Geometric And Functional Analysis
Vol. 5, No. 2 (1995)
1016-443X/95/0200387-15$1.50+0.20/0
© 1995 Birkhäuser Verlag, Basel

ON SELBERG'S EIGENVALUE CONJECTURE

W. LUO, Z. RUDNICK AND P. SARNAK

1. Introduction

Let $\Gamma \subset SL_2(\mathbf{Z})$ be a congruence subgroup, and $\lambda_0 = 0 < \lambda_1 < \dots$ be the eigenvalues of the non-euclidean Laplacian on $L^2(\Gamma\backslash\mathbf{H}^2)$. A fundamental conjecture of Selberg ([Se]) asserts that the smallest nonzero eigenvalue $\lambda_1(\Gamma) \geq 1/4 = 0.25$. In the same paper Selberg proved that $\lambda_1(\Gamma) \geq 3/16 = 0.1875$. Gelbart and Jacquet ([GJ]), using very different methods, improved this to $\lambda_1(\Gamma) > 3/16$. Iwaniec ([I]) showed that for almost all Hecke congruence groups $\Gamma_0(p)$ with a certain multiplier χ_p, one has $\lambda_1(\Gamma_0(p), \chi_p) \geq 44/225 = 0.19555\dots$. In [I], he also established a density theorem for possible exceptional eigenvalues as above, which while not giving any improvement on 3/16 for an individual Γ, is sufficiently strong to substitute for Selberg's conjecture in many applications to number theory. Selberg's conjecture is the archimedean analogue of the "Ramanujan conjectures" on the Fourier coefficients of Maass forms. For these, much progress has been made in improving the relevant estimates, beginning with Serre ([Ser]) and later on Shahidi ([Sh2]) and Bump-Duke-Hoffstein-Iwaniec ([BDHI]). In this paper we restore the balance and establish in part for the archimedean place what is known at the finite places. The method on the face of it is quite different, but the quality of the results coincide (the reason will be made clear later).

THEOREM 1.1. *For any congruence subgroup $\Gamma \subset SL_2(\mathbf{Z})$, we have*

$$\lambda_1(\Gamma) \geq \frac{21}{100} .$$

There are numerous applications of this result. We present some immediate ones. The first is towards the Linnik-Selberg conjecture ([L],[Se]) on cancellation in sums of Kloosterman sums. For $m, n, c \geq 1$ one defines the Kloosterman sum as

$$S(m,n,c) = \sum_{\substack{x \bmod c \\ x\bar{x} \equiv 1 \bmod c}} e\left(\frac{mx + n\bar{x}}{c}\right) .$$

Supported by NSF grants DMS-9304580, DMS-9400163 and DMS-9102082

COROLLARY 1.1. *Fix N, n, m. Then as $x \to \infty$,*

$$\sum_{\substack{c \le x \\ c \equiv 0 \bmod N}} \frac{S(m,n,c)}{c} \ll x^{2/5} .$$

Note that Weil's bound ([We]) $S(m,n,c) \ll_\epsilon c^{1/2+\epsilon}$ would give a bound of $x^{1/2+\epsilon}$ for the sum so that Corollary 1.1 indicates that there is considerable cancellation due to the signs of the Kloosterman sums along any progression $c \equiv 0 \bmod N$. Indeed this Corollary is the first such result on cancellation of Kloosterman sums on a general progression.

A second application is to the remainder term in the "Prime Geodesic Theorem" for congruence subgroups Γ of $SL_2(\mathbf{Z})$. Let $\pi_\Gamma(x)$ be the number of prime closed geodesics of length $\ell \le \log x$ on $\Gamma \backslash \mathbf{H}^2$.

COROLLARY 1.2. *For any congruence group $\Gamma \subset SL_2(\mathbf{Z})$,*

$$\pi_\Gamma(x) = Li(x) + O(x^{7/10})$$

where $Li(x) = \int_2^x \frac{dt}{\log t}$.

A remainder term of the form $O(x^{3/4})$ has been known for a some time (see [S]). The natural conjecture here is that as in the theory of primes, the remainder term is $O_\epsilon(x^{1/2+\epsilon})$ for any $\epsilon > 0$.

Our proof of Theorem 1.1 is based on the Gelbart-Jacquet lift ([GJ]) and so is naturally concerned with the cuspidal spectrum of GL_m (in what follows $m \ge 2$). To describe our results we need to introduce various L-functions. For this we assume some familiarity with the adelic language. Let $\mathbf{A}$ be the adeles of $\mathbf{Q}$, and $\pi = \otimes_{p \le \infty} \pi_p$ be an irreducible cuspidal automorphic representation of $GL_m(\mathbf{A})$, which we normalize to have unitary central character. Assume that the archimedean component π_∞ is spherical, so that one associates to it a semi-simple conjugacy class $diag(\mu_\infty(1), \dots, \mu_\infty(m))$ in $GL_m(\mathbf{C})$. The gamma factor for the principal L-function $L(s,\pi)$ associated to π ([GoJ],[J]) is

$$L(s, \pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbf{R}}\big(s - \mu_\infty(j)\big) \tag{1.1}$$

where $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma\big(\frac{s}{2}\big)$. The analogue of Selberg's conjecture for GL_m is that π_∞ is *tempered*, i.e. for $j = 1, \dots, m$

$$\operatorname{Re}\big(\mu_\infty(j)\big) = 0 . \tag{1.2}$$

For $m = 2$, Selberg's bound $\lambda_1 \ge 3/16$ is equivalent to $|\operatorname{Re}(\mu_\infty(j))| \le 1/4$. For $m \ge 3$ the only known bound toward (1.2) is the (local) result of

Jacquet-Shalika ([JSha1,2]), which asserts that for generic unitary representations

$$\left|\operatorname{Re}(\mu_\infty(j))\right| < \tfrac{1}{2} \tag{1.3}$$

(see [BR] for a proof using Vogan's classification of the unitary dual of $GL_m(\mathbf{R})$).

