( \vec{a} \in \overline{\mathbb{F}_q}^{n-1} \) which satisfy the system (3.5) for some \( \rho \in \overline{\mathbb{F}_q}^2 \).

We consider the affine variety (possibly reducible) defined by these equations

\[
Z = \{ (\rho, a) \in \mathbb{A}^2 \times \mathbb{A}^{n-1} : M(\rho)a = b(\rho) \} \subset \mathbb{A}^2 \times \mathbb{A}^{n-1}.
\]

Let \( \phi : Z \to \mathbb{A}^{n-1} \) be the restriction to \( Z \) of the projection \( \mathbb{A}^2 \times \mathbb{A}^{n-1} \to \mathbb{A}^{n-1} \) and \( \pi : Z \to \mathbb{A}^2 \) be the restriction to \( Z \) of the projection \( \mathbb{A}^2 \times \mathbb{A}^{n-1} \to \mathbb{A}^2 \).

From the above, the solution set of \( R(\vec{a}) = 0 \) is precisely \( \phi(Z) \).

We will show that \( Z \) has dimension \( n - 2 \), and hence the dimension of \( \{ R = 0 \} = \phi(Z) \) cannot exceed \( n - 2 \) and hence is not all of \( \mathbb{A}^{n-1} \). Thus, \( R \) is not the zero polynomial, proving Proposition 3.1.

To do so, we study the dimensions of the fibres \( \pi^{-1}(\rho) \), which are affine linear subspaces. We first assume that \( n > 3 \). In this case, we will show that \( \pi(Z) \) is dense in \( \mathbb{A}^2 \) and generically, that is,
if \( \rho_1 \neq \rho_2 \), the fibres \( \pi^{-1}(\rho) \) have dimension \( n - 4 \). Moreover, there are at most \( \deg \alpha \) non-generic fibres, each of dimension \( n - 2 \). This will show that \( \dim Z = n - 2 \).

We rewrite the system (3.5) as

\[
\begin{align*}
\cdots + 3a_3\rho_1^2 + 2a_2\rho_1 + a_1 &= -n\rho_1^{-1}, \\
\cdots + 3a_3\rho_2^2 + 2a_2\rho_2 + a_1 &= -\alpha'(\rho_2) - n\rho_2^{-1}, \\
\cdots + a_3(\rho_3^2 - \rho_1^2) + a_2(\rho_2^2 - \rho_1^2) + a_1(\rho_2 - \rho_1) &= -\alpha(\rho_2) - (\rho_2^n - \rho_1^n).
\end{align*}
\]

(3.8)

To find the rank of the matrix \( M(\rho) \), we compute that

\[
\det \begin{pmatrix}
3\rho_1^2 & 2\rho_1 & 1 \\
3\rho_2^2 & 2\rho_2 & 1 \\
\rho_3^2 - \rho_1^3 & \rho_2^2 - \rho_1^2 & \rho_2 - \rho_1
\end{pmatrix} = (\rho_1 - \rho_2)^4,
\]

(3.9)

and thus \( M(\rho) \) has full rank 3 unless \( \rho_1 = \rho_2 \), and so the generic fibres \( \pi^{-1}(\rho) \) have dimension \( n - 1 - 3 = n - 4 \).

In the non-generic case \( \rho_1 = \rho_2 \), the matrix has rank 1 and we need \( \alpha'(\rho_2) = 0 = \alpha(\rho_2) \), which constrains us to have at most finitely many fibres (the number bounded by \( \deg \alpha/2 \)), each of which has dimension \( n - 1 - 1 = n - 2 \).

Finally, the cases \( n = 2, 3 \) are handled similarly, except that the image of the map \( \pi : Z \to \mathbb{A}^2 \) is no longer dense, due to algebraic conditions constraining \( \rho_1, \rho_2 \). We omit the (tedious) details. \( \square \)

4. Non-vanishing of the discriminant

We wish to show that the condition for being in \( A_n \) is not always satisfied. Without loss of generality, we can assume \( \alpha_i = 0 \). We first study a couple of small degree cases.

For \( n = 2 \), \( \text{disc}(x^2 + ax + t) = a^2 - 4t \) is linear and hence has no repeated roots (recall \( q \) is odd), hence \( A_n \) is empty. When \( n = 3 \), we have

\[
D_f(t) = \text{disc}_x(x^3 + ax^2 + bx + t) = (a^2b^2 - 4b^3) + (18ab - 4a^3)t - 27t^2.
\]

(4.1)

If \( 3 \mid q \), then \( D_f(t) = (a^2b^2 - 4b^3) - 4a^3t \) has degree 1 for \( a \neq 0 \); if \( 3 \nmid q \), then \( D(t) \) has degree 2 and we compute that

\[
\text{disc}_t \text{disc}_x(x^3 + ax^2 + bx + t) = -16(a^2 - 3b)^3,
\]

(4.2)

which is clearly not identically zero. So we may assume \( n \geq 4 \).

