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A METRIC THEORY OF MINIMAL GAPS

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Abstract. We study the minimal gap statistic for fractional parts of sequences of the form $\mathcal{A}^\alpha = \{\alpha a(n)\}$, where $\mathcal{A} = \{a(n)\}$ is a sequence of distinct integers. Assuming that the additive energy of the sequence is close to its minimal possible value, we show that for almost all α, the minimal gap $\delta_{\min}^\alpha(N) = \min\{\alpha a(m) - \alpha a(n) \bmod 1 : 1 \leqslant m \neq n \leqslant N\}$ is close to that of a random sequence.

We start with a sequence of points $\mathcal{X} = \{x_n : n = 1, 2, \ldots\} \subset \mathbb{R}/\mathbb{Z}$ in the unit interval/circle, which we assume is asymptotically uniformly distributed: for any subinterval $\subset \mathbb{R}/\mathbb{Z}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leqslant N : x_n \in I\} = |I|. \tag{1}$$

In particular, the mean spacing between the points lying in any subinterval is $1/N$. Our goal is to understand the *minimal gap*

$$\delta_{\min}(\mathcal{X}, N) = \min(|x_n - x_m| : n, m \leqslant N, n \neq m)$$

(with a suitable modification for wrapping around).

For *random* points, namely N independent uniform points in the unit interval (Poisson process), the minimal gap is almost surely of size $1/N^2$ [**9**]. In this note we study the metric theory of the minimal gap statistic for a class of deterministic sequences of fractional parts, such as fractional parts of polynomials. The case of quadratic polynomials $x_n = \alpha n^2$ has its roots in the recent paper [**3**], which studies the more complicated case of the minimal gap statistic for the sequence of eigenvalues of the Laplacian on a rectangular billiard, namely the points $\{\alpha m^2 + n^2 : m, n \geqslant 1\}$ on the real line.

We fix a sequence $\mathcal{A} = \{a(n) : n = 1, 2, \ldots\} \subset \mathbb{Z}$ of distinct integers ($a(n) \neq a(m)$ if $m \neq m$), and study the minimal gap statistic of fractional parts of the set

$$\mathcal{A}^\alpha = \{\alpha a(n) \bmod 1 : n = 1, 2, \ldots\} \subset \mathbb{R}/\mathbb{Z}.$$

(it is an old result of Weyl that $\mathcal{A}^\alpha$ satisfies (1) for almost all α). We want to know under which conditions we can show that for *almost all* α, the minimal gap statistics

$$\delta_{\min}^\alpha(N) = \delta_{\min}(\mathcal{A}^\alpha, N)$$

follows that of the random case, that is of size about $1/N^2$ for almost all α. It is easy to see that we cannot have much smaller minimal gaps.

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THEOREM 1. *Assume that $\mathcal{A}$ consists of distinct integers. Then for all $\eta > 0$, for almost all α,*

$$\delta^{\alpha}_{\min}(N) > \frac{1}{N^{2+\eta}} \quad \textit{for all } N > N_0(\alpha).$$

To make the minimal gap small, we give a criterion in terms of the "additive energy" $E(\mathcal{A}, N)$ of the sequence

$$E(\mathcal{A}, N) := \#\{(n_1, n_2, n_3, n_4) \in [1, N]^4 : a(n_1) + a(n_2) = a(n_3) + a(n_4)\}.$$

Note that $N^2 \leqslant E(\mathcal{A}, N) \leqslant N^3$. The result is the following.

THEOREM 2. *Assume that $\mathcal{A}$ consists of distinct integers, and that the additive energy satisfies*

$$E(\mathcal{A}, N) \ll N^{2+o(1)} \quad \textit{for all } N \gg 1.$$

Then for all $\eta > 0$, for almost all α,

$$\delta^{\alpha}_{\min}(N) < \frac{1}{N^{2-\eta}} \quad \textit{for all } N > N_0(\alpha).$$

Examples: for $a(n) = n^d$, $d \geqslant 2$, it is shown in [**11**] that the $E(\mathcal{A}, N) \ll N^{2+o(1)}$. For lacunary sequences, we have $E(\mathcal{A}, N) \ll N^2$ [**12**]. Hence, Theorem 2 applies to these sequences.