THEOREM 1.2. *Let π be a cuspidal automorphic representation of $GL_m/\mathbf{Q}$ with π_∞ spherical. Then*

$$\left|\operatorname{Re}(\mu_\infty(j))\right| \le \frac{1}{2} - \frac{1}{m^2+1}\ .$$

Theorem 1.1 follows from Theorem 1.2 via the Gelbart-Jacquet lift. Indeed if $\lambda = 1/4 - r^2$, $r > 0$, is an exceptional eigenvalue for $\Gamma\backslash\mathbf{H}^2$ then there is a cuspidal automorphic representation π on $GL_2/\mathbf{Q}$ such that π_∞ is parametrized by $\mu_\infty(1) = r$, $\mu_\infty(2) = -r$. Since π cannot be monomial (as these have $\lambda \ge 1/4$), it lifts to a cuspidal automorphic representation Π on GL_3 whose archimedean component Π_∞ is also spherical and is parametrized by $diag(2r, 0, -2r)$. Now apply Theorem 1.2.

Our proof of Theorem 1.2 makes heavy use of the by now well developed Rankin-Selberg theory on GL_m. The key to the proof is the following observation: If π on GL_m is as above, the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi})$ has as its gamma factor

$$L(s, \pi_\infty \times \tilde{\pi}_\infty) = \prod_{j,k=1}^{m} \Gamma_{\mathbf{R}}\big(s - \mu_\infty(j) - \overline{\mu_\infty(k)}\big)\ .$$

Let $\beta_0 = 2\max\operatorname{Re}(\mu_\infty(j))$, then $L(s, \pi_\infty \times \tilde{\pi}_\infty)$ is holomorphic for $\operatorname{Re} s > \beta_0$ and has a pole at $s = \beta_0$. If χ is a primitive even Dirichlet character then the same is true for the gamma factor of $L(s, (\pi \otimes \chi)_\infty \times \tilde{\pi}_\infty)$ - in fact the gamma factor is still equal to $L(s, \pi_\infty \times \tilde{\pi}_\infty)$. For χ even primitive of sufficiently large (prime) conductor q we have $\pi \otimes \chi \not\simeq \pi$ and so $L(s, \pi_\infty \times \tilde{\pi}_\infty) L(s, (\pi \otimes \chi) \times \tilde{\pi})$ is entire. Hence β_0 is a *trivial* zero of $L(s, (\pi \otimes \chi) \times \tilde{\pi})$, that is

$$L\big(\beta_0, (\pi \otimes \chi) \times \tilde{\pi}\big) = 0 \tag{1.4}$$

for all such χ. In this way the problem becomes the familiar one of proving that certain twists of L-functions do not vanish at a given point. Theorem 1.2 follows from

$$\sum_{q \sim Q} \sum_{\substack{\chi \ne \chi_0 \\ \text{even}}} L\big(\beta, (\pi \otimes \chi) \times \tilde{\pi}\big) \gg \frac{Q^2}{\log Q} \tag{1.5}$$

for $\operatorname{Re}\beta > 1 - \frac{2}{m^2+1}$, with Q large, the implied constants depending only on π and β.

To prove (1.5) we use the functional equation for $L(s,(\pi\otimes\chi)\times\tilde{\pi})$ to approximate $L(\beta,(\pi\otimes\chi)\times\tilde{\pi})$. This brings in Gauss sums and in order to optimize the analysis we use Deligne's bounds on hyper-Kloosterman sums ([D]); these arise for similar reasons in Duke-Iwaniec ([DuI]), Rohrlich ([R]) and Barthel-Ramakrishnan ([BR]). We note however that an improvement of (1.3) and hence of Selberg's 3/16 bound would result even without use of Deligne's bound.

To end the Introduction we make some further remarks. Firstly one can treat the finite places in a similar way and reduce the problem of bounding the size of Fourier coefficients to one of non-vanishing of twists. That is, fix a prime p at which π is unramified. The local L-factor $L(s,\pi_p\times\tilde{\pi}_p)=\prod_{j,k=1}^{m}(1-\alpha_j(p)\overline{\alpha_k(p)}p^{-s})^{-1}$ has a pole at the point β_0 defined via $p^{\beta_0}=\max_j|\alpha_j(p)|^2$. Hence the partial L-function $L^{(p)}(s,\pi\times\tilde{\pi}):=L(s,\pi_p\times\tilde{\pi}_p)^{-1}L(s,\pi\times\tilde{\pi})$ has a "trivial" zero at $s=\beta_0$. The same is true for the twists $L^{(p)}(s,(\pi\otimes\chi)\times\tilde{\pi})$ for any χ of conductor q for which $\chi(p)=1$ (this being the analogue of $\chi(-1)=1$ which we needed in (1.4)). By choosing special q's (as in Rohrlich ([R])) one can modify the arguments in this paper and obtain similar results towards Ramanujan at p. This puts the finite and infinite places on the same footing and explains the comment before Theorem 1.1. In this connection we note that the full Selberg conjecture (or as above, the Ramanujan conjecture) would follow from the following statement: Given π an irreducible cuspidal automorphic representation and β with $\mathrm{Re}(\beta)>0$, there is an even Dirichlet character such that $L(\beta,\pi\otimes\chi)\neq 0$. Such problems have been studied by many authors ([Shi], [R], [BR]).

In the special case of $m=3$ an improvement of Theorem 1.2 would result from the theory of the symmetric square L-function $L(s,\pi,\mathrm{Sym}^2)$ on GL_3. The point is that by using $L(s,\pi\otimes\chi,\mathrm{Sym}^2)$ instead of $L(s,(\pi\otimes\chi)\times\tilde{\pi})$, the conductor dependence in χ is reduced from q^9 to q^6. On the other hand the location of the trivial zeros remains unchanged. The result would be an improvement in the RHS of Theorem 1.2, with 2/5 being replaced by 5/14 and correspondingly $\lambda_1(\Gamma)\geq\frac{171}{784}=0.21811\ldots$ in Theorem 1.1. Unfortunately, the archimedean theory (even in the unramified case) for $L(s,\pi,\mathrm{Sym}^2)$ is not well understood at present and so we cannot carry out the above analysis.[1] However, in view of [PP-S],[BuGi], this is not a problem at the finite unramified places, and so for these one can carry out the above. This is the analogue of [BDHI].