4.1.

Similarly to our approach in the previous section, it suffices to show that outside a set of \( \bar{\alpha} \)'s of codimension at least 1 in the parameter space \( \mathbb{A}^{n-1} \), \( D_f(t) \) is of positive degree, and is square-free, that is, with non-zero discriminant.
We conclude from (2.16) and (2.17) that if \( n \geq 4 \) and \( \vec{a} \) is in the ‘bad’ set (but \( a_{n-1} \neq 0 \) if \( \gcd(n, q) \neq 1 \)), then at least one of the following occurs:

1. There is some \( \rho \in \overline{\mathbb{F}_q} \) for which \( f'(x) \) has a double zero at \( x = \rho \), that is, there is some \( \rho \in \overline{\mathbb{F}_q} \) for which
   \[
   f'(\rho) = 0, \quad f''(\rho) = 0. \tag{4.3}
   \]

2. There are two distinct \( \rho_1 \neq \rho_2 \) so that \( f(\rho_1) = f(\rho_2) \) and so that \( f'(x) \) vanishes at both \( x = \rho_1 \) and \( x = \rho_2 \), that is,
   \[
   f'(\rho_1) = 0, \quad f'(\rho_2) = 0, \quad f(\rho_1) = f(\rho_2). \tag{4.4}
   \]

We want to show that the set of \( \vec{a} \in \overline{\mathbb{F}_q}^{n-1} \), which solves at least one of (4.3) and (4.4), has dimension at most \( n - 2 \).

4.2.

We first look at \( f \) for which (4.3) happens. This gives a pair of equations for \( \vec{a} \in \overline{\mathbb{F}_q}^{n-1} \):

\[
\cdots + 2\rho a_2 + a_1 = -n\rho^{n-1}, \\
\cdots + 2a_2 + 0 = -(n(n-1))\rho^{n-2}. \tag{4.5}
\]

Defining
\[
W = \{(\rho, \vec{a}) \in \mathbb{A}^1 \times \mathbb{A}^{n-1} : (4.3) \text{ holds}\},
\]
we have a fibration of \( W \) over the \( \rho \) line \( \mathbb{A}^1 \) and a map \( \phi : W \to \mathbb{A}^{n-1} \), the restriction of the projection \( \mathbb{A}^1 \times \mathbb{A}^{n-1} \to \mathbb{A}^{n-1} \),

\[
\begin{array}{ccc}
\mathbb{A}^1 & \xrightarrow{\pi} & \mathbb{A}^1 \times \mathbb{A}^{n-1} \\
& \xleftarrow{\phi} & \\
\mathbb{A}^{n-1}
\end{array}
\]

and the solutions of (4.3) are precisely \( \phi(W) \).

The system (4.5) is non-singular (rank 2) and hence \( \pi : W \to \mathbb{A}^1 \) is surjective and for each \( \rho \) the dimension of the solution set is \( n-1-2 = n-3 \). We find that \( \dim W = n-2 \) and hence \( \dim \phi(W) \leq n-2 \).

4.3.

Next we consider the system (4.4) which given \( \rho_1 \neq \rho_2 \) is a linear system for \( \vec{a} \in \overline{\mathbb{F}_q}^{n-1} \) of the form

\[
\cdots + 3\rho_1^2 a_3 + 2\rho_1 a_2 + a_1 = -n\rho_1^{n-1}, \\
\cdots + 3\rho_2^2 a_3 + 2\rho_2 a_2 + a_1 = -n\rho_2^{n-1}, \\
\cdots + (\rho_2^3 - \rho_1^3) a_3 + (\rho_2^2 - \rho_1^2) a_2 + (\rho_2 - \rho_1) a_1 = -\rho_2^n + \rho_1^n. \tag{4.8}
\]
This system shares the matrix part of (3.8), and hence has rank 3 for every \( \rho_1 \neq \rho_2 \). Thus, the arguments of the previous section show that

\[
\{ \vec{a} \in \mathbb{k}^{n-1} : \exists \rho_1 \neq \rho_2 \text{ s.t. (4.4) holds} \}
\] (4.9)

is of dimension at most \( n - 2 \). This shows that (2.10) holds, thus concluding the proof of Proposition 2.1.

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**References**