Relaxing the required bound on the additive energy will give a weaker result on the minimal spacing, basically that $\delta^{\alpha}_{\min}(N) < E(\mathcal{A}, N)/N^{4-\eta}$ almost surely. For this to be non-trivial, we need the additive energy to be no bigger than $E(\mathcal{A}, N) \ll N^{3-\eta'}$ for some $\eta' > 0$. A notable case where the additive energy is bigger is that of $\mathcal{A} = \mathcal{P}$ being the sequence of primes, where $E(\mathcal{P}, N) \approx N^3/\log N$. In this case, we cannot have gaps much larger than the average gap: a simple argument shows that given any $\varepsilon > 0$, for almost all α, we have $\delta_{\min}(\mathcal{P}^{\alpha}, N) \gg 1/(N(\log N)^{2+\varepsilon})$; see §3.

§1. *A bilinear statistic.* To study the minimal gap, we introduce statistics counting all possible gaps: we start with a smooth, compactly supported window function $f \in C_c^{\infty}([-1/2, 1/2])$, which is non-negative: $f \geqslant 0$, and of unit mass $\int f(x)\,dx = 1$, and define

$$F_M(x) = \sum_{j \in \mathbb{Z}} f(M(x + j)),$$

which is localized on the scale of $1/M$, and periodic: $F_M(x + 1) = F_M(x)$. We then set

$$D_{\mathcal{A}}(N, M)(\alpha) = \sum_{\substack{1 \leqslant m,n \leqslant N \\ m \neq n}} F_M(\alpha a(n) - \alpha a(m)).$$

The expected value of $D_{\mathcal{A}}(N, M)$ is easily seen to equal

$$\int_0^1 D_{\mathcal{A}}(N, M)(\alpha)\, d\alpha = \frac{N(N-1)}{M} \sim \frac{N^2}{M}. \tag{2}$$

This already suffices to show that minimal gaps cannot typically be small (Theorem 1); see §2.

We will bound the variance of $D_{\mathcal{A}}(N, M)$, from which Theorem 2 will follow.

PROPOSITION 3.

$$\mathrm{Var}\, D_{\mathcal{A}}(N, M) \ll \frac{1}{M} N^{\epsilon} E(\mathcal{A}, N).$$

The statistic $D_{\mathcal{A}}(N, M)(\alpha)$ is related to the pair correlation function of the sequence $\mathcal{A}^{\alpha}$, which in our notation is $D_{\mathcal{A}}(N, N)(\alpha)/N$. Pair correlation measures gaps on the scale of the mean spacing, assumed here to be $1/N$, corresponding to $M = N$, here we are looking at much smaller scales of M close to N^2.

The metric theory of the pair correlation function of fractional parts was initiated in [**11**], where the sequences $a(n) = \alpha n^d$ were shown to almost surely have Poissonian pair correlation for $d \geqslant 2$ (see [**7**, **10**] for different proofs of the quadratic case $d = 2$). The problem has since been studied in several other cases and has recently been revived in an abstract setting [**1**, **4**, **8**, **14**]. In particular, a convenient criterion for almost sure Poisson pair correlation has been formulated by Aistleitner, Larcher and Lewko [**1**] in terms of the additive energy $E(\mathcal{A}, N)$ of the sequence. The proof of Proposition 3 is close to that of the analogous statement for the pair correlation function in [**1**], which in turn is based on [**11**, **12**].

Proof. The Fourier expansion of $F_M(x)$ is

$$F_M(x) = \sum_{k \in \mathbb{Z}} \frac{1}{M} \widehat{f}\left(\frac{k}{M}\right) e(kx), \tag{3}$$

where $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi \mathrm{i} x y}\, dx$. Inserting into the definition of $D(N, M)$ gives

$$D(N, M)(\alpha) = \sum_{k \in \mathbb{Z}} \frac{1}{M} \widehat{f}\left(\frac{k}{M}\right) \sum_{\substack{1 \leqslant m,n \leqslant N \\ m \neq n}} e(k\alpha(a(m) - a(n))). \tag{4}$$

Integrating over α gives the expected value (we assume $a(m) \neq a(n)$ if $n \neq m$)

$$\int_0^1 D(N, M)(\alpha)\, d\alpha = \widehat{f}(0) \frac{1}{M} N(N-1) = \frac{N(N-1)}{M}.$$

The variance is the second moment of the sum over non-zero frequencies:

$$\operatorname{Var} D(N, M) = \int_0^1 \left| \sum_{0 \neq k \in Z} \frac{1}{M} \widehat{f}\left(\frac{k}{M}\right) \sum_{\substack{1 \leqslant m,n \leqslant N \\ m \neq n}} e(k\alpha(a(m) - a(n))) \right|^2 d\alpha.$$