The results above, both at the infinite and finite places, can be estab-

[1] D. Ramakrishnan has pointed out to us a device using [BuGi] and the functional equation in [Sh1] to overcome this difficulty.

lished with $\mathbf{Q}$ replaced by a number field F with no loss in the quality of the estimates. The point being that the size of the conductor (in the character aspect) of $L(s,(\pi\otimes\chi)\times\tilde{\pi})$ is independent of the number field. The analysis is made more difficult by the presence of units which restrict the choice of χ. A similar difficulty appears, and is overcome, in the work of Rohrlich ([R]; see also [BR]). A complete proof of the results with $\mathbf{Q}$ replaced by a number field will appear in a forthcoming article.

Acknowledgement. We thank H. Iwaniec for several illuminating discussions on this work.

2. Background on Rankin-Selberg L-functions

2.1 Rankin-Selberg theory. We recall the Rankin-Selberg theory as developed by Jacquet, Piatetski-Shapiro and Shalika ([JP-SS]), Shahidi ([Sh1]) and Mœglin-Waldspurger ([MW]). The Rankin-Selberg L-function associated to a pair of cuspidal automorphic representations π' on GL_m, π'' on GL_n is given by an Euler product

$$L(s,\pi'\times\pi'')=\prod_{p<\infty}L(s,\pi'_p\times\pi''_p)\ .$$

For primes p where both π' and π'' are unramified the local factors are given by

$$L(s,\pi'_p\times\pi''_p)=\det(I-A'_p\otimes A''_p p^{-s})^{-1} \tag{2.1}$$

where A'_p (respectively A''_p) are the Satake parameters associated to π'_p (respectively to π''_p). At finite primes where one of π', π'' are ramified, the local factor is still of the form $L(s,\pi'_p\times\pi''_p)=P_p(p^{-s})^{-1}$, where $P_p(x)$ is a polynomial of degree at most mn with $P(0)=1$. The Euler product is absolutely convergent for $\operatorname{Re}s>1$ [JSha1].

In case π'_∞, π''_∞ are spherical, the local factor at infinity is given by

$$L(s,\pi'_\infty\times\pi''_\infty)=\prod_{j=1}^{m}\prod_{k=1}^{n}\Gamma_{\mathbf{R}}\big(s-\mu'_\infty(j)-\mu''_\infty(k)\big)\ . \tag{2.2}$$

The completed L-function $\Lambda(s,\pi'\times\pi'')=L(s,\pi'_\infty\times\pi''_\infty)L(s,\pi'\times\pi'')$ has a meromorphic continuation and satisfies a functional equation

$$\Lambda(s,\pi'\times\pi'')=\epsilon(s,\pi'\times\pi'')\Lambda(1-s,\tilde{\pi}'\times\tilde{\pi}'') \tag{2.3}$$

where the ϵ-factor is of the form

$$\epsilon(s,\pi'\times\pi'')=\tau(\pi'\times\pi'')\mathfrak{f}(\pi'\times\pi'')^{-s} \tag{2.4}$$

with $\mathfrak{f}(\pi'\times\pi'')>0$ and $\tau(\pi'\times\pi'')\in\mathbf{C}^*$. It can be written as a product of local factors by fixing an additive character $\psi=\prod\psi_p$ of $\mathbf{A}/\mathbf{Q}$ (which we will assume to be everywhere normalized):

$$\epsilon(s,\pi'\times\pi'')=\prod_p \epsilon_p(s,\pi'_p\times\pi''_p,\psi_p) \tag{2.5}$$

and each local factor is 1 if both π'_p and π''_p are unramified and ψ_p is normalized, and otherwise it is of the form

$$\epsilon_p(s,\pi'_p\times\pi''_p,\psi_p)=\tau(\pi'_p\times\pi''_p)p^{-c(\pi'_p\times\pi''_p)s} \tag{2.6}$$

with $c(\pi'_p\times\pi''_p)\in\mathbf{Z}$ and $\tau(\pi'_p\times\pi''_p)$ a "Gauss sum". The "conductor" $\mathfrak{f}(\pi'\times\pi'')$ is the product $\prod_p p^{c(\pi'_p\times\pi''_p)}$. At infinity, since we are assuming that both π_∞ and π''_∞ are unramified, one has $\epsilon(s,\pi'_\infty\times\pi''_\infty,\psi_\infty)=1$.

The completed L-function $\Lambda(s,\pi'\times\pi'')$ is entire unless $\pi''\simeq\tilde{\pi}'\otimes|\cdot|^t$ for some $t\in\mathbf{C}$, and $\Lambda(s,\pi\times\tilde{\pi})$ is holomorphic except for simple poles at $s=0,1$ ([MW]). Moreover it is easily deduced from [JSha1,2] and [MW] that $L(s,\pi'\times\pi'')$ is of order one, see [RuS]. If we express $L(s,\pi\times\tilde{\pi})$ as a Dirichlet series

$$L(s,\pi\times\tilde{\pi})=\sum_{n=1}^{\infty}\frac{b(n)}{n^s} \tag{2.7}$$

then the coefficients $b(n)\geq 0$ (see [RuS] for the verification at the ramified primes) and the following is a simple consequence:

$$\sum_{n\leq x} b(n)\sim c_\pi x,\qquad x\to\infty \tag{2.8}$$

for some $c_\pi>0$.

2.2 Twists. Let χ be a primitive Dirichlet character mod q. As is well known, χ corresponds to a Hecke character of the idele class group $\mathbf{A}^\times/\mathbf{Q}^\times$, trivial on $\mathbf{R}_+^\times$, so χ is of the form $\chi=\otimes\chi_p$. The Dirichlet character being even (i.e. $\chi(-1)=1$) is equivalent to $\chi_\infty\equiv 1$. For $q>2$ prime, there are $(q-1)/2$ such characters mod q.

