On the Robin spectrum for the equilateral triangle[*]

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Abstract

The equilateral triangle is one of the few planar domains where the Dirichlet and Neumann eigenvalue problems were explicitly determined, by Lamé in 1833, despite not admitting separation of variables. In this paper, we study the Robin spectrum of the equilateral triangle, which was determined by McCartin in 2004 in terms of a system of transcendental coupled secular equations. We give uniform upper bounds for the Robin–Neumann gaps, showing that they are bounded by their limiting mean value, which is hence an almost sure bound. The spectrum admits a systematic double multiplicity, and after removing it we study the gaps in the resulting desymmetrized spectrum. We show a spectral gap property, that there are arbitrarily large gaps, and also arbitrarily small ones, moreover that the nearest neighbour spacing distribution of the desymmetrized spectrum is a delta function at the origin. We show that for sufficiently small Robin parameter, the desymmetrized spectrum is simple.

Keywords: Robin boundary condition, equilateral triangle, Robin–Neumann gaps, level spacing distribution, Laplacian

(Some figures may appear in colour only in the online journal)

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1. Introduction

The equilateral triangle is one of the few planar domains where the Dirichlet and Neumann eigenvalue problems are explicitly solved, despite not admitting separation of variables. The solution was found by Lamé in 1833 [13], who also investigated the Robin[3] boundary value problem: denoting by T an equilateral triangle and by ∂T its boundary as in figure 1, the Robin problem is to solve

$$\Delta f + \lambda f = 0 \quad \text{on } T, \qquad \frac{\partial f}{\partial n} + \sigma f = 0 \text{ on } \partial T,$$

where $\frac{\partial}{\partial n}$ is the derivative in the outward pointing normal direction, and $\sigma > 0$ is the Robin parameter (which we take to be constant).

Lamé only determined the Robin eigenfunctions possessing $120°$ rotational symmetry, and it is only in 2004 that McCartin [16, 18] completely determined the eigenproblem, showing that all of the eigenfunctions are trigonometric polynomials[4], and that the eigenvalues are determined by a system of transcendental coupled secular equations as follows: define auxiliary parameters $L \in (-\pi/2, 0]$, $M, N \in [0, \pi/2)$, which are required to satisfy the coupled system of equations

$$
\begin{aligned}
(2L - M - N - (m+n)\pi)\tan L &= 3r\sigma \\
(2M - N - L + m\pi)\tan M &= 3r\sigma \\
(2N - L - M + n\pi)\tan N &= 3r\sigma.
\end{aligned}
\tag{1.1}
$$

The corresponding Robin eigenvalues are

$$\Lambda_{m,n}(\sigma) = \frac{4\pi^2}{27r^2}(\mu^2 + \nu^2 + \mu\nu), \tag{1.2}$$

where

$$\mu = \frac{2M - N - L}{\pi} + m, \qquad \nu = \frac{2N - L - M}{\pi} + n.$$

Note that there is a systematic multiplicity of order 2 coming from the symmetry $\Lambda_{m,n} = \Lambda_{n,m}$, and we will refer to $\{\Lambda_{m,n}(\sigma)\}_{0 \leqslant m \leqslant n}$ as the *desymmetrized* Robin spectrum.

For $\sigma \geqslant 0$, let λ_n^σ denote the nth eigenvalue of the Robin Laplacian on the equilateral triangle, arranged by size and repeated with multiplicities (the case $\sigma = 0$ are the Neumann eigenvalues). We will study a number of aspects of the Robin spectrum of the equilateral triangle.

In the first part of the paper, we study the Robin–Neumann gaps

$$d_n(\sigma) := \lambda_n^\sigma - \lambda_n^0,$$

see figure 2. As is the case for any bounded piecewise smooth planar domain, the RN gaps have a limiting mean value [23], which equals

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[3] The term 'Robin boundary condition' came much later, see [6] for a historical discussion.

[4] The only polygonal domains where all Dirichlet or Neumann eigenfunctions are trigonometric are rectangles, and the equilateral, hemi-equilateral and right isosceles triangles [17], see [20] for a higher dimensional version.
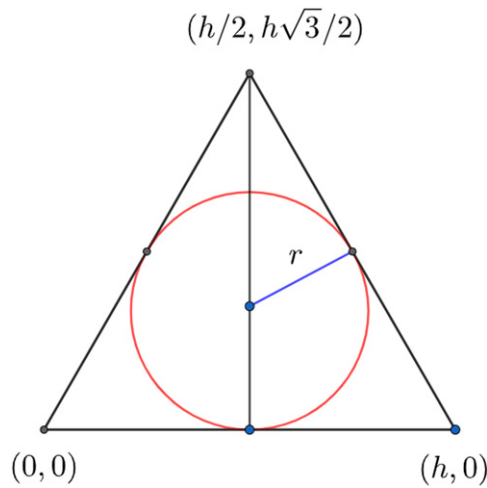
Figure 1. An equilateral triangle of side length h. The inscribed circle has radius $r = h/(2\sqrt{3})$.
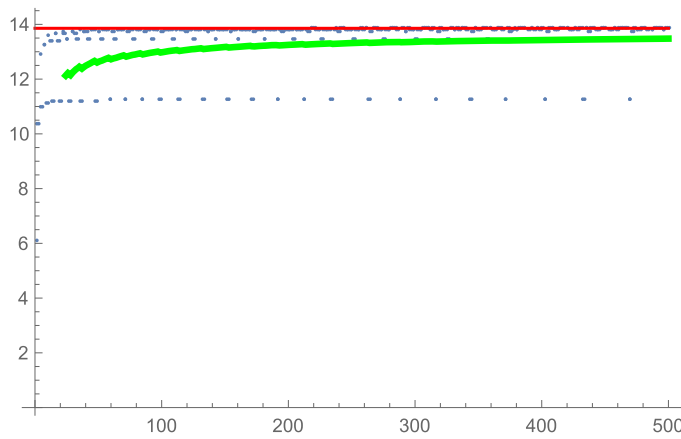

Figure 2. The first 500 RN gaps for the equilateral triangle with side length 1, with $\sigma = 1$. The solid (red) line is the limiting mean value $2\,\text{length}(\partial T)/\,\text{area}$ $(T) = 8\sqrt{3} = 13.8564\ldots$. Note that all the gaps are below the limiting mean value, as is proved in theorem 1.1.

$$\bar{d} := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} d_n(\sigma) = \frac{2 \text{ length } \partial T}{\text{area } T}\sigma = \frac{4}{r}\sigma,$$

where r is the radius of the inscribed circle. Remarkably, for the equilateral triangle, the limiting mean value is also an upper bound:

Theorem 1.1. *We have $d_n(\sigma) < \bar{d}$ for all n.*

For most domains, we do not expect a uniform upper bound, e.g. we expect that for the disk there are arbitrarily large RN gaps, but cannot prove this for any example of a planar domain, see [21] for the hemisphere.

As a consequence of theorem 1.1, we show:

Corollary 1.2. *The RN gaps tend to the mean value along a density one sequence of eigenvalues.*

We use the results on the RN gaps to deduce information on the asymptotics of the Robin spectrum of the equilateral triangle by comparing it to the Neumann spectrum:

Corollary 1.3. *For fixed $\sigma > 0$, there are arbitrarily large gaps in the Robin spectrum $\{\lambda_n^\sigma\}$.*

This is sometimes called the 'spectral gap property' and is useful in a variety of applications. An example is to show the existence of inertial manifolds in dissipative reaction–diffusion equations [5, 14]

$$\frac{\partial u}{\partial t} = \nu \Delta u + g(u),$$

for u on a domain satisfying suitable boundary conditions, with g a suitable nonlinear function, and where $\nu > 0$ is a parameter, see [12] for the case of the equilateral triangle. There are very few instances of planar domains where the existence of arbitrarily large gaps in the spectrum (with any boundary condition) is known. The question is open even for the Dirichlet spectrum of the rectangle having the golden mean as its aspect ratio.

We can also show that there are arbitrarily small nonzero gaps in the spectrum. In fact, we have a stronger result:

Theorem 1.4. *The distribution of nearest neighbour gaps in the desymmetrized spectrum is a delta function at the origin, i.e. for any fixed $x > 0$,*

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \leqslant N : \lambda_{n+1}^\sigma - \lambda_n^\sigma \leqslant x\} = 1.$$

We emphasize that in this paper, σ is fixed; it is of great interest to study the spacings when σ grows with the eigenvalue, see the discussion by Sieber *et al* [24], and by Berry and Dennis [1]. In the second part of the paper we examine spectral multiplicities (or 'modal degeneracies'). For the Dirichlet or Neumann spectrum of the equilateral triangle, there are large multiplicities of arithmetic origin; the same holds for the hemi-equilateral (half of an equilateral triangle) and right isosceles triangles, but there are other triangles with accidental degeneracies, see the paper of Berry and Wilkinson [2] for an exploration of these 'diabolical points'. Hillairet and Judge [7] showed that for *almost all*[5] triangles the Dirichlet spectrum is simple, but their method does not give a single explicit example. For the Robin spectrum on the equilateral triangle, there is a systematic doubling due to the symmetry $(m, n) \mapsto (n, m)$ in (1.2). McCartin [16, section 8] observed that there are additional degeneracies for $\sigma \gg 1$. We will show that for small $\sigma > 0$, there are no other degeneracies:

Theorem 1.5. *There is some $\sigma_0 > 0$ so that there are no multiplicities in the Robin spectrum for $0 < \sigma < \sigma_0$ except for the systematic doubling.*

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[5] In the sense of Lebesgue measure on the space of triangles of fixed area, which can be parameterized by triples of angles which sum to π.