Squaring out and integrating gives

$$\operatorname{Var} D(N, M) = \sum_{k_1, k_2 \neq 0} \frac{1}{M^2} \widehat{f}\left(\frac{k_1}{M}\right) \widehat{f}\left(\frac{k_2}{M}\right) \times \#\{m_1 \neq n_1, m_2 \neq n_2 : k_1(a(m_1) - a(n_1)) = k_2(a(m_2) - a(n_2))\}.$$

We now follow [1, Lemma 3] to convert this to "greatest common divisor (GCD) sums". Let

$$R(v) = \#\{1 \leqslant m \neq n \leqslant N : a(m) - a(n) = v\}.$$

Then

$$\operatorname{Var} D(N, M) = \sum_{v_1, v_2 \neq 0} \sum_{k_1, k_2 \neq 0} c(k_1) c(k_2) R(v_1) R(v_2) \delta(k_1 v_1 = k_2 v_2),$$

where we set

$$c(k) := \frac{1}{M} \widehat{f}\left(\frac{k}{M}\right).$$

LEMMA 4. *Let $f \in C_c^\infty(\mathbb{R})$. For any non-zero integers $v_1, v_2 \neq 0$,*

$$\sum_{k_1, k_2 \neq 0} c(k_1) c(k_2) \delta(k_1 v_1 = k_2 v_2) \ll_f \frac{1}{M} \frac{\gcd(v_1, v_2)}{\sqrt{|v_1 v_2|}}. \tag{5}$$

Proof. For $k_1, k_2 \neq 0$, we have $k_1 v_1 = k_2 v_2$ if and only if

$$(k_1, k_2) = \ell \left(\frac{v_2}{\gcd(v_1, v_2)}, \frac{v_1}{\gcd(v_1, v_2)} \right)$$

for some non-zero integer $0 \neq \ell \in \mathbb{Z}$. Abbreviating $a_i = v_i / (M \gcd(v_1, v_2))$, we find

$$\sum_{k_1, k_2 \neq 0} c(k_1) c(k_2) \delta(k_1 v_1 = k_2 v_2) = \sum_{0 \neq \ell \in \mathbb{Z}} c\left(\ell \frac{v_2}{\gcd(v_1, v_2)}\right) c\left(\ell \frac{v_1}{\gcd(v_1, v_2)}\right) = \frac{1}{M^2} \sum_{0 \neq \ell \in \mathbb{Z}} \widehat{f}(a_2 \ell) \widehat{f}(a_1 \ell)$$

so that it suffices to show that

$$\sum_{0 \neq \ell \in \mathbb{Z}} \widehat{f}(a_1 \ell) \widehat{f}(a_2 \ell) \ll_f \frac{1}{\sqrt{|a_1 a_2|}} = \frac{M \gcd(v_1, v_2)}{\sqrt{|v_1 v_2|}}. \tag{6}$$

Applying Cauchy–Schwarz we get

$$\sum_{0\neq\ell\in\mathbb{Z}} \widehat{f}(\ell a_1)\widehat{f}(\ell a_2) \leqslant \Big(\sum_{\ell\neq 0}\widehat{f}(a_1\ell)^2\Big)^{1/2}\Big(\sum_{\ell\neq 0}\widehat{f}(a_2\ell)^2\Big)^{1/2}.$$

We will obtain (6) if we show that for any $a > 0$,

$$\sum_{\ell\neq 0}\widehat{f}(a\ell)^2 \ll_f \frac{1}{a}.$$

Indeed, if $0 < a \ll 1$ then we get a Riemann sum for $(\widehat{f}\,)^2$:

$$\sum_{\ell\neq 0}\widehat{f}(a\ell)^2 \sim \frac{1}{a}\int_{-\infty}^{\infty}\widehat{f}(y)^2\,dy = \frac{1}{a}\int_{-\infty}^{\infty} f(x)^2\,dx.$$

If $a \gg 1$ then use the decay rate of the Fourier transform: for $y \neq 0$,

$$|\widehat{f}(y)| \leqslant \frac{1}{2\pi|y|}\int_{-\infty}^{\infty}|f'(x)|\,dx,$$

to obtain

$$\sum_{\ell\neq 0}\widehat{f}(a\ell)^2 \ll \sum_{\ell\neq 0}\left(\frac{\int_{-\infty}^{\infty}|f'(x)|\,dx}{|a\ell|}\right)^2 \ll_f \frac{1}{a^2},$$

which for $a \gg 1$ is $\ll 1/a$.