We apply the Rankin-Selberg theory described above to the following situation: Fix π on GL_m, and let χ be an even *primitive* Dirichlet character mod q, where q is a prime not dividing the conductor $\mathfrak{f}(\pi)$ of π. Take $\pi'=\pi(\chi):=\pi\otimes\chi$ and $\pi''=\tilde{\pi}$. To describe the exact functional equation in this case, we recall the Gauss sum

$$\tau(\chi)=\sum_{x \bmod q}\chi(x)e(\frac{x}{q})\ . \tag{2.9}$$

LEMMA 2.1. *Let q be a prime, $q\nmid\mathfrak{f}(\pi)$, and let χ be a primitive even Dirichlet character* mod$q$.

i) *If we write $L(s,\pi\times\tilde{\pi})=\sum_{n=1}^{\infty}b(n)n^{-s}$, then (recall $\chi(n)$=0 if $(q,n)\neq 1$) :*

$$L(s,\pi(\chi)\times\tilde{\pi}) = \sum_{n=1}^{\infty} \frac{\chi(n)b(n)}{n^s}$$

ii) $\Lambda(s,\pi(\chi)\times\tilde{\pi}) = L(s,\pi_\infty\times\tilde{\pi}_\infty)L(s,\pi(\chi)\times\tilde{\pi})$ *satisfies the functional equation*

$$\Lambda(s,\pi(\chi)\times\tilde{\pi}) = \epsilon(s,\pi(\chi)\times\tilde{\pi})\Lambda(1-s,\pi(\bar{\chi})\times\tilde{\pi}) \tag{2.10}$$

where the global ϵ-factor is given by

$$\begin{aligned}\epsilon(s,\pi(\chi)\times\tilde{\pi}) &= \chi(\mathfrak{f}(\pi\times\tilde{\pi}))\epsilon(s,\pi\times\tilde{\pi})\epsilon(s,\chi)^{m^2}\\ &= \chi(\mathfrak{f}(\pi\times\tilde{\pi}))\tau(\chi)^{m^2}q^{-m^2 s}\epsilon(s,\pi\times\tilde{\pi})\ .\end{aligned} \tag{2.11}$$

Proof: If $p \not| q\mathfrak{f}(\pi)$ then

$$\begin{aligned}&L(s,\pi(\chi)_p\times\tilde{\pi}_p) = \det\left(I-\chi(p)A(p)\otimes\bar{A}(p)p^{-s}\right)^{-1}\\ &\epsilon(s,\pi(\chi)_p\times\tilde{\pi}_p,\psi_p) = 1\ .\end{aligned} \tag{2.12}$$

To describe the local factors in the case $p \mid q\mathfrak{f}(\pi)$, we begin with $p = q$: If $\chi \neq 1$ then the local L-factors are given by

$$L(s,\pi(\chi)_q\times\tilde{\pi}_q) = 1\ .$$

Indeed, at the prime q, $\pi_q = Ind(GL_m,B;\mu_1,\dots,\mu_m)$ is an unramified principal series representation, where $\mu_j(x) = |x|^{u_j}$ are unramified characters. Likewise $\tilde{\pi}_q = Ind(GL_m,B;\mu_1^{-1},\dots,\mu_m^{-1})$ is unramified. Then $\pi_q\otimes\chi = Ind(GL_m,B;\chi\mu_1,\dots,\chi\mu_m)$. Hence (see [JP-SS])

$$\begin{aligned}L(s,\pi(\chi)_q\times\tilde{\pi}_q) &= \prod_{j=1}^{m} L(s,\pi_q\otimes\chi_q\otimes\mu_j^{-1})\\ &= \prod_{j,k=1}^{m} L(s,\chi\mu_k\mu_j^{-1})\end{aligned}$$

and since χ_q is ramified, each factor above is 1. As for the epsilon factor, we have by [JP-SS]

$$\begin{aligned}\epsilon(s,\pi(\chi)_q\times\tilde{\pi}_q,\psi_q) &= \prod_{j=1}^{m}\epsilon(s,\pi_q\otimes\chi\mu_j^{-1},\psi_q)\\ &= \prod_{j,k=1}^{m}\epsilon(s,\chi\mu_k\mu_j^{-1},\psi_q)\\ &= \prod_{j,k=1}^{m}\epsilon(s+u_k-u_j,\chi,\psi_q)\end{aligned}$$

where the abelian ϵ-factor (for χ primitive) is given by

$$\epsilon(s,\chi,\psi_q) = \tau(\chi)q^{-s} .$$

Therefore we have

$$\begin{aligned}\epsilon\big(s,\pi(\chi)_q \times \tilde{\pi}_q,\psi_q\big) &= \prod_{j,k=1}^{m} \tau(\chi)q^{-(s+u_k-u_j)} \\ &= \tau(\chi,\psi_q)^{m^2} q^{-m^2 s} .\end{aligned}$$

Since the local ϵ-factor $\epsilon(s,\pi_q \times \tilde{\pi}_q,\psi_q) = 1$, we see that

$$\epsilon\big(s,\pi(\chi)_q \times \tilde{\pi}_q,\psi_q\big) = \epsilon(s,\chi,\psi_q)^{m^2}\epsilon(s,\pi_q \times \tilde{\pi}_q,\psi_q) . \tag{2.13}$$

Now suppose that $p \mid \mathfrak{f}(\pi)$. Then (with $P_p(p^{-s}) := L(s,\pi_p \times \tilde{\pi}_p)$)

$$\begin{aligned}L\big(s,\pi(\chi)_p \times \tilde{\pi}_p\big) &= P_p\big(\chi(p)p^{-s}\big)^{-1} \\ \epsilon\big(s,\pi(\chi)_p \times \tilde{\pi}_p,\psi_p\big) &= \chi(p^{c(\pi_p\times\tilde{\pi}_p)})\epsilon(s,\pi_p \times \tilde{\pi}_p,\psi_p)\end{aligned} \tag{2.14}$$