A similar result holds for the square; however, for rectangles whose squared aspect ratio is irrational, there are multiplicities for arbitrarily small $\sigma > 0$ [22], showing the special arithmetic nature of the result. For the proof of theorem 1.5, we partition the spectrum into clusters

$$\mathcal{C}_R(\sigma) = \{\Lambda_{m,n}(\sigma) : m, n \geqslant 0, m^2 + mn + n^2 = R^2\},$$

consisting of all Robin eigenvalues given by (1.2), that, for $\sigma = 0$, correspond to the common Neumann eigenvalue satisfying

$$\frac{4\pi^2}{27r^2}(m^2 + mn + n^2) = \Lambda_{m,n}(0) = \frac{4\pi^2}{27r^2}R^2$$

with some $m, n \geqslant 0$ integers, for the given $R > 0$. At $\sigma = 0$, these clusters are well separated, as the Neumann eigenvalues are multiples by $\frac{4\pi^2}{27r^2}$ of integers. As σ varies, different clusters remain separated for small σ due to our upper bound on the Robin–Neumann gaps (theorem 1.1). This reduces the problem to showing that there is some $\sigma_0 > 0$ for which all of the clusters break up completely (except for a systematic double multiplicity) for all $0 < \sigma < \sigma_0$. To prove this requires a detailed study of the secular equation (1.1) governing the eigenvalues, which takes up sections 9, 10 and 11.

2. Background on the equilateral triangle

We consider an equilateral triangle T of side length h. Denote by

$$r = \frac{h}{2\sqrt{3}}$$

the radius of the inscribed circle. The area of T is then

$$\text{area}(T) = \frac{\sqrt{3}h^2}{4} = 3\sqrt{3}r^2.$$

We use Cartesian coordinates (x, y) so that the vertices are located at $\{(0,0), (0,h), (h/2, h\sqrt{3}/2)\}$ (figure 1).

2.1. Neumann eigenfunctions

The eigenfunctions are either symmetric or antisymmetric w.r.t the altitude of the triangle, that is the line $x = h/2$. A complete set of orthogonal Neumann eigenfunctions is

$$T_{m,n}^{s/a}(x,y) = \cos\left(\frac{\pi\ell}{3r}(3r-y)\right)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}\left(\frac{\sqrt{3}\pi(m-n)}{9r}\left(x-\sqrt{3}r\right)\right)$$

$$+ \cos\left(\frac{\pi m}{3r}(3r-y)\right)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}\left(\frac{\sqrt{3}\pi(n-\ell)}{9r}\left(x-\sqrt{3}r\right)\right)$$

$$+ \cos\left(\frac{\pi n}{3r}(3r-y)\right)\begin{Bmatrix}\cos\\\sin\end{Bmatrix}\left(\frac{\sqrt{3}\pi(\ell-m)}{9r}\left(x-\sqrt{3}r\right)\right),$$

where for the symmetric eigenfunctions $T_{m,n}^s$ we take $0 \leqslant m \leqslant n$ and cosine, and for the antisymmetric ones $T_{m,n}^a$ we take $0 \leqslant m < n$ and sine. Here the $m, n \geqslant 0$ are integers, and m, n, ℓ satisfy

$$m + n + \ell = 0$$

with the corresponding eigenvalue being

$$\Lambda_{m,n}(0) := \frac{2\pi^2}{27r^2}\left(m^2 + n^2 + \ell^2\right) = \frac{4\pi^2}{27r^2}\left(m^2 + mn + n^2\right).$$

There are high multiplicities in the Neumann spectrum of the equilateral triangle, coming from the fact that for integers which can be written in the form $m^2 + mn + n^2$ there are 'typically' many ways to do so. This is a well-understood number theoretic issue, completely similar to the problem of representation as a sum of two squares. The squared L^2 norm of the eigenfunctions is [15, section 8.1]

$$\|T_{m,n}^{a/s}\|_2^2 = \int_T (T_{m,n}^{s/a})^2 = \frac{9\sqrt{3}r^2}{4}, \quad m < n$$

and

$$\|T_{m,m}^s\|_2^2 = \frac{9\sqrt{3}r^2}{2}, \quad m > 0.$$

2.2. Robin eigenfunctions

The eigenfunctions are either symmetric or antisymmetric w.r.t the altitude of the triangle, that is the line $x = h/2$. McCartin showed that a complete set of orthogonal eigenfunctions is

$$T_{m,n}^{s/a}(x,y) = \cos\left(\frac{\pi\lambda}{3r}(3r-y) - \delta_1\right)\{\substack{\cos\\\sin}\}\left(\frac{\sqrt{3}\pi(\mu-\nu)}{9r}\left(x - \sqrt{3}r\right)\right)$$

$$+ \cos\left(\frac{\pi\mu}{3r}(3r-y) - \delta_2\right)\{\substack{\cos\\\sin}\}\left(\frac{\sqrt{3}\pi(\nu-\lambda)}{9r}\left(x - \sqrt{3}r\right)\right)$$

$$+ \cos\left(\frac{\pi\nu}{3r}(3r-y) - \delta_3\right)\{\substack{\cos\\\sin}\}\left(\frac{\sqrt{3}\pi(\lambda-\mu)}{9r}\left(x - \sqrt{3}r\right)\right)$$

with some $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$, where for the symmetric eigenfunctions $T_{m,n}^s$ we take $0 \leqslant m \leqslant n$ and cosine, and for the antisymmetric ones $T_{m,n}^a$ we take $0 \leqslant m < n$ and sine. Here μ, ν, λ (depending on m, n and the Robin constant σ) are chosen subject to

$$\mu + \nu + \lambda = 0$$

and $\mu, \nu \geqslant 0$ are determined by a set of transcendental equations (imposed by requiring that the corresponding eigenfunctions satisfy the Robin condition on the boundary): define auxiliary parameters

$$L \in (-\pi/2, 0], \quad M, N \in [0, \pi/2) \tag{2.1}$$

and set

$$\lambda = \frac{2L - M - N}{\pi} - m - n, \qquad \mu = \frac{2M - N - L}{\pi} + m, \qquad \nu = \frac{2N - L - M}{\pi} + n.$$
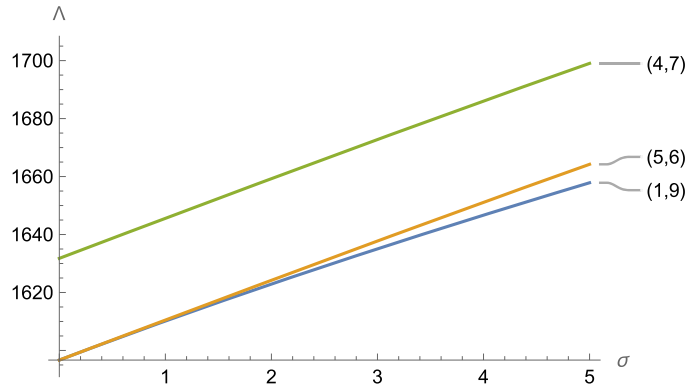
Figure 3. Plots of $\Lambda_{m,n}(\sigma)$ with $(m,n) = (1,9),(5,6),(4,7)$. Note that $\Lambda_{1,9}(0) = \Lambda_{5,6}(0)$.

Then L, M, N are required to satisfy the coupled system of equations

$$(2L - M - N - (m+n)\pi) \tan L = 3r\sigma$$

$$(2M - N - L + m\pi) \tan M = 3r\sigma$$

$$(2N - L - M + n\pi) \tan N = 3r\sigma, \tag{2.2}$$

see [16] for existence and uniqueness of solutions.

The corresponding eigenvalues are

$$\Lambda_{m,n}(\sigma) = \frac{2\pi^2}{27r^2}(\mu^2 + \nu^2 + \lambda^2) = \frac{4\pi^2}{27r^2}(\mu^2 + \nu^2 + \mu\nu). \tag{2.3}$$

One may find some examples of plots of $\Lambda_{m,n}(\cdot)$ in figure 3. Note that there is a systematic multiplicity of order 2 coming from the symmetry $\Lambda_{m,n} = \Lambda_{n,m}$. We refer to [4] for a computation of the L^2 norm of the eigenfunctions.

3. A uniform upper bound for the RN gaps: proof of theorem 1.1

Our goal is to show that for the equilateral triangle, the Robin–Neumann gaps are bounded above by their limiting mean value, which we recall equals

$$\bar{d} := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} d_n(\sigma) = \frac{2 \text{ length } \partial\, T}{\text{area } T} \sigma = \frac{4}{r}\sigma.$$

Proof. We will show that

$$0 < \Lambda_{m,n}(\sigma) - \Lambda_{m,n}(0) < \bar{d} = \frac{4}{r}\sigma. \tag{3.1}$$

Given that, to pass from the $\Lambda_{m,n}(\sigma)$ to its analogue for the *ordered* eigenvalues is the same argument as for the rectangle (see [23, section 8.2]) that we reproduce here for the sake of completeness, albeit briefly. Namely, recall that $\{\lambda_n^\sigma\}_{n \geqslant 0}$ is the Robin spectrum corresponding to the Robin parameter σ (in non-decreasing order), and, given $k \geqslant 1$, consider the closed interval

$$I_k := [0, \lambda_k^0 + \bar{d}] \subseteq \mathbb{R}.$$

Then, thanks to the inequality (3.1) to be proved immediately below, I_k is bound to contain all of $\lambda_n^\sigma < \lambda_n^0 + \overline{d}$, for $n \leqslant k$, i.e. I_k contains at least $(k+1)$ of the eigenvalues $\{\lambda_n^\sigma\}$, implying, in particular, that

$$\lambda_k^\sigma \leqslant \lambda_k^0 + \overline{d},$$

sufficient to deduce the claimed analogue of (3.1) for the ordered Robin eigenvalues.