Hence,

$$\operatorname{Var} D(N, M) \ll \frac{1}{M}\sum_{v_1,v_2\neq 0} R(v_1)R(v_2)\frac{\gcd(v_1, v_2)}{\sqrt{|v_1 v_2|}}.$$

According to the GCD bounds of [6],

$$\sum_{v_1,v_2\neq 0} R(v_1)R(v_2)\frac{\gcd(v_1, v_2)}{\sqrt{|v_1 v_2|}} \ll \exp\left(\frac{10\log N}{\log\log N}\right)\sum_{v} R(v)^2$$

(see [5] for an essentially optimal refinement). Now

$$\sum R(v)^2 = \#\{m_i, n_j \leqslant N, m_1 \neq n_1, m_2 \neq m_2 : a(m_1)-a(n_1) = a(m_2)-a(n_2)\}$$

is at most the additive energy $E(\mathcal{A}, N)$. Thus,

$$\operatorname{Var} D(N, M) \ll \frac{1}{M}N^{\epsilon}E(\mathcal{A}, N)$$

as claimed. □

COROLLARY 5. *Assume that the additive energy satisfies* $E(\mathcal{A}, N) < N^{2+o(1)}$. *If* $M < N^{2-\eta}$, *then for almost all* α,

$$D_{\mathcal{A}}(N, M)(\alpha) \sim \frac{N^2}{M}.$$

Proof. Take $N_k = \lfloor k^{4/\eta} \rfloor$, so that $\sum_k N_k^{-\eta/2} < \infty$, and pick any $M_k < N_k^{2-\eta}$, then we find from Proposition 3,

$$\sum_k \int_0^1 \left| \frac{D(N_k, M_k)(\alpha)}{N_k(N_k-1)/M_k} - 1 \right|^2 d\alpha = \sum_k \frac{\operatorname{Var} D(N_k, M_k)}{(N_k(N_k-1)/M_k)^2}$$
$$< \sum_k N_k^{o(1)} \frac{E(N_k)M_k}{N_k^4} \ll \sum_k \frac{1}{N_k^{\eta/2}} < \infty,$$

and so for almost all α,

$$D(N_k, M_k)(\alpha) \sim \frac{N_k^2}{M_k} \quad \text{for all } k > k_0(\alpha). \tag{7}$$

A priori the set depends on the test function f, but that can be taken care of by a standard diagonalization procedure; for our purposes we only need one test function.

Given $N \gg 1$, there is a unique value of k so that $N_k \leqslant N < N_{k+1}$. Note that $N/N_k = 1 + O(N^{-\eta/4})$. Since $M < N^{2-\eta} \sim N_k^{2-\eta} < N_{k+1}^{2-\eta}$, we know from (7) that almost surely $D(N_k, M)/(N_k^2/M) \to 1$.

Note that

$$D(N_k, M) \leqslant D(N, M) \leqslant D(N_{k+1}, M).$$

This is because the sums $D(N, M)$ consist of non-negative terms, and hence,

$$D(N, M) = \sum_{1 \leqslant m \neq n \leqslant N} F_M(\alpha(a(m) - a(n)))$$
$$\geqslant \sum_{1 \leqslant m \neq n \leqslant N_k} F_M(\alpha(a(m) - a(n))) = D(N_k, M)$$

(we dropped all pairs (m, n) where $\max(m, n) > N_k$).

Since $N/N_k = 1 + O(N^{-\eta/4})$, we have

$$\frac{D(N_k, M)}{N_k^2/M} \leqslant \frac{D(N, M)}{N^2/M(1 + O(N^{-\eta/4}))} = \frac{D(N, M)}{N^2/M}(1 + O(N^{-\eta/4}))$$

and likewise

$$\frac{D(N, M)}{N^2/M} \leqslant \frac{D(N_{k+1}, M)}{N_{k+1}^2/M}(1 + O(N^{-\eta/4})).$$

Since we know that almost surely $D(N_k, M)/(N_k^2/M) \to 1$, we deduce that almost surely also $D(N, M)/(N^2/M) \to 1$. □

COROLLARY 6. *Theorem 2 holds.*

Proof. Fix $\eta > 0$, and let $M = 1/2N^{2-\eta}$. Since $D(N, M)(\alpha) \sim N^2/M > 1$ by Corollary 5, we have a gap of size at most $1/(2M) = 1/N^{2-\eta}$, that is $\delta^\alpha_{\min}(N) < 1/N^{2-\eta}$ almost surely. □