Indeed, $\chi_p(x) = |x|^{v_p}$ is unramified. We claim that $L(s,\pi(\chi)_p \times \tilde{\pi}_p) = L(s+v_p,\pi_p \times \tilde{\pi}_p)$ and similarly for the ϵ-factor. This can be seen from the local Rankin-Selberg integrals of [JP-SS]. With this given, we have

$$L\big(s,\pi(\chi)_p \times \tilde{\pi}_p\big) = P_p(p^{-s-v_v})^{-1} = P_p\big(\chi(p)p^{-s}\big)^{-1}$$

while

$$\begin{aligned}\epsilon\big(s,\pi(\chi)_p \times \tilde{\pi}_p,\psi_p\big) &= \tau(\pi_p \times \tilde{\pi}_p)p^{-c(\pi_p\times\tilde{\pi}_p)(s+v_p)} \\ &= \chi(p^{c(\pi_p\times\tilde{\pi}_p)})\tau(\pi_p \times \tilde{\pi}_p)p^{-c(\pi_p\times\tilde{\pi}_p)s} \\ &= \chi(p^{c(\pi_p\times\tilde{\pi}_p)})\epsilon(s,\pi_p \times \tilde{\pi}_p,\psi_p)\end{aligned}$$

Since $\chi_\infty = 1$, $\epsilon(s,\chi_\infty,\psi_\infty) = 1$ and so we find for the global ϵ-factor

$$\begin{aligned}\epsilon\big(s,\pi(\chi) \times \tilde{\pi}\big) &= \prod_p \epsilon\big(s,\pi(\chi)_p \times \tilde{\pi}_p,\psi_p\big) \\ &= \epsilon(s,\pi_\infty \times \tilde{\pi}_\infty)\epsilon(s,\chi,\psi_q)^{m^2}\epsilon(s,\pi_q \times \tilde{\pi}_q,\psi_q) \\ &\quad \cdot \prod_{p\mid\mathfrak{f}(\pi)} \chi(p^{c(\pi_p\times\tilde{\pi}_p)})\epsilon(s,\pi_p \times \tilde{\pi}_p,\psi_p) \\ &= \chi\big(\mathfrak{f}(\pi \times \tilde{\pi})\big)\tau(\chi)^{m^2} q^{-m^2 s}\epsilon(s,\pi \times \tilde{\pi})\end{aligned}$$

as required. □

3. The Proofs

For $\chi \neq \chi_0$ a primitive even Dirichlet character mod$q$, $q \not| \mathfrak{f}(\pi)$, let

$$L(s,\chi) := L\big(s,\pi(\chi)\times\tilde{\pi}\big) = \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^s} \ . \tag{3.1}$$

We write the functional equation for $L(s,\chi)$ as

$$L(s,\chi) = \epsilon\big(s,\pi(\chi)\times\tilde{\pi}\big)G(s)L(1-s,\bar{\chi}) \tag{3.2}$$

with $\epsilon(s,\pi(\chi)\times\tilde{\pi})$ is given by Lemma 2.1 and where we set

$$G(s) = \frac{L(1-s,\pi_\infty\times\tilde{\pi}_\infty)}{L(s,\pi_\infty\times\tilde{\pi}_\infty)} \ . \tag{3.3}$$

We investigate the averages

$$\sum_{q\sim Q}\sum_{\substack{\chi\neq\chi_0\\ \text{even}}} L(\beta,\chi) \tag{3.4}$$

where $\sum_{q\sim Q}$ means we sum over primes $Q \le q \le 2Q$.

PROPOSITION 3.1. *For* $0 < \operatorname{Re}\beta < 1$, *and* $\epsilon > 0$

$$\sum_{q\sim Q}\sum_{\substack{\chi\neq\chi_0\\ \text{even}}} L(\beta,\chi) = \tfrac{1}{2}\sum_{q\sim Q} q + O_{\beta,\epsilon}\big(Q^{1+\frac{m^2+1}{2}(1-\operatorname{Re}\beta)+\epsilon}\big) \ . \tag{3.5}$$

Proof of Theorem 1.2: As noted in the introduction, Theorem 1.2 follows from noting that if π_∞ is spherical and parametrized by $diag(\mu_\infty(1),\ldots,\mu_\infty(m))$ then for all even Dirichlet characters χ the Rankin-Selberg L-function $L(s,\pi(\chi)\times\tilde{\pi})$ has a trivial zero at $\beta_0 = 2\max\operatorname{Re}\mu_\infty(j)$. Note that since π_∞ is unitary, $\{\overline{\mu_\infty(j)}\} = \{-\mu_\infty(k)\}$ and so to prove Theorem 1.2 it suffices to show that $\beta_0 \le 1-2/(m^2+1)$. However if $\operatorname{Re}\beta > 1-2/(m^2+1)$ then in (3.5) the O-term is of smaller order than $\frac{1}{2}\sum_{q\sim Q} q \sim \frac{3}{4}Q^2/\log Q$ while the left-hand side is zero. This gives a contradiction and so proves Theorem 1.2.

AN APPROXIMATE FUNCTIONAL EQUATION. To prove Proposition 3.1, we need an appropriate series representation of $L(\beta,\chi)$. The following is such a representation which is gotten by a well known use of the functional equation (3.2). For $f \in C_c^\infty(0,\infty)$ with $\int_0^\infty f(x)dx = 1$, set

$$k(s) = \int_0^\infty f(y)y^s\frac{dy}{y} \ . \tag{3.6}$$

Thus $k(s)$ is entire, rapidly decreasing in vertical strips and $k(0) = 1$. For $x > 0$ set

$$
\begin{aligned}
F_1(x) &= \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} k(s)x^{-s}\frac{ds}{s} \\
F_2(x) &= \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} k(-s)G(-s+\beta)x^{-s}\frac{ds}{s} .
\end{aligned} \tag{3.7}
$$

Recall that $\beta_0 = 2\max \mathrm{Re}\,\mu_\infty(j)$ and we assume $0 < \mathrm{Re}\,\beta < 1$.