We now turn to proving (3.1). To this end we rewrite the equation (2.2) in a compact form as follows: set

$$m_1 = m, \quad m_2 = n, \quad m_3 = -(m+n),$$
$$\mu_1 = \mu, \quad \mu_2 = \nu, \quad \mu_3 = \lambda,$$
$$M_1 = M, \quad M_2 = N, \quad M_3 = L$$

so that

$$\mu_1 + \mu_2 + \mu_3 = 0 = m_1 + m_2 + m_3 \tag{3.2}$$

and

$$\mu_j = m_j + \frac{1}{\pi}(2M_j - M_i - M_k), \tag{3.3}$$

where $\{i, j, k\} = \{1, 2, 3\}$, and the system (2.2) becomes

$$\mu_j \tan M_j = \frac{3r\sigma}{\pi}, \quad j = 1, 2, 3.$$

Therefore, since $|M_j| < \pi/2$,

$$|\mu_j M_j| < |\mu_j \tan M_j| = \frac{3r}{\pi}\sigma. \tag{3.4}$$

Now consider the difference (compare (2.3))

$$\Lambda_{m,n}(\sigma) - \Lambda_{m,n}(0) = \frac{2\pi^2}{27r^2}\sum_{j=1}^{3}(\mu_j^2 - m_j^2).$$

We have

$$\begin{aligned}
\mu_j^2 - m_j^2 &= (\mu_j - m_j)(2\mu_j - (\mu_j - m_j)) \\
&= 2\mu_j(\mu_j - m_j) - (\mu_j - m_j)^2 \\
&\leqslant 2\mu_j(\mu_j - m_j) \\
&= \frac{2}{\pi}\mu_j(2M_j - M_i - M_k)
\end{aligned}$$

on inserting (3.3). Therefore

$$0 < \Lambda_{m,n}(\sigma) - \Lambda_{m,n}(0) = \frac{2\pi^2}{27r^2} \sum_{j=1}^{3} (\mu_j^2 - m_j^2)$$

$$\leqslant \frac{4\pi}{27r^2} \sum_{j=1}^{3} \mu_j (2M_j - M_i - M_k).$$

Recalling that $\{i, j, k\} = \{1, 2, 3\}$ and using (3.2) gives

$$\sum_{j=1}^{3} \mu_j (M_i + M_k) = \sum_j M_j(\mu_i + \mu_k) = -\sum_j M_j \mu_j$$

and so

$$\sum_{j=1}^{3} \mu_j (2M_j - M_i - M_k) = 2 \sum_{j=1}^{3} \mu_j M_j - \sum_{j=1}^{3} \mu_j (M_i + M_k) = 3 \sum_{j=1}^{3} M_j \mu_j.$$

Inserting (3.4) gives that this is $\leqslant 27 \frac{r}{\pi} \sigma$ and hence

$$0 < \Lambda_{m,n}(\sigma) - \Lambda_{m,n}(0) \leqslant \frac{4\pi}{27r^2} \cdot \frac{27r}{\pi} \sigma = \frac{4}{r} \sigma$$

proving (3.1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Almost sure convergence of the RN gaps: proof of corollary 1.2

A tautological consequence of theorem 1.1, which says that all RN gaps (which are positive) are bounded by their limiting mean value, is that *almost all* RN gaps converge to the limiting mean value $\bar{d}$.

Proof. Indeed, let $d_n \geqslant 0$ be a sequence of non-negative numbers, which has a limiting mean value

$$\bar{d} := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} d_n,$$

and assume that for all n we have $d_n \leqslant \bar{d}$. Then we claim that necessarily, for almost all n, we have $d_n = \bar{d} + o(1)$ as $n \to \infty$. On the contrary, assume that there is some $\delta > 0$ so that the set

$$\mathcal{N}_\delta := \{n : d_n \leqslant \bar{d} - \delta\}$$

satisfies:

$$\lim \sup\nolimits_{N \to \infty} \frac{1}{N} \# \mathcal{N}_\delta \cap [1, N] = c > 0.$$

Thus we are guaranteed an infinite sequence $\mathcal{S}$ of N's satisfying

$$\# \mathcal{N}_\delta \cap [1, N] > cN/2.$$

For all $N \geqslant 1$, we can compute the mean value as

$$\frac{1}{N}\sum_{n=1}^{N} d_n = \frac{1}{N}\sum_{\substack{n \leqslant N \\ n \in \mathcal{N}_\delta}} d_n + \frac{1}{N}\sum_{\substack{n \leqslant N \\ n \notin \mathcal{N}_\delta}} d_n \leqslant \frac{1}{N}\sum_{\substack{n \leqslant N \\ n \in \mathcal{N}_\delta}} (\bar{d} - \delta) + \frac{1}{N}\sum_{\substack{n \leqslant N \\ n \notin \mathcal{N}_\delta}} \bar{d},$$

where we have used $d_n \leqslant \bar{d}$ for $n \notin \mathcal{N}_\delta$. In particular, for $N \in \mathcal{S}$,

$$\frac{1}{N}\sum_{n=1}^{N} d_n \leqslant (\bar{d} - \delta)\frac{1}{N}\#\mathcal{N}_\delta \cap [1, N] + \bar{d}\frac{1}{N}\#\{n \notin \mathcal{N}_\delta, n \leqslant N\}$$

$$= \bar{d} - \delta\frac{1}{N}\#\mathcal{N}_\delta \cap [1, N] \leqslant \bar{d} - \delta\frac{c}{2}$$

and so

$$\bar{d} = \lim_{\substack{N \to \infty \\ N \in \mathcal{S}}} \frac{1}{N}\sum_{n=1}^{N} d_n \leqslant \bar{d} - \delta\frac{c}{2} < \bar{d}$$

which is a contradiction.   □

5. Large gaps in the Robin spectrum: proof of corollary 1.3

Proof. Since the Robin spectrum clusters at a bounded distance around the Neumann spectrum $\{\frac{4\pi^2}{27r^2}(m^2 + mn + n^2) : m, n \geqslant 0\}$, it suffices to observe that the Neumann spectrum has arbitrarily large gaps. This well known arithmetic fact admits a quick proof by noting that for integers of the form $m^2 + mn + n^2$, the prime decomposition can only contain primes of the form $p = 3k + 2$ to an even power (see e.g. [8, chapter 9.1]). Let $p_1 = 2, p_2 = 5, \ldots, p_K$ be the first K primes congruent to $2 \bmod 3$. Using the Chinese remainder theorem we find n satisfying $n = -j + p_j \bmod p_j^2$ for $j = 1, \ldots, K$. Then $n + 1, n + 2, \ldots, n + K$ are not of the form $x^2 + xy + y^2$ because $n + j = 0 \bmod p_j$ while $n + j \neq 0 \bmod p_j^2$. Thus we found a gap of size $\geqslant \frac{4\pi^2}{27r^2} \cdot K$ in the Neumann spectrum.   □

We can extract qualitative results from the finer results known about gaps between values of binary quadratic forms:

In 1982, Richards [19] proved that the maximal gap $g(x)$ among integers of the form $m^2 + mn + n^2$ up to x is at least $(\frac{1}{3} - o(1))\log x$ as $x \to \infty$, see [3, 10] for improvement to the constant. Hence, if we denote by

$$g_\sigma(x) = \max\left(\lambda_{n+1}^\sigma - \lambda_n^\sigma : \lambda_n^\sigma \leqslant x\right),$$

then, with the help of theorem 1.1, we obtain the bound

$$g_\sigma(x) \gg \log x.$$

6. Spacings: proof of theorem 1.4

Fix $\sigma > 0$ and denote by λ_n^σ the Robin spectrum ($\sigma = 0$ being the Neumann spectrum) and let

$$x_\sigma(n) = c(\lambda_{n+1}^\sigma - \lambda_n^\sigma), \quad c = \frac{4\pi}{\text{area } T}$$

be the normalized nearest neighbour gaps, whose mean value is unity by Weyl's law. Let

$$\tilde{P}_\sigma(t, N) := \frac{1}{N} \#\{n \leqslant N : x_\sigma(n) < t\}$$

be the cumulative distribution function of the $x_\sigma(n)$.

Note that if for a pair of tuples (m, n) and (m', n') representing consecutive Neumann energies one has

$$m^2 + n^2 + mn = m'^2 + n'^2 + m'n',$$

then $\Lambda_{m,n}(0) = \Lambda_{m',n'}(0)$, so that the corresponding nearest neighbour gap vanishes. Hence for the Neumann spectrum, all the nearest gaps vanish except for at most the number of integers representable by the form $m^2 + n^2 + mn$, whose number of those energies $\leqslant X$ is $O\left(\frac{X}{\sqrt{\log X}}\right)$ by [9]. On the other hand, by Weyl's law, the total number of energies $\leqslant X$ is proportional to X, hence most of the gaps vanish precisely, implying, in particular, that the limiting spacing distribution is the delta function. Hence, in the Neumann case, the limiting spacing distribution is a delta-function at the origin: we have for any $t > 0$,

$$\lim_{N \to \infty} \tilde{P}_0(t, N) = 1. \tag{6.1}$$

Proof of theorem 1.4. By corollary 1.2, the bulk of Robin spectrum is obtained from the Neumann spectrum by an approximately constant shift, therefore the spacing distribution remains unchanged. Denote by $d_\sigma(n) = \lambda_n^\sigma - \lambda_n^0$ the Robin–Neumann gaps and $\bar{d}$ their limiting mean value. Fix $\epsilon > 0$ and let $\mathcal{S}$ be the set of integers n so that

$$\bar{d} - \epsilon < d_\sigma(n)$$

(and also $d_n(\sigma) < \bar{d}$). We showed that $\mathcal{S}$ has density one (corollary 1.2). Therefore, the set

$$\mathcal{S}_2 := \{n \geqslant 1 : n \in \mathcal{S} \text{ and } n + 1 \in \mathcal{S}\}$$

also has density one.