§2. *Lower bounds: proof of Theorem 1.* We take any sequence of integers $\mathcal{A} = \{a(n)\}$ with distinct elements. We want to show that for any $\eta > 0$, almost surely,

$$\delta^\alpha(N) > 1/N^{2+\eta} \quad \text{for all } N > N_0(\alpha).$$

Let $N_k = \lfloor k \rfloor^{2/\eta}$. We claim that it suffices to show that, for almost all α,

$$\delta^\alpha_{\min}(N_k) > 2/N_k^{2+\eta} \quad \text{for all } k > k_0(\alpha). \tag{8}$$

Indeed, note that if $N_k \leqslant N < N_{k+1}$ then $\delta^\alpha_{\min}(N) \geqslant \delta^\alpha_{\min}(N_{k+1})$. Since $N_{k+1} \sim N$, by (8) we have, for almost all α

$$\delta^\alpha_{\min}(N) \geqslant \delta^\alpha_{\min}(N_{k+1}) > 2/N_{k+1}^{2+\eta} > 1/N^{2+\eta}$$

for $N > N_0(\alpha)$.

To prove (8), it suffices, by the Borel–Cantelli lemma, to show that

$$\sum_k \mathrm{Prob}(\delta^\alpha_{\min}(N_k) \leqslant 2/N_k^{2+\eta}) < \infty. \tag{9}$$

In the definition of $D(N, M) = D_f(N, M)$, choose f so that $f(x) \geqslant 1$ if $|x| \leqslant 1/4$ (and in addition, $f \geqslant 0$ is non-negative, $\int_{-\infty}^{\infty} f(x)\,dx = 1$, f is smooth and supported in $[-1/2, 1/2]$). Now note that for such f, if $D_f(N, M) < 1$ then $\delta^\alpha_{\min}(N) > 1/(4M)$. This is because $D_f(N, M)$ is a sum of non-negative terms, and if there is one gap of size $\leqslant 1/(4M)$ then the corresponding term $F_M(\alpha(a(m) - a(n))) = \sum_j f(M(\alpha(a(m) - a(n)) + j)) \geqslant 1$ by the choice of f, so that $D_f(N, M) \geqslant 1$. Thus, we find that

$$\delta^\alpha_{\min}(N) \leqslant \frac{1}{4M} \quad \Rightarrow \quad D_f(N, M) \geqslant 1,$$

and hence,

$$\mathrm{Prob}\left(\delta^\alpha_{\min}(N) \leqslant \frac{1}{4M}\right) \leqslant \mathrm{Prob}(D_f(N, M) \geqslant 1). \tag{10}$$

Now since $D_f \geqslant 0$,

$$\mathrm{Prob}(D_f(N, M) \geqslant 1) \leqslant \int_0^1 D_f(N, M)(\alpha)\,d\alpha$$

so that by (2), for $M_k = \frac{1}{8} N_k^{2+\eta}$,

$$\int_0^1 D(N_k, M_k)(\alpha)\, d\alpha \sim 8N_k^{-\eta} \ll \frac{1}{k^2},$$

which together with (10) proves (9), and hence, (8). This proves Theorem 1.

§3. *Minimal gaps for the primes.* Let $a(n) = p_n$, the nth prime. By Khinchin's theorem, for all $\varepsilon > 0$ there is a set of full measure of the α so that $\|q\alpha\| \geqslant 1/(q(\log q)^{1+\varepsilon})$ for any integer $q \geqslant q_0(\alpha)$. In particular, for such α, the gap between fractional parts of αp_n mod 1 are

$$\|\alpha(p_m - p_n)\| \gg \frac{1}{|p_m - p_n|(\log|p_m - p_n|)^{1+\varepsilon}} \geqslant \frac{1}{N(\log N)^{2+\varepsilon}},$$

since $|p_m - p_n| \leqslant p_N \sim N \log N$ for $m < n \leqslant N$. Hence, for such α, the minimal gap satisfies $\delta^{\alpha}_{\min}(N) > 1/N(\log N)^{2+\varepsilon}$.

A similar argument applies to other dense cases, such as the sequence of square-free integers. An extreme case is that when $\mathcal{A} = \mathbb{N}$ is the sequence of all natural numbers. The argument above gives the minimal gap here is, for almost all α, at least $\delta^{\alpha}_{\min}(N) \gg 1/(N(\log N)^{1+\varepsilon})$. Note that in this case the "three-gap" theorem shows that there are at most three distinct gaps between the fractional parts $\{\alpha n \bmod 1 : n \leqslant N\}$. Concerning other "dense" sequences, it is known that for any sequence of integers $\mathcal{A} \subset [1, M]$, the fractional parts $\alpha a(m)$ mod 1 have at most $O(\sqrt{M})$ distinct gaps [**2**, **13**].

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