LEMMA 3.1. i) *$F_1(x)$ and $F_2(x)$ are rapidly decreasing as $x \to \infty$.*
ii) *$F_1(x) = 1 + O(x^N)$ for all $N \geq 1$ as $x \to 0$.*
iii) *$F_2(x) \ll 1 + x^{1-\beta_0-\mathrm{Re}\,\beta-\epsilon}$ as $x \to 0$.*

Proof: The asymptotics of $F_1(x)$ follow upon shifting the contour of integration to the right (for $x \to \infty$) and left (for $x \to 0$). As for $F_2(x)$, by Stirling's formula, $G(s)$ is of moderate growth in vertical strips and so we may shift contours. To get the behaviour as $x \to \infty$, shift the contour to the right. For the behaviour as $x \to 0$, shift to the left. If $\mathrm{Re}\,\beta + \beta_0 - 1 < 0$ then we pick up a simple pole at $s = 0$ which gives $F_2(x) = O(1)$; otherwise we pick up the first pole at $s = \beta + \beta_0 - 1$ and none to its right. In this case we get the bound

$$F_2(x) \ll x^{1-\beta-\mathrm{Re}\,\beta}(-\log x)^{d-1} , \qquad \text{as } x \to 0$$

where $d \leq m^2$ is the maximal order of a pole of $L(s, \pi_\infty \times \tilde{\pi}_\infty)$ on the line $\mathrm{Re}\, s = \beta_0$. □

In the rest of this section we set

$$\mathfrak{f} = \mathfrak{f}(\pi \times \tilde{\pi}) . \tag{3.8}$$

LEMMA 3.2 [Approximate Functional Equation]. *If $\chi \neq \chi_0$ is an even primitive Dirichlet character* mod$q$, *with $q \nmid \mathfrak{f}(\pi)$, and $0 < \mathrm{Re}\,\beta < 1$ then for any $Y > 1$,*

$$
\begin{aligned}
L(\beta, \chi) &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} F_1\left(\frac{n}{Y}\right) \\
&\quad + \tau(\pi \times \tilde{\pi})(q^{m^2}\mathfrak{f})^{-\beta} \sum_{n=1}^{\infty} \frac{b(n)\bar{\chi}(n)}{n^{1-\beta}} \chi(\mathfrak{f})\tau(\chi)^{m^2} F_2\left(\frac{nY}{\mathfrak{f}q^{m^2}}\right) .
\end{aligned} \tag{3.9}
$$

Proof: Consider the integral

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} k(s)L(s+\beta,\chi)Y^s\frac{ds}{s} &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta}\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=2} k(s)\left(\frac{Y}{n}\right)^s\frac{ds}{s} \\
&= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} F_1\left(\frac{n}{Y}\right) .
\end{aligned} \tag{3.10}
$$

Both the fact that this converges absolutely and the justification of the contour shifts follow from the comments at the end of section 2.1. On the other hand, shifting the contour to $\operatorname{Re} s = -1$, since $L(s,\chi)$ is entire for $\chi \neq \chi_0$,

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=2} k(s)L(s+\beta,\chi)Y^s\frac{ds}{s} = L(\beta,\chi)+\frac{1}{2\pi i}\int_{\operatorname{Re} s=-1} k(s)L(s+\beta,\chi)Y^s\frac{ds}{s}\,.$$

On applying the functional equation (3.2), this gives

$$L(\beta,\chi) + \frac{1}{2\pi i}\int_{\operatorname{Re} s=-1} k(s)\tau(\pi\times\tilde{\pi})\chi(\mathfrak{f})\tau(\chi)^{m^2}(\mathfrak{f}q^{m^2})^{-s-\beta}G(s+\beta)L(1-s-\beta,\bar{\chi})Y^s\frac{ds}{s}\,.$$

On changing variable $s \to -s$ this gives

$$\begin{aligned} &= L(\beta,\chi) \\ &\quad - \frac{1}{2\pi i}\int_{\operatorname{Re} s=1} k(-s)\tau(\pi\times\tilde{\pi})\chi(\mathfrak{f})\tau(\chi)^{m^2}(\mathfrak{f}q^{m^2})^{s-\beta} \\ &\quad \cdot G(-s+\beta)L(s+1-\beta,\bar{\chi})Y^{-s}\frac{ds}{s} \\ &= L(\beta,\chi) - \tau(\pi\times\tilde{\pi})\chi(\mathfrak{f})\tau(\chi)^{m^2}(\mathfrak{f}q^{m^2})^{-\beta}\sum_{n=1}^{\infty}\frac{b(n)\bar{\chi}(n)}{n^{1-\beta}}F_2\left(\frac{nY}{\mathfrak{f}q^{m^2}}\right)\,. \end{aligned}$$

Comparing with (3.10) we recover (3.9). □

Proof of Proposition 3.1: We study the average (3.4) by using the approximate functional equation (3.9) with $Q \ll Y \ll Q^{m^2}$. On using

$$\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}{}' \chi(n) = \begin{cases} 0\,, & n\equiv 0 \bmod q \\ \frac{q-1}{2}-1\,, & n\equiv \pm 1 \bmod q \\ -1\,, & \text{otherwise}\end{cases} \tag{3.11}$$

we find that the contribution of the first sum on the RHS of (3.9) to the average is

$$\begin{aligned}\sum_{q\sim Q}\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}\sum_{n}\frac{b(n)\chi(n)}{n^\beta}F_1\left(\frac{n}{Y}\right) &= \sum_{q\sim Q}\frac{q-1}{2}\sum_{n\equiv\pm1 \bmod q}\frac{b(n)}{n^\beta}F_1\left(\frac{n}{Y}\right) \\ &\quad - \sum_{q\sim Q}\sum_{\substack{(n,q)=1\\ n\not\equiv\pm1 \bmod q}}\frac{b(n)}{n^\beta}F_1\left(\frac{n}{Y}\right)\,. \end{aligned}\tag{3.12}$$

We single out the contribution from $n=1$ in the first term above:

$$\sum_{q\sim Q}\frac{q-1}{2}F_1\left(\frac{1}{Y}\right) = \frac{1}{2}\sum_{q\sim Q} q + O(Q) + O(Q^2Y^{-N})\,. \tag{3.13}$$

We will choose $Y \sim Q^{\frac{m^2+1}{2}}$ and so we use $F_1(x) \to 1$ as $x \to 0$. Note that $\sum_{q\sim Q} q \sim \frac{3}{2}Q^2/\log Q$.