For $n \in \mathcal{S}_2$, we compute the difference of the normalized gaps $x_\sigma(n)$ and $x_0(n)$:

$$\frac{1}{c}(x_\sigma(n) - x_0(n)) = (\lambda_{n+1}^\sigma - \lambda_n^\sigma) - (\lambda_{n+1}^0 - \lambda_n^0)$$

$$= (\lambda_{n+1}^\sigma - \lambda_{n+1}^0) - (\lambda_n^\sigma - \lambda_n^0) = d_\sigma(n+1) - d_\sigma(n).$$

Since $d_\sigma(n+1), d_\sigma(n) \in (\bar{d} - \epsilon, \bar{d})$ we obtain $d_\sigma(n+1) - d_\sigma(n) \in (-\epsilon, \epsilon)$ so that for all $n \in \mathcal{S}_2$,

$$x_\sigma(n) - x_0(n) \in (-c\epsilon, c\epsilon).$$

Fix $t > 0$, and take $\epsilon < t/c$. Then for $n \in \mathcal{S}_2$, if $x_\sigma(n) < t$ then $x_0(n) < t + c\epsilon$, while $x_0(n) < t - c\epsilon$ implies that $x_\sigma(n) < t$. Thus

$$\{n \in \mathcal{S}_2 : x_\sigma(n) < t\} \subseteq \{n \in \mathcal{S}_2 : x_0(n) < t + c\epsilon\}$$

and

$$\{n \in \mathcal{S}_2 : x_\sigma(n) < t\} \supseteq \{n \in \mathcal{S}_2 : x_0(n) < t - c\epsilon\}.$$

On the other hand, we have

$$0 \leqslant \tilde{P}_\sigma(t, N) - \frac{1}{N} \#\{n \leqslant N, n \in \mathcal{S}_2 : x_\sigma(n) < t\}$$
$$\leqslant \frac{1}{N} \#\{n \leqslant N : n \notin \mathcal{S}_2\} = o(1),$$

and likewise for $\sigma = 0$, hence

$$\tilde{P}_0(t - c\epsilon, N) + o(1) \leqslant \tilde{P}_\sigma(t, N) \leqslant \tilde{P}_0(t + c\epsilon, N) + o(1).$$

Since $\epsilon > 0$ is arbitrary, for any fixed $t > 0$, we obtain by (6.1)

$$\lim_{N \to \infty} \tilde{P}_\sigma(t, N) = 1$$

which gives our claim.　　　　　　　　　　　　　　　　　　　　　　　　　　□

7. Simplicity of the desymmetrized spectrum: overview of the proof of theorem 1.5

7.1. Review of notation

We recall notation: for integers $m, n \geqslant 0$, and $\sigma \geqslant 0$, we defined the variables

$$L = L_{m,n}(\sigma) \in \left(-\frac{\pi}{2}, 0\right], \; M = M_{m,n}(\sigma), N = N_{m,n}(\sigma) \in \left[0, \frac{\pi}{2}\right) \tag{7.1}$$

given by solutions of the system

$$\begin{cases} (2L - M - N - (m+n)\pi)\tan L = 3r\sigma \\ (2M - N - L + m\pi)\tan M \quad = 3r\sigma. \\ (2N - L - M + n\pi)\tan N \quad = 3r\sigma \end{cases} \tag{7.2}$$

The variables μ, ν were defined as

$$\mu = \frac{2M - N - L}{\pi} + m, \qquad \nu = \frac{2N - L - M}{\pi} + n,$$

and, finally, the Robin eigenvalues with parameter σ are:

$$\Lambda_{m,n}(\sigma) := \frac{4\pi^2}{27r^2}\left(\mu^2 + \nu^2 + \mu\nu\right).$$

To prove that the desymmetrized spectrum is simple (theorem 1.5), it is needed to show that there exists $\sigma_0 > 0$ so that for all $\sigma \in (0, \sigma_0)$, one has $\Lambda_{m,n}(\sigma) \neq \Lambda_{m',n'}(\sigma)$ for all pairs $(m, n) \neq (m', n')$ with $0 \leqslant m \leqslant n, 0 \leqslant m' \leqslant n'$. We adopt the notation

$$R^2 = R^2(m, n) := \frac{27r^2}{4\pi^2}\Lambda_{m,n}(0) = m^2 + n^2 + mn, \tag{7.3}$$

and

$$F_R(m, n) = \frac{R^4}{m^2 n^2 (m+n)^2} = \frac{1}{m^2} + \frac{1}{n^2} + \frac{1}{(m+n)^2}.$$

7.2. Key propositions

Proposition 7.1. *For $\sigma > 0$ sufficiently small:*

(a) For $1 \leqslant m \leqslant n$,

$$\Lambda_{m,n}(\sigma) = \Lambda_{m,n}(0) + \frac{4}{r} \cdot \sigma - \frac{4F_R(m, n)}{\pi^2} \cdot \sigma^2(1 - r\sigma) + O\left(\frac{1}{m^4} \cdot \sigma^3\right),$$

(7.4)

with the implied constant absolute.

(b) For $m = 0 < n$, $\Lambda_{0,n}(\cdot)$ satisfies[6]

$$\Lambda_{0,n}(\sigma) = \Lambda_{0,n}(0) + \frac{10}{3r} \cdot \sigma + O(\sigma^{3/2}),$$

(7.5)

with the implied constant absolute.

(c) The function $\Lambda_{0,0}(\sigma)$ is continuous[7] at $\sigma = 0$.

Proposition 7.2. *If (m, n) and (m', n') are two integer points on the ellipse*

$$X^2 + XY + Y^2 = R^2$$

with $1 \leqslant m < m' \leqslant n' < n$, then $F_R(m, n) > F_R(m', n')$ and as $R \to \infty$ we have a lower bound for the difference

$$F_R(m, n) - F_R(m', n') \gg \frac{1}{m^4}$$

with the implied constant absolute.

7.3. Proof of theorem 1.5 assuming propositions 7.1 and 7.2

Proof. We assert that for any integers $m, n, m', n' \geqslant 0$, one has:

(a) If

$$m^2 + n^2 + mn < m'^2 + n'^2 + m'n',$$

then $\Lambda_{m,n}(\sigma) < \Lambda_{m',n'}(\sigma)$ for $\sigma \in (0, \pi^2/(27r))$.

---

[6] By general perturbation theory [11, chapter VII] (see in particular remark 4.22 on page 408), the functions $\Lambda_{0,n}(\cdot)$ are analytic, at least in some neighbourhood of the origin. Therefore the remainder term in (7.5) can be replaced by $O_n(\sigma^2)$.

[7] In fact, $\Lambda_{0,0}(\cdot)$ is analytic, by [11, chapter VII].

(b) For $\sigma > 0$ sufficiently small (absolute), if $0 \leqslant m < m' \leqslant n' < n$ satisfy

$$m^2 + n^2 + mn = m'^2 + n'^2 + m'n'$$

then $\Lambda_{m,n}(\sigma) > \Lambda_{m',n'}(\sigma)$.

The desymmetrized Neumann spectrum $\{\Lambda_{m,n}(0) : 0 \leqslant m \leqslant n\}$ is partitioned into clusters of coinciding eigenvalues

$$\mathcal{C}_R = \{\Lambda_{m,n}(0) : 0 \leqslant m \leqslant n, m^2 + mn + n^2 = R^2\}.$$

Part (a) deals with the situation that at $\sigma = 0$, we start from different clusters $\mathcal{C}_R$, $\mathcal{C}_{R'}$ with $R < R'$, and the claim is that there is some σ_0 so that for $\sigma \in (0, \sigma_0)$, the evolved clusters remain separate. Since distinct integers are spaced at least one apart from each other, the distance between different Neumann clusters ($\sigma = 0$) is at least $4\pi^2/(27r^2)$. We use our upper bound (theorem 1.1) on the Robin–Neumann gaps:

$$\lambda_n(\sigma) - \lambda_n(0) < \frac{4}{r}\sigma,$$

so that if $4\sigma/r < 4\pi^2/(27r^2)$ then different Robin clusters cannot mix, that is

$$\Lambda_{m,n}(\sigma) < \Lambda_{m',n'}(\sigma)$$

for $\sigma \in (0, \pi^2/(27r))$.

Now take integers $0 \leqslant m < m' \leqslant n' < n$, so that

$$m^2 + n^2 + mn = m'^2 + n'^2 + m'n'$$

(equivalently, $\Lambda_{m,n}(0) = \Lambda_{m',n'}(0)$). If $m = 0$, then we invoke proposition 7.1 (a) and (b) to write

$$\Lambda_{m',n'}(\sigma) - \Lambda_{0,n}(\sigma) = \frac{2}{3r} \cdot \sigma + O(\sigma^{3/2}) > 0,$$

since $F(m', n') < 3$.

Otherwise, if $m \geqslant 1$, then we invoke proposition 7.1 (a) to yield

$$\Lambda_{m',n'}(\sigma) - \Lambda_{m,n}(\sigma) = \left(F_R(m,n) - F_R(m',n')\right) \cdot \frac{4\sigma^2(1 - r\sigma)}{\pi^2} + O\left(\frac{\sigma^3}{m^4}\right)$$

which along with proposition 7.2 show that for $\sigma > 0$ sufficiently small,

$$\Lambda_{m',n'}(\sigma) - \Lambda_{m,n}(\sigma) \gg \frac{\sigma^2}{m^4} + O\left(\frac{\sigma^3}{m^4}\right) \gg \frac{\sigma^2}{m^4} > 0$$

in particular this difference is nonzero. In either case, $m = 0$ or $m \geqslant 1$, (b) is proved. $\qquad\square$

8. Asymptotic expansion of the eigenvalue curves

8.1. Some auxiliary results

We state some lemmas on the properties of the auxiliary parameters M, N and L, which we then use to prove proposition 7.1.

Lemma 8.1. *For every $0 \leqslant m \leqslant n$ and $\sigma \geqslant 0$ there exists a unique solution (L, M, N) to (7.2) in the prescribed range (7.1). These solutions satisfy:*

(a) *For $0 \leqslant m < n$, $\sigma > 0$ one has $L, N = O\left(\frac{\sigma}{n}\right)$, with the implied constant absolute.*

(b) *In addition to the above, $M = O\left(\frac{\sigma}{m}\right)$ uniformly for $1 \leqslant m \leqslant n$, $\sigma > 0$. Otherwise, for $m = 0 < n$, one has $M = O\left(\sqrt{\sigma}\right)$ for $\sigma > 0$ sufficiently small, with the implied constant absolute.*

(c) *For $m = n = 0$ one has $|L|, |M|, |N| \ll \sqrt{\sigma}$, so, in particular the functions L, M, N are continuous at $\sigma = 0$.*

Lemma 8.1 implies in particular, that, as $\sigma \to 0$, one has

$$L(\sigma), M(\sigma), N(\sigma) \to 0$$

uniformly w.r.t. $m, n \geqslant 0$. Therefore $\mu_{m,n}(\sigma) \to \mu_{m,n}(0) = m$ and $\nu_{m,n}(\sigma) \to \nu_{m,n}(0) = n$ uniformly. It will also follow *a fortiori* from our analysis below that uniformly in m, n,

$$\lim_{\sigma \to 0} \Lambda_{m,n}(\sigma) = \Lambda_{m,n}(0) = \frac{4\pi^2}{27r^2}\left(m^2 + n^2 + mn\right).$$

Lemma 8.2. *For $m = 0$, $n \geqslant 1$ the functions L, M, N are analytic on $\sigma > 0$ and continuous at $\sigma = 0$, with L, N continuously differentiable on $\mathbb{R}_{\geqslant 0}$. Further, L, M, N satisfy the following asymptotics around the origin, with all the implied constants absolute:*

$$N, -L = \frac{3r}{n\pi} \cdot \sigma + O\left(\frac{\sigma^{3/2}}{n^2}\right), \qquad M = \sqrt{3r/2} \cdot \sqrt{\sigma} + O(\sigma^{3/2}).$$

To state the next lemmas, we use the uniform notation as in section 3, where we rewrite the coupled system (7.2) in compact form as follows: set

$$m_1 = m, \quad m_2 = n, \quad m_3 = -(m+n),$$

$$\mu_1 = \mu, \quad \mu_2 = \nu, \quad \mu_3 = -(\mu_1 + \mu_2),$$

$$M_1 = M, \quad M_2 = N, \quad M_3 = L,$$

so that

$$\mu_1 + \mu_2 + \mu_3 = 0 = m_1 + m_2 + m_3$$

and

$$\mu_j = m_j + \frac{1}{\pi}(2M_j - M_i - M_k),$$

where $\{i, j, k\} = \{1, 2, 3\}$. Thus for each $0 \leqslant m_1 \leqslant m_2$, we obtain a coupled system for the variables M_1, M_2, M_3 with $M_1, M_2, -M_3 \in [0, \pi/2)$

$$\mu_j \tan M_j = \frac{3r\sigma}{\pi}, \quad j = 1, 2, 3. \tag{8.1}$$

Lemma 8.3. *If* $1 \leqslant m_1 \leqslant m_2$ *then the derivatives at* $\sigma = 0$ *are*

$$M_j'(0) = \frac{3r}{\pi m_j}, \tag{8.2}$$

and

$$M_j''(0) = -\frac{18r^2}{\pi^3} \frac{m_j^2 + 2m_i m_k}{m_j^3 m_i m_k}. \tag{8.3}$$

We next give bounds for the derivatives of M_j:

Lemma 8.4. *There is some* $\sigma_0 > 0$ *so that if* $1 \leqslant m_1 \leqslant m_2$ *then* $M_j(\sigma)$ *are analytic in* $[0, \sigma_0]$ *and satisfy* (*uniformly in* $[0, \sigma_0]$)

$$M_j' = \frac{3r}{\pi m_j} \left(1 + O\left(\frac{1}{m_j} \right) \right), \tag{8.4}$$

$$|M_j''| \ll \frac{1}{m_1 m_j^2}, \tag{8.5}$$

$$M_j''' = 2\left(M_j'\right)^3 + O\left(\frac{1}{m_1^3 m_j^2} \right) = \frac{54r^3}{\pi^3 m_j^3} + O\left(\frac{1}{m_1^3 m_j^2} \right). \tag{8.6}$$

Lemma 8.5. *The values of the first three derivatives of* $\Lambda_{m,n}(\cdot)$ *at the origin are:*

$$\Lambda_{m,n}'(0) = \frac{4}{r} \tag{8.7}$$

$$\Lambda_{m,n}''(0) = -\frac{8}{\pi^2} F_R(m, n) \tag{8.8}$$

$$\Lambda_{m,n}^{(3)}(\sigma) = \frac{24r}{\pi^2} F_R(m, n) + O\left(\frac{1}{m^4} \right). \tag{8.9}$$

The proofs of these lemmas will be given in sections 9, 10 and 11.

8.2. Proof of proposition 7.1

Proof. Proposition 7.1 (a) is a direct consequence of (8.7) and (8.8) via a three-term Taylor expansion around $\sigma = 0$, invoking the Lagrange form of the remainder appealing to the estimate (8.9): for every $\sigma > 0$ sufficiently small, one has the estimate

$$\Lambda_{m,n}(\sigma) = \Lambda_{m,n}(0) + \frac{4}{r} \cdot \sigma - \frac{4F_R(m, n)}{\pi^2} \cdot \sigma^2 + \left(\frac{4rF_R(m, n)}{\pi^2} + O\left(\frac{1}{m^4} \right) \right) \cdot \sigma^3,$$

which yields (7.4). Part (c) of proposition 7.1 is a direct consequence of lemma 8.1 (c).

To prove proposition 7.1 (b), namely that

$$\Lambda_{0,n}(\sigma) = \Lambda_{0,n}(0) + \frac{10}{3r} \cdot \sigma + O(\sigma^{3/2})$$

we write

$$\Lambda_{0,n}(\sigma) = \frac{2\pi^2}{27r^2} \sum_{j=1}^{3} \mu_j^2.$$

Using $m = 0$, $n \geqslant 1$ we have

$$\mu_1 = \frac{1}{\pi}(2M_1 - M_2 - M_3)$$

$$\mu_2 = n + \frac{1}{\pi}(2M_2 - M_3 - M_1), \quad \mu_3 = -n + \frac{1}{\pi}(2M_3 - M_1 - M_2)$$

so that

$$\sum_{j=1}^{3} \mu_j^2 = 2n^2 + \frac{6n}{\pi}(M_2 - M_3) + \frac{1}{\pi^2} \sum_{j=1}^{3} (2M_j - M_i - M_k)^2.$$

Inserting lemma 8.2 which asserts that

$$M_1 = \sqrt{\frac{3r\sigma}{2}} + O(\sigma^{3/2}), \quad M_2, -M_3 = \frac{3r}{n\pi}\sigma + O\left(\frac{\sigma^{3/2}}{n^2}\right)$$

gives

$$\sum_{j=1}^{3} \mu_j^2 = 2n^2 + \frac{45r}{\pi^2}\sigma + O(\sigma^{3/2})$$

so that

$$\Lambda_{0,n}(\sigma) = \frac{2\pi^2}{27r^2} \sum_{j=1}^{3} \mu_j^2 = \Lambda_{0,n}(0) + \frac{10r}{3}\sigma + O(\sigma^{3/2})$$

as claimed. $\square$

9. Proofs of lemmas 8.1 and 8.2

9.1. Proof of lemma 8.1

Proof. The existence and uniqueness of the solutions (L, M, N) to (7.2) was established in [16, section 6]. We observe that $L \leqslant 0$ and $M, N \geqslant 0$ forces $2L - M - N \leqslant 0$, and so

$$2L - M - N - (m + n)\pi \leqslant -(m + n)\pi.$$

Hence, the first equation of (7.2) implies that if $(m, n) \neq (0, 0)$ (which, for $m \leqslant n$ is equivalent to $n \neq 0$), then

$$|\tan L| \ll \frac{\sigma}{m + n} \leqslant \frac{\sigma}{n},$$

and so

$$L \ll \frac{\sigma}{n}.$$

The proof of $N \ll \frac{\sigma}{n}$ is similar to the above, except that we focus on the 3rd equation of (7.2) and notice that

$$2N - L - M + n\pi \geqslant n\pi - M \geqslant (n - 1/2)\pi \gg n.$$

This concludes the proof of lemma 8.1 (a), and the same argument yields the case $m > 0$ of lemma 8.1 (b), by exploiting the 2nd equation of (7.2).

If $m = 0$ but $n > 0$, then, the 2nd equation of (7.2) reads

$$(2M - L - N)\tan M = 3r\sigma. \tag{9.1}$$

First, assume by contradiction that $M < 10 \cdot (|L| + N)$ (say). Then, assuming that $\sigma > 0$ is sufficiently small, so that, with the help from the (readily established) part (a) of lemma 8.1, $|L|, |N| < 1/100$ (so that $0 \leqslant M < 1/5$), the lhs of (9.1) is bounded above by

$$(2M - L - N)\tan M \ll \frac{\sigma}{n} \cdot M \ll \frac{\sigma^2}{n^2},$$

so the equality (9.1) cannot hold with $\sigma > 0$ sufficiently small. Hence we may as well assume that

$$M \geqslant 10 \cdot (|L| + N).$$

But then we may deduce from (9.1):

$$M^2 \ll (2M - M/10) \cdot M \leqslant (2M - N) \cdot M$$
$$\ll (2M - N)\tan M \leqslant (2M - L - N)\tan M = 3r\sigma,$$

so that $M \ll \sqrt{\sigma}$ as in the 2nd assertion of lemma 8.1 (b).