The sum over $n \equiv 1 \bmod q$, $n \neq 1$ contributes

$$\sum_{q\sim Q} \frac{q-1}{2} \sum_{d\geq 1} \frac{b(1+dq)}{(1+dq)^\beta} F_1\left(\frac{1+dq}{Y}\right) \ll Q \sum_m \frac{b(m)m^\epsilon}{m^{\mathrm{Re}\,\beta}} \left|F_1\left(\frac{m}{Y}\right)\right| \quad (3.14)$$

where we use the fact that for $n \neq 1$, the number of different representations $n = 1+dq = 1+d'q'$ is $O(n^\epsilon)$. Now apply (2.8) and $F_1(x) \sim 1$ as $x \to 0$ to find that

$$\sum_{q\sim Q} \frac{q-1}{2} \sum_{\substack{n\equiv 1 \bmod q\\ n\neq 1}} \frac{b(n)}{n^\beta} F_1\left(\frac{n}{Y}\right) \ll QY^{1-\mathrm{Re}\,\beta+\epsilon} \quad (3.15)$$

(recall that $\mathrm{Re}\,\beta < 1$). Similarly we find that

$$\sum_{q\sim Q} \frac{q-1}{2} \sum_{n\equiv -1 \bmod q} \frac{b(n)}{n^\beta} F_1\left(\frac{n}{Y}\right) \ll QY^{1-\mathrm{Re}\,\beta+\epsilon}\,. \quad (3.16)$$

The last sum in (3.12) is bounded by

$$\sum_{q\sim Q} \sum_{(n,q)=1} \frac{b(n)}{n^{\mathrm{Re}\,\beta}} \left|F_1\left(\frac{n}{Y}\right)\right| \ll QY^{1-\mathrm{Re}\,\beta+\epsilon}\,. \quad (3.17)$$

To treat the contribution of the second term in (3.9) we first note that if $q \nmid n$ then

$$\sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \bar{\chi}(n)\chi(\mathfrak{f})\tau(\chi)^{m^2} \ll q^{\frac{m^2+1}{2}}\,. \quad (3.18)$$

Indeed, setting $r \equiv n\bar{\mathfrak{f}} \bmod q$ (with $\mathfrak{f}\bar{\mathfrak{f}} \equiv 1 \bmod q$) we have

$$\sum_{\substack{\chi\neq\chi_0\\ \text{even}}} \bar{\chi}(r)\tau(\chi)^{m^2} = \frac{q-1}{2}\{\mathrm{Kl}_{m^2}(r,q) + \mathrm{Kl}_{m^2}(-r,q)\} - (-1)^{m^2} \quad (3.19)$$

where for $r \not\equiv 0 \bmod q$ the hyper-Kloosterman sum $\mathrm{Kl}_n(r,q)$ is defined by

$$\mathrm{Kl}_n(r,q) = \sum_{x_1\cdot\ldots\cdot x_n \equiv r \bmod q} e\left(\frac{x_1+\ldots+x_n}{q}\right)\,. \quad (3.20)$$

Using Deligne's bound $\mathrm{Kl}_n(r,q) \ll q^{(n-1)/2}$ ([D]), we get (3.18).

Now sum over q to find

$$\sum_{q\sim Q}(\mathfrak{f}q^{m^2})^{-\beta}\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}\sum_{n}\frac{b(n)\bar{\chi}(n)}{n^{1-\beta}}\chi(\mathfrak{f})\tau(\chi)^{m^2}F_2\left(\frac{nY}{\mathfrak{f}q^{m^2}}\right)$$
$$=\sum_{q\sim Q}(\mathfrak{f}q^{m^2})^{-\beta}\sum_{(n,q)=1}\frac{b(n)}{n^{1-\beta}}\left[\frac{q-1}{2}\operatorname{Kl}_{m^2}(n\bar{\mathfrak{f}},q)\right.$$
$$\left.+\frac{q-1}{2}\operatorname{Kl}_{m^2}(-n\bar{\mathfrak{f}},q)-(-1)^{m^2}\right]F_2\left(\frac{nY}{\mathfrak{f}q^{m^2}}\right)$$
$$\ll\sum_{q\sim Q}(\mathfrak{f}q^{m^2})^{-\operatorname{Re}\beta}\sum_{(n,q)=1}\frac{b(n)}{n^{1-\operatorname{Re}\beta}}q^{\frac{m^2+1}{2}}\left|F_2\left(\frac{nY}{\mathfrak{f}q^{m^2}}\right)\right|$$
$$\ll\sum_{q\sim Q}(\mathfrak{f}q^{m^2})^{-\operatorname{Re}\beta}q^{\frac{m^2+1}{2}}\int_1^\infty\left|F_2\left(\frac{xY}{\mathfrak{f}q^{m^2}}\right)\right|\frac{dx}{x^{1-\operatorname{Re}\beta}}\ll Q^{1+\frac{m^2+1}{2}}Y^{-\operatorname{Re}\beta}$$

on using the bound for $F_2(x)$ in Lemma 3.1 and $\operatorname{Re}\beta>0$, $\beta_0<1$. Thus

$$\sum_{q\sim Q}(\mathfrak{f}q^{m^2})^{-\beta}\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}\sum_{n}\frac{b(n)\bar{\chi}(n)}{n^{1-\beta}}\chi(\mathfrak{f})\tau(\chi)^{m^2}F_2\left(\frac{nY}{\mathfrak{f}q^{m^2}}\right)\ll Q^{1+\frac{m^2+1}{2}}Y^{-\operatorname{Re}\beta}. \tag{3.21}$$

Collecting together (3.13), (3.15), (3.16), (3.17) and (3.21) we find

$$\sum_{q\sim Q}\sum_{\substack{\chi\neq\chi_0\\ \text{even}}}L(\beta,\chi)=\sum_{q\sim Q}\frac{q-1}{2}+O(QY^{1-\operatorname{Re}\beta+\epsilon}+Q^{1+\frac{m^2+1}{2}}Y^{-\operatorname{Re}\beta})\,. \tag{3.22}$$

On taking $Y\sim Q^{(m^2+1)/2}$ we prove Proposition 3.1.