Finally we show lemma 8.1 (c): since $m = n = 0$, then the equality $M = N$ is forced by the symmetry between these two. (If, by contradiction, $M > N$, then the lhs of the 2nd equation of (7.2) is strictly bigger than the lhs of the 3rd equation of (7.2).) Then the system (7.2) reads

$$\begin{cases} 2(L - M)\tan L = 3r\sigma \\ (M - L)\tan M = 3r\sigma \end{cases}. \tag{9.2}$$

Then, by the 1st equation of (9.2), and recalling that $L, \tan L \leqslant 0$ and $M \geqslant 0$, it forces $2L\tan L \leqslant 3r\sigma$, and so $L \ll \sqrt{\sigma}$, as above. Further, either $M = L \ll \sqrt{\sigma}$ or we may divide the equations in (9.2), and so

$$M \ll \tan M = -2\tan L \ll \sqrt{\sigma},$$

as we have already seen. This concludes the proof of lemma 8.1 (c). □

9.2. Proof of lemma 8.2

Recall (7.2) (with $m = 0$), namely

$$(2M - N - L)\tan M - 3r\sigma = 0$$
$$(\pi n + 2N - L - M)\tan N - 3r\sigma = 0$$
$$(-\pi n + 2L - M - N)\tan L - 3r\sigma = 0. \tag{9.3}$$

Consider the third equation of (9.3): thanks to lemma 8.1 (a) we may write

$$\tan L = L + O(L^3) = L + O(\sigma^3/n^3),$$

and, in addition, from lemma 8.1 (b), one has

$$n\pi L = -3r\sigma + O(\sigma^{3/2}/n),$$

and then

$$L = -\frac{3r}{n\pi} \cdot \sigma + O(\sigma^{3/2}/n^2). \tag{9.4}$$

Analogously,

$$N = \frac{3r}{n\pi} \cdot \sigma + O(\sigma^{3/2}/n^2). \tag{9.5}$$

Next we focus on the first equation of (9.3): we write

$$\tan(M) = M + \frac{1}{3}M^3 + O(M^5) = M + \frac{1}{3}M^3 + O(\sigma^{5/2}),$$

and feed (9.4) and (9.5) into it to derive:

$$2M^2 = 3r\sigma + M(L + N) - \frac{2}{3}M^4 - \frac{1}{3}M^3(L + N) + O(\sigma^3) = 3r\sigma + O(\sigma^2)$$
$$= 3r\sigma(1 + O(\sigma)),$$

so that

$$M = \sqrt{\frac{3r}{2}} \cdot \sqrt{\sigma}(1 + O(\sigma)) = \sqrt{\frac{3r}{2}} \cdot \sqrt{\sigma} + O(\sigma^{3/2}).$$

The continuity of L, M, N at $\sigma = 0$ follows directly from lemma 8.1, and here we deal with the analyticity of L, M, N for $\sigma > 0$ sufficiently small. We want to use the analytic implicit function theorem for the system (9.3). To do that we evaluate the Jacobian of (9.3) (with $m = 0$) as

$$J_{0,n}(\sigma) = \begin{pmatrix} \dfrac{2M - N - L}{\cos^2 M} + 2\tan M & -\tan M & -\tan M \\[2mm] -\tan N & \dfrac{\pi n + 2N - L - M}{\cos^2 N} & -\tan N \\[2mm] -\tan L & -\tan L & \dfrac{-\pi n + 2L - M - N}{\cos^2 L} + 2\tan L \end{pmatrix}.$$

For $\sigma \to 0$, we have $M, N, L \to 0$ so

$$\tan M \sim M = O\left(\sqrt{\sigma}\right), \qquad \tan N \sim N = O\left(\frac{\sigma}{n}\right),$$

and likewise $\tan L = O(\frac{\sigma}{n})$. Also

$$\frac{1}{\cos^2 M} = 1 + \tan^2 M = 1 + O(\sigma), \qquad \frac{1}{\cos^2 N} = 1 + O\left(\frac{\sigma^2}{n^2}\right),$$

and likewise $1/\cos^2 L = 1 + O(\sigma^2/n^2)$. Thus for small $\sigma > 0$,

$$J_{0,n}(\sigma) = \begin{pmatrix} 4M + O(\sigma) & O(\sqrt{\sigma}) & O(\sqrt{\sigma}) \\ O(\sigma/n) & \pi n + O(\sqrt{\sigma}) & O(\sigma/n) \\ O(\sigma/n) & O(\sigma/n) & -\pi n + O(\sqrt{\sigma}) \end{pmatrix}$$

when $\sigma \to 0$, we obtain

$$|\det J_{0,n}(\sigma)| = 4\pi^2 n^2 \cdot M + O(\sigma) \gg \sqrt{\sigma}$$

because $M \approx \sqrt{\sigma}$. Thus we found $|\det J_{0,n}(\sigma)| \gg \sqrt{\sigma} \neq 0$ so that by the analytic implicit function theorem, M, N, L are analytic in σ near $\sigma = 0$.

10. Proofs of lemmas 8.3 and 8.4

10.1. The derivatives of M_j at $\sigma = 0$: proof of lemma 8.3

Proof.   We compute derivatives: from the definition of μ_j we obtain

$$\mu_j' = \frac{1}{\pi}\left(2M_j' - M_i' - M_k'\right).$$

From (8.1) we obtain after one differentiation

$$\mu_j' \tan M_j + \mu_j(\tan M_j)' = \frac{3r}{\pi} \tag{10.1}$$

and differentiating again

$$\mu_j'' \tan M_j + 2\mu_j'(\tan M_j)' + \mu_j(\tan M_j)'' = 0. \tag{10.2}$$

We also recall that

$$\mu_j(0) = m_j, \qquad M_j(0) = 0 \tag{10.3}$$

so that $\tan M_j(0) = 0$, $\cos M_j(0) = 1$. Now $(\tan M_j)' = M_j'/(\cos^2 M_j)$ and therefore

$$(\tan M_j)'(0) = M_j'(0).$$

Substituting in (10.1) and evaluating at $\sigma = 0$ using (10.3) gives $m_j M_j'(0) = \frac{3r}{\pi}$, or

$$M_j'(0) = \frac{3r}{\pi m_j},$$

which is (8.2). We also obtain

$$\mu'_j(0) = \frac{1}{\pi}\left(2M_j(0)' - M'_i(0) - M'_k(0)\right) = \frac{3r}{\pi^2}\left(\frac{2}{m_j} - \frac{1}{m_i} - \frac{1}{m_k}\right)$$

$$= \frac{3r}{\pi^2}\frac{2m_im_k - m_jm_i - m_jm_k}{m_jm_im_k} = \frac{3r}{\pi^2}\frac{m_j^2 + 2m_im_k}{m_jm_im_k}$$

on using $m_i + m_k = -m_j$.

The second derivative of $\tan M_j$ is

$$(\tan M_j)'' = \left(\frac{M'_j}{\cos^2 M_j}\right)' = \frac{M''_j}{\cos^2 M_j} - \frac{2(M'_j)^2 \tan M_j}{\cos^2 M_j} \tag{10.4}$$

and at $\sigma = 0$ we obtain

$$(\tan M_j)''(0) = M''_j(0).$$

Inserting in (10.2) yields

$$2\mu'_j(0)M'_j(0) + m_jM''_j(0) = 0$$

or

$$m_jM''_j(0) = -2\frac{3r}{\pi m_j} \cdot \frac{3r}{\pi^2}\frac{m_j^2 + 2m_im_k}{m_jm_im_k} = -\frac{18r^2}{\pi^3}\frac{m_j^2 + 2m_im_k}{m_j^2m_im_k}$$

which gives

$$M''_j(0) = -\frac{18r^2}{\pi^3}\frac{m_j^2 + 2m_im_k}{m_j^3m_im_k}$$

as claimed in (8.3). $\square$

10.2. Bounding derivatives of M_j: proof of lemma 8.4

Analyticity of M_j near $\sigma = 0$ is proved analogously to the case $m = 0$, $n \geqslant 1$ in lemma 8.2.

We will prove the bounds on the derivatives. Before proceeding, we formulate a standard fact from linear algebra (we leave the verification to the reader):

Lemma 10.1. *Suppose we have a system of the form $(I + B)x = y$, $x, y \in \mathbb{R}^n$ with B a rank one matrix*

$$B = \beta \cdot \alpha^T, \quad \beta, \alpha \in \mathbb{R}^n,$$

where $\|\alpha\| \cdot \|\beta\| < 1$. Then

$$x = y - \frac{\langle \alpha, y \rangle}{1 + \langle \alpha, \beta \rangle}\beta$$

and so

$$x_j = y_j + O(\|y\| \cdot \|\alpha\| \cdot |\beta_j|).$$

We now proceed with the proof of lemma 8.4:

Proof.   For the first derivative we use (10.1) and rearrange it as

$$M'_j \left(1 + \frac{6r\sigma \cos^2 M_j}{\pi^2} \frac{1}{\mu_j^2} \right) - \frac{3r\sigma \cos^2 M_j}{\pi^2 \mu_j^2} (M'_i + M'_k) = \frac{3r \cos^2 M_j}{\pi \mu_j}$$

that is, the vector $\vec{M} = (M_1, M_2, M_3)^{\mathrm{T}}$ satisfies a matrix equation of the form

$$(I + B)\vec{M}' = y$$

with

$$B = \frac{3r\sigma}{\pi^2} \begin{pmatrix} \dfrac{2\cos^2 M_1}{\mu_1^2} & -\dfrac{\cos^2 M_1}{\mu_1^2} & -\dfrac{\cos^2 M_1}{\mu_1^2} \\[2ex] -\dfrac{\cos^2 M_2}{\mu_2^2} & \dfrac{2\cos^2 M_2}{\mu_2^2} & -\dfrac{\cos^2 M_2}{\mu_2^2} \\[2ex] -\dfrac{\cos^2 M_3}{\mu_3^2} & -\dfrac{\cos^2 M_3}{\mu_3^2} & \dfrac{2\cos^2 M_3}{\mu_3^2} \end{pmatrix} = \beta \cdot \alpha^{\mathrm{T}},$$

where

$$\alpha^{\mathrm{T}} = \frac{3r\sigma}{\pi^2} (2, -1, -1), \qquad \beta^{\mathrm{T}} = \left(\frac{\cos^2 M_1}{\mu_1^2}, \frac{\cos^2 M_2}{\mu_2^2}, \frac{\cos^2 M_3}{\mu_3^2} \right),$$

and

$$y_j = \frac{3r \cos^2 M_j}{\pi \mu_j}.$$

We use lemma 10.1, noting that

$$|\langle \alpha, y \rangle| \leqslant |\alpha| \cdot |y| \ll \frac{1}{m_1}, \qquad |\beta_j| \ll \frac{1}{m_j^2}$$

to find

$$M'_j = \frac{3r \cos^2 M_j}{\pi \mu_j} + O\left(\frac{1}{m_1 m_j^2} \right) = \frac{3r}{\pi m_j} \left(1 + O\left(\frac{1}{m_j} \right) \right)$$

locally uniformly in σ.

For the second derivative, use

$$\cos^2 M_j (\tan M_j)'' = M''_j - 2(M'_j)^2 \frac{3r\sigma}{\pi \mu_j} \tag{10.5}$$

(which is a rewriting of (10.4) using (8.1)) and insert into (10.2) (again using (8.1)) to obtain (recalling $(\tan M_j)' \cos^2 M_j = M'_j$)

$$\mu_j \left(M''_j - \frac{6r\sigma}{\pi} \frac{(M'_j)^2}{\mu_j} \right) = -2\mu'_j M'_j - \mu''_j \cos^2 M_j \tan M_j$$

or, after using (8.1)

$$M_j'' + \frac{3r\sigma\cos^2 M_j}{\pi^2\mu_j^2}\left(2M_j'' - M_i'' - M_k''\right) = -\frac{2\mu_j'M_j'}{\mu_j} + \frac{6r\sigma}{\pi}\frac{(M_j')^2}{\mu_j}. \tag{10.6}$$

The rhs of (10.6) is $O(1/(m_1m_j^2))$ by our bounds on the first derivative while the lhs of (10.6) is of the form $(I + B)\vec{M}''$ with $B = \beta\alpha^{\mathrm{T}}$,

$$\alpha^{\mathrm{T}} = \frac{3r\sigma}{\pi^2}(2, -1, -1), \qquad \beta^{\mathrm{T}} = \left(\frac{\cos^2 M_1}{\mu_1^2}, \frac{\cos^2 M_2}{\mu_2^2}, \frac{\cos^2 M_3}{\mu_3^2}\right).$$

Applying lemma 10.1 we find

$$M_j'' = -\frac{2\mu_j'M_j'}{\mu_j} + \frac{6r\sigma}{\pi}\frac{(M_j')^2}{\mu_j} + O\left(\frac{1}{m_1^3}\cdot\frac{1}{m_j^2}\right) \ll \frac{1}{m_1 m_j^2}.$$

For the third derivative we have:

Lemma 10.2.

$$M_j''' = 2(M_j')^3 + O\left(\frac{1}{m_1^3 m_j^2}\right) = \frac{54r^3}{\pi^3 m_j^3} + O\left(\frac{1}{m_1^3 m_j^2}\right).$$

Proof.   Differentiate (10.2) to obtain

$$\mu_j(\tan M_j)''' + 3\mu_j'(\tan M_j)'' + 3\mu_j''(\tan M_j)' + \mu_j'''\tan M_j = 0. \tag{10.7}$$

Recall (10.4) which gives

$$\cos^2 M_j(\tan M_j)'' = M_j'' - 2(M_j')^2\tan M_j \tag{10.8}$$

which in particular we now know to be $O(1/m_1 m_j^2)$ using (8.1). Differentiating (10.8) gives

$$-2\tan M_j(\cos^2 M_j)M_j'(\tan M_j)'' + \cos^2 M_j(\tan M_j)'''$$
$$= M_j''' - 4M_j'M_j''\tan M_j - 2(M_j')^2(\tan M_j)'$$

so that by (8.5)

$$\cos^2 M_j(\tan M_j)''' = M_j''' - 2(M_j')^3 + O\left(\frac{1}{m_1 m_j^4}\right) \tag{10.9}$$

on using (8.1) and

$$(\tan M_j)' = \frac{M_j'}{\cos^2 M_j} = M_j'\left(1 + \left(\tan M_j\right)^2\right) = M_j' + O\left(\frac{1}{m_j^3}\right).$$

Multiplying (10.7) by $(\cos^2 M_j)/\mu_j$ and inserting (10.9) gives

$$M_j''' + \mu_j'''\tan M_j\frac{\cos^2 M_j}{\mu_j} = 2(M_j')^3 + O\left(\frac{1}{m_1 m_j^4}\right)$$

and thus we get

$$M_j''' + O\left(\frac{1}{m_j^2}\right)(2M_j''' - M_i''' - M_k''') = 2(M_j')^3 + O\left(\frac{1}{m_1 m_j^4}\right).$$

Applying lemma 10.1 with $\alpha^T = (2, -1, -1)$, $\beta_j = O(1/m_j^2)$ and $y_j = 2(M_j')^3 + O(\frac{1}{m_1 m_j^4})$ (so that $\langle \alpha, y \rangle = O(1/m_1^3)$) gives

$$M_j''' = 2(M_j')^3 + O\left(\frac{1}{m_1^3 m_j^2}\right).$$

We then use (8.4)

$$M_j' = \frac{3r}{\pi m_j}\left(1 + O\left(\frac{\sigma}{m_j}\right)\right)$$

to obtain

$$M_j''' = \frac{54r^3}{\pi^3 m_j^3} + O\left(\frac{1}{m_1^3 m_j^2}\right).$$

11. Proof of lemma 8.5

Recall that

$$\Lambda_{m,n} = \frac{2\pi^2}{27r^2}\sum_{j=1}^{3}\mu_j^2$$

and we set

$$R^2 = m^2 + mn + n^2 = \frac{1}{2}\sum_{j=1}^{3}m_j^2,$$

$$F_R(m, n) = \frac{R^4}{m^2 n^2 (m+n)^2} = \frac{1}{m^2} + \frac{1}{n^2} + \frac{1}{(m+n)^2}.$$

We want to show

Lemma 11.1. *Assume that $1 \leqslant m \leqslant n$. Then*

$$\Lambda_{m,n}'(0) = \frac{4}{r}, \quad \Lambda_{m,n}''(0) = -8F_R(m, n)$$

$$\Lambda_{m,n}^{(3)}(\sigma) = 24 \cdot r \cdot F_R(m, n) + O\left(\frac{1}{m^4}\right).$$

11.1. The first derivative of $\Lambda_{m,n}$

Differentiating gives

$$\Lambda'_{m,n}(0) = \frac{4\pi^2}{27r^2}\sum_{j=1}\mu_j(0)\mu'_j(0).$$

We have

$$\sum_{j=1}\mu_j(0)\mu'_j(0) = \frac{1}{\pi}\sum_{j=1}^{3} m_j\left(2M'_j(0) - M'_i(0) - M'_k(0)\right)$$

$$= \frac{3r}{\pi^2}\sum_{j=1}^{3} m_j\left(\frac{2}{m_j} - \frac{1}{m_i} - \frac{1}{m_k}\right)$$

since $\mu_j(0) = m_j$, and $M'_j(0) = 3r/(\pi m_j)$. Since

$$\sum_{j=1}^{3} m_j\left(\frac{2}{m_j} - \frac{1}{m_i} - \frac{1}{m_k}\right) = 6 - \sum_{j=1}^{3} m_j\left(\frac{1}{m_i} + \frac{1}{m_k}\right),$$

and

$$-\sum_{j=1}^{3} m_j\left(\frac{1}{m_i} + \frac{1}{m_k}\right) = -\sum_{j=1}^{3}\frac{m_j}{m_i} - \sum_{j=1}^{3}\frac{m_j}{m_k}$$

$$= -\sum_{k=1}^{3}\frac{1}{m_k}\left(m_j + m_i\right) = \sum_{k=1}^{3} 1 = 3,$$

we obtain

$$\sum_{j=1}\mu_j(0)\mu'_j(0) = \frac{27r}{\pi^2}$$

which gives

$$\Lambda'_{m,n}(0) = \frac{4\pi^2}{27r^2}\cdot\frac{27r}{\pi^2} = \frac{4}{r}.$$

11.2. The second derivative of $\Lambda_{m,n}$

Using Leibnitz's rule gives

$$\Lambda''_{m,n} = \frac{4\pi^2}{27r^2}\sum_{j=1}^{3}\mu_j\mu''_j + (\mu'_j)^2.$$

We have

$$\mu_j(0) = m_j, \qquad M'_j(0) = \frac{3r}{\pi m_j}, \qquad M''_j(0) = -\frac{18r^2}{\pi^3}\left(\frac{1}{m_i m_j m_k} + \frac{2}{m_j^3}\right)$$

so that

$$\mu_j'(0) = \frac{1}{\pi}(2M_j'(0) - M_i'(0) - M_k'(0)) = \frac{3r}{\pi^2}\left(\frac{2}{m_j} - \frac{1}{m_i} - \frac{1}{m_k}\right)$$

and

$$\mu_j(0)\mu_j''(0) = m_j\frac{1}{\pi}(2M_j''(0) - M_i''(0) - M_k''(0))$$

$$= -\frac{18r^2}{\pi^4}\left(2\left(\frac{1}{m_im_j} + \frac{2}{m_j^2}\right) - \left(\frac{1}{m_jm_k} + \frac{2m_j}{m_i^3}\right)\right.$$

$$\left. - \left(\frac{1}{m_jm_i} + \frac{2m_j}{m_k^3}\right)\right)$$

giving

$$\sum_{j=1}^{3}\mu_j(0)\mu_j''(0) = -\frac{36r^2}{\pi^4}\sum_{j=1}^{3}\left(\frac{2}{m_j^2} - \frac{m_j}{m_i^3} - \frac{m_j}{m_k^3}\right).$$

Likewise,

$$\sum_{j=1}^{3}\mu_j'(0)^2 = \frac{9r^2}{\pi^4}\sum_{j=1}^{3}\left(\frac{2}{m_j} - \frac{1}{m_i} - \frac{1}{m_k}\right)^2.$$

A straightforward computation reveals that

$$\sum_{j=1}^{3}\left(\frac{2}{m_j^2} - \frac{m_j}{m_i^3} - \frac{m_j}{m_k^3}\right) = \frac{3R^4}{(m_1m_2m_3)^2} = 3F_R(m,n)$$

and

$$\sum_{j=1}^{3}\left(\frac{2}{m_j} - \frac{1}{m_i} - \frac{1}{m_k}\right)^2 = \frac{6R^4}{(m_1m_2m_3)^2} = 6F_R(m,n).$$

These give

$$\frac{4\pi^2}{27r^2}\sum_{j=1}^{3}\mu_j(0)\mu_j''(0) = -\frac{16}{\pi^2}F_R(m,n), \quad \frac{4\pi^2}{27r^2}\sum_{j=1}^{3}\mu_j'(0)^2 = \frac{8}{\pi^2}F_R(m,n).$$

Altogether we find

$$\Lambda_{m,n}''(0) = -\frac{8}{\pi^2}F_R(m,n).$$

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11.3. The third derivative

We have

$$\Lambda_{m,n}^{(3)} = \frac{2\pi^2}{27r^2} \sum_{j=1}^{3} 2\mu_j \mu_j''' + 6\mu_j' \mu_j''.$$