APPLICATIONS. We sketch how Corollaries 1.1 and 1.2 follow from Theorem 1.1. For Corollary 1.1, we use the result of Goldfeld and Sarnak ([GolS]) which asserts that if $\lambda_j=s_j(1-s_j)<\frac{1}{4}$, are the exceptional eigenvalues for $\Gamma_0(N)\backslash\mathbf{H}^2$, then

$$\sum_{\substack{1\le c\le x\\ c\equiv 0 \bmod N}}\frac{S(m,n,c)}{c}=\sum_{s_j}\tau_j(m,n)x^{2s_j-1}+O_\epsilon(x^{\frac{1}{6}+\epsilon})\,. \tag{3.23}$$

Since Theorem 1.1 gives $\frac{1}{2}\le s_j<\frac{7}{10}$, we recover Corollary 1.1.

Corollary 1.2 was established in the recent work of Luo and Sarnak ([LuS]) for the full modular group $\Gamma=SL_2(\mathbf{Z})$. It was pointed out there that the only obstruction to establishing Corollary 1.2 for any congruence subgroup is the presence of small eigenvalues $\lambda_j=s_j(1-s_j)$ with $s_j>\frac{7}{10}$. Theorem 1.1 asserts precisely that these do not exist.

References

[BR] L. BARTHEL, D. RAMAKRISHNAN, A nonvanishing result for twists of L-functions of $GL(n)$, Duke Math. Jour. 74 (1994), 681–700.

[BDHI] D. BUMP, W. DUKE, J. HOFFSTEIN, H. IWANIEC, An estimate for the Hecke eigenvalues of Maass forms, Inter. Math. Res. Notices 4 (1992), 75–81.

[BuGi] D. BUMP, D. GINZBURG, Symmetric square L-functions on $GL(r)$, Ann. of Math. 136 (1992), 137–205.

[D] P. DELIGNE, SGA $4\frac{1}{2}$ Cohomologie Etale, Lecture Notes in Math. 569, Springer-Verlag 1977.

[DuI] W. DUKE, H. IWANIEC, Estimates for coefficients of L-functions, Proc. C.R.M. Conf. on Automorphic Forms and Analytic Number Theory, Montréal (1989), 43–48.

[GJ] S. GELBART, H. JACQUET, A relation between automorphic representations of $GL(2)$ and $GL(3)$, Ann. Sci. Ecole Norm. Sup. 4^e série 11 (1978), 471–552.

[GoJ] R. GODEMENT, H. JACQUET, Zeta Functions of Simple Algebras, Lecture Notes in Math. 260, Springer-Verlag, 1972.

[GolS] D. GOLDFELD, P. SARNAK, Sums of Kloosterman sums, Invent. Math. 71 (1983), 243–250.

[I] H. IWANIEC, Small eigenvalues of the Laplacian on $\Gamma_0(N)$, Acta Arith. 56 (1990), 65–82.

[J] H. JACQUET, Principal L-functions of the linear group, Proc. Symp. Pure Math. 33:2, 63–86, American Math. Soc., Providence 1979.

[JP-SS] H. JACQUET, I.I. PIATETSKI-SHAPIRO, J.A. SHALIKA, Rankin-Selberg convolutions, Amer. Jour. of Math 105 (1983), 367–464.

[JSha1] H. JACQUET, J.A. SHALIKA, On Euler products and the classification of automorphic representations I, Amer. Jour. of Math. 103 (1981), 499–558.

[JSha2] H. JACQUET, J.A. SHALIKA, Rankin-Selberg convolutions: Archimedean theory, Israel Math. Conf. Proceedings 2 (1990), 125–208.

[L] YU.V. LINNIK, Additive problems and eigenvalues of modular operators, Proc. I.C.M. Stockholm (1962), 270–284.

[LuS] W. LUO, P. SARNAK, Quantum Ergodicity of Eigenfunctions on $PSL_2(\mathbf{Z})\backslash\mathbf{H}^2$, preprint (1994).

[MW] C. MŒGLIN, J.-L. WALDSPURGER, Le spectre résiduel de $GL(n)$, Ann. Sci. Ecole Norm. Sup. 4^e série 22 (1989), 605–674.

[PP-S] S.J. PATTERSON, I.I. PIATETSKI-SHAPIRO, The symmetric square L-function attached to a cuspidal automorphic representation of $GL(3)$, Math. Ann. 283 (1989), 551–572.

[R] D. ROHRLICH, Nonvanishing of L-functions for $GL(2)$, Invent. Math. 97 (1989), 383-401.

[RuS] Z. RUDNICK, P. SARNAK, Zeros of Principal L-functions and Random Matrix Theory, preprint.

[S] P. SARNAK, Prime Geodesic Theorems, Ph.D. Thesis, Stanford University (1980).

[Se] A. SELBERG, On the estimation of Fourier coefficients of modular forms, Proc. Symp. in Pure Math. 8, Amer. Math. Soc., Providence (1965), 1–15.

[Ser] J.-P. SERRE, letter to J.-M. Deshouillers (1981).

[Sh1] F. SHAHIDI, On certain L-functions, Amer. Jour. of Math. 103 (1981), 297–355.

[Sh2] F. SHAHIDI, On the Ramanujan conjectures and finiteness of poles for certain L-functions, Ann. of Math. 127 (1988), 547–584.

[Shi] G. SHIMURA, On the periods of modular forms, Math. Ann. 229 (1977), 211–221.

[We] A. WEIL, On some exponential sums, Proc. Nat. Acad. Sci. 34 (1948), 204–207.

Wenzhi Luo
School of Mathematics
Institute for Advanced Studies
Princeton, NJ 08540
USA
Current address:
MSRI
Berkeley, CA 94720
USA

Zeév Rudnick
Dept. of Math.
Princeton University
Princeton, NJ 08544
USA
Current address:
School of Math. Sci.
Tel Aviv University
Tel Aviv 69978
Israel

Peter Sarnak
Dept. of Math.
Princeton University
Princeton, NJ 08544
USA

Submitted: September 1994