We use $\mu_j' \ll 1/m_1$, $\mu_j'' \ll 1/m_1^3$ to deduce that

$$\Lambda_{m,n}^{(3)} = \frac{4\pi^2}{27r^2} \sum_{j=1}^{3} \mu_j \mu_j''' + O\left(\frac{1}{m_1^4}\right).$$

Now

$$\sum_{j=1}^{3} \mu_j \mu_j''' = \frac{1}{\pi} \sum_{j=1}^{3} \mu_j(2M_j''' - M_i''' - M_k''') = \frac{1}{\pi} \sum_{j=1}^{3} M_j'''(2\mu_j - \mu_i - \mu_k)$$

after reordering the sum. We use a simple lemma:

Lemma 11.2. *If* $\{i, j, k\} = \{1, 2, 3\}$*, and* $b_1 + b_2 + b_3 = 0$*, then*

$$\sum_{j=1}^{3} a_j(2b_j - b_i - b_k) = 3\sum_{j=1}^{3} a_j b_j.$$

Apply lemma 11.2 to $a_j = M_j'''$, $b_j = \mu_j$, to obtain

$$\sum_{j=1}^{3} \mu_j \mu_j''' = \frac{3}{\pi} \sum_{j=1}^{3} \mu_j M_j'''.$$

Using $\mu_j = m_j + O(1/m_j)$ and (8.6) which states that

$$M_j''' = \frac{54r^3}{\pi^3 m_j^3} + O\left(\frac{1}{m_1^3 m_j^2}\right)$$

gives

$$\mu_j M_j''' = \frac{54r^3}{\pi^3 m_j^2} + O\left(\frac{1}{m_1^3 m_j}\right) = \frac{54r^3}{\pi^3 m_j^2} + O\left(\frac{1}{m_1^4}\right).$$

We obtain

$$\sum_{j=1}^{3} \mu_j \mu_j''' = \frac{3}{\pi} \sum_{j=1}^{3} \frac{54r^3}{\pi^3 m_j^2} + O\left(\frac{1}{m_1^4}\right)$$

and so

$$\Lambda_{m,n}^{(3)} = \frac{4\pi^2}{27r^2} \sum_{j=1}^{3} \mu_j \mu_j''' + O\left(\frac{1}{m_1^4}\right) = \frac{4\pi^2}{27r^2} \frac{3}{\pi} \sum_{j=1}^{3} \frac{54r^3}{\pi^3 m_j^2} + O\left(\frac{1}{m_1^4}\right)$$

$$= \frac{24r}{\pi^2} \sum_{j=1}^{3} \frac{1}{m_j^2} + O\left(\frac{1}{m_1^4}\right) = 24r F_R(m,n) + O\left(\frac{1}{m_1^4}\right).$$

This concludes the proof of lemma 8.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

12. Proof of proposition 7.2

Proof. We first assume that

$$m' > 10m.$$

Then use

$$F_R(m,n) = \frac{1}{m^2} + \frac{1}{n^2} + \frac{1}{(m+n)^2} > \frac{1}{m^2},$$

$$F_R(m',n') = \frac{1}{m'^2} + \frac{1}{n'^2} + \frac{1}{(m'+n')^2} < \frac{3}{m'^2} < \frac{3}{(10m)^2}$$

to obtain

$$F_R(m,n) - F_R(m',n') > \frac{1}{m^2} - \frac{3}{(10m)^2} \gg \frac{1}{m^2}$$

which is certainly sufficient.

Now assume that for $\delta > 0$ very small (but fixed),

$$\delta R < m < m' \leqslant 10m.$$

We will show that

$$F_R(m,n) - F_R(m',n') = \frac{1}{R^2}\left(f\left(\frac{m}{R}\right) - f\left(\frac{m'}{R}\right)\right)$$
$$> \frac{243}{4}\frac{1}{R^4} + O\left(\frac{1}{R^5}\right) \gg \frac{1}{R^4}$$

and since $m > \delta R$, we obtain

$$F_R(m,n) - F_R(m',n') \gg \frac{\delta^4}{m^4} \gg \frac{1}{m^4}$$

as required.

Given R, and $1 \leqslant m < n$, with $m^2 + mn + n^2 = R^2$ we can express n in terms of m as

$$n = \sqrt{R^2 - 3\left(\frac{m}{2}\right)^2} - \frac{m}{2}.$$

Therefore we can write

$$F_R(m,n) = \frac{1}{R^2}f\left(\frac{m}{R}\right),$$

where

$$f(x) = \frac{1}{x^2} + \frac{1}{\left(\sqrt{1 - 3\left(\frac{x}{2}\right)^2} - \frac{x}{2}\right)^2} + \frac{1}{\left(\sqrt{1 - 3\left(\frac{x}{2}\right)^2} + \frac{x}{2}\right)^2}$$

which simplifies to

$$f(x) = \frac{1}{x^2(1 - x^2)^2}.$$

The derivative of f is

$$f'(t) = -\frac{2(1-3t^2)}{t^3(1-t^2)^3}$$

which is negative for $0 < t < 1/\sqrt{3}$, so that f is decreasing in that range. Moreover, the second derivative is

$$f''(t) = \frac{6\left(7t^4 - 4t^2 + 1\right)}{t^4\left(t^2-1\right)^4} = \frac{6\left(3t^4 + (1-2t^2)^2\right)}{t^4\left(t^2-1\right)^4} > 0$$

which is positive, hence $f'(t)$ is increasing (and negative) and $-f'(t)$ is positive and decreasing.

Let $x = m/R$, $x' = m'/R$. Then

$$F_R(m,n) - F_R(m',n') = \frac{1}{R^2}\left(f(x) - f(x')\right).$$

We separate two cases: $m' = n'$, or $m' < n'$.

If $m' = n'$ then $x' = 1/\sqrt{3}$ (since then $R^2 = 3(m')^2$), and $f'(1/\sqrt{3}) = 0$. We expand $f(x)$ around $x' = 1/\sqrt{3}$ to first order with Lagrange remainder term

$$f(x) - f\left(\frac{1}{\sqrt{3}}\right) = f'\left(\frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right) + \frac{f''(t)}{2}\left(x - \frac{1}{\sqrt{3}}\right)^2$$

for some $t \in (x, 1/\sqrt{3}) \subset [0, \frac{1}{\sqrt{3}}]$. Hence using $\frac{1}{\sqrt{3}} - x = \frac{m'-m}{R} \geqslant \frac{1}{R}$ and a numerical finding that $\min\limits_{[0,1/\sqrt{3}]} f''(t) = 119.167\ldots$, we obtain

$$f(x) - f\left(\frac{1}{\sqrt{3}}\right) \geqslant \frac{1}{2}\left(\min_{[0,1/\sqrt{3}]} f''(t)\right)\left(x - \frac{1}{\sqrt{3}}\right)^2 > \frac{59}{R^2}. \tag{12.1}$$

If $m' < n'$ then we use the mean value theorem, obtaining that for some $x < t < x'$ we have

$$F_R(m,n) - F_R(m',n') = \frac{1}{R^2}\left(f(x) - f(x')\right) = \frac{1}{R^2}(x' - x)\cdot(-f'(t)).$$

We want to give lower bounds for $x' - x$ and for $-f'(t)$.

We have $x' - x = (m' - m)/R \geqslant 1/R$. Moreover, we claim that

$$x' \leqslant \frac{1}{\sqrt{3}} - \frac{1}{2R}.$$

Indeed, since $m'^2 + m'n' + (n')^2 = R^2$ with $n' > m'$ so that by integrality $n' \geqslant m' + 1$, we have

$$4R^2 = 3(m')^2 + (m' + 2n')^2 \geqslant 3(m')^2 + (m' + 2(m' + 1))^2$$
$$= 4(1 + 3m' + 3(m')^2)$$

so that $x' = m'/R$ satisfies $3(x')^2 + \frac{3}{R}x' + \frac{1}{R^2} \leqslant 1$, giving

$$x' \leqslant \frac{1}{6}\left(\sqrt{12 - \frac{3}{R^2}} - \frac{3}{R}\right) < \frac{1}{\sqrt{3}} - \frac{1}{2R}.$$

Hence

$$-f'(t) > -f'(x') > -f'\left(\frac{1}{\sqrt{3}} - \frac{1}{2R}\right) = \frac{243}{4R} + O\left(\frac{1}{R^2}\right).$$

Therefore

$$f(x) - f(x') = (x' - x) \cdot (-f'(t)) \geqslant \frac{243}{4}\frac{1}{R^2} + O\left(\frac{1}{R^3}\right) \tag{12.2}$$

in this case.

Combining (12.1) and (12.2) gives that in both cases,

$$F_R(m, n) - F_R(m', n') = \frac{1}{R^2}\left(f\left(\frac{m}{R}\right) - f\left(\frac{m'}{R}\right)\right) > \frac{59}{R^4} + O\left(\frac{1}{R^5}\right)$$

as claimed.

Finally assume

$$m < m' \leqslant 10m \quad \text{and} \quad m \leqslant \delta R,$$

so that in particular, $m' < 10\delta R$: then

$$F_R(m, n) - F_R(m', n') = \frac{1}{R^2}(f(x) - f(x')) = \frac{1}{R^2}(x' - x) \cdot (-f'(t))$$

for some $t \in (x, x')$. Note that $0 < t < x' < 10x < 10\delta$ so that

$$-f'(t) = \frac{2(1 - 3t^2)}{t^3(1 - t^2)^3} > \frac{2(1 - (10\delta)^2)}{t^3} > \frac{1}{t^3}$$

for $\delta > 0$ sufficiently small, so since $tR < m' < 10m$ we obtain

$$F_R(m, n) - F_R(m', n') = \frac{m' - m}{R^3}(-f'(t)) > \frac{m' - m}{t^3 R^3} > \frac{1}{(10m)^3}$$

which is consistent with the assertion of proposition 7.2 in this case.   □

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Data availability statement

No new data were created or analysed in this study.

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