

A Central Limit Theorem for the Spectrum of the Modular Domain*

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The statistics of the high-lying eigenvalues of the Laplacian on a Riemannian manifold have been intensively studied in the past few years by physicists working on “quantum chaos”. A number of fundamental insights have emerged from these studies, though to date these have yet to be set on rigorous footing. In the case the manifold at hand is of arithmetic origin, these studies are related to some profound number theoretical problems and as such may be more amenable to investigation. In this note I make use of the arithmetic structure of the modular domain to establish Gaussian fluctuations in its spectrum for certain smooth counting functions.

1 Background

To set the stage, I start with describing some of what is currently believed to hold for the statistics of the eigenvalues. Given a compact Riemannian surface M , Weyl’s law for the eigenvalues E_j of the Laplacian says that the number of eigenvalues below x grows linearly with x :

$$\#\{E_j \leq x\} \sim \frac{\text{area}(M)}{4\pi}x, \quad \text{as } x \rightarrow \infty.$$

Let $n(E, \mathcal{L})$ be the number of levels in a window around E for which the leading term in Weyl’s law predicts \mathcal{L} levels:

$$n(E, \mathcal{L}) = \#\left\{E - \frac{2\pi}{\text{area}(M)} \cdot \mathcal{L} < E_j < E + \frac{2\pi}{\text{area}(M)} \cdot \mathcal{L}\right\}$$

and more generally for a test function f define

$$n_f(E, \mathcal{L}) := \sum_j f\left(\frac{\text{area}(M)}{4\pi} \cdot \frac{(E_j - E)}{\mathcal{L}}\right) \quad (1.1)$$

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which counts the levels lying in a “soft” window of length $\frac{4\pi}{\text{area}(M)}\mathcal{L}$ about E . In the above $\mathcal{L} = \mathcal{L}(E)$ depends on the location E . In what follows we will usually write $n(\mathcal{L})$ for $n(E, \mathcal{L})$, the dependence on E implicitly understood.

To study the statistical behavior of $n(\mathcal{L})$ we need to consider E as random, drawn from a certain distribution on the line. We denote by $\langle \cdot \rangle$ this kind of energy averaging, e.g., $\langle F \rangle = \frac{1}{E} \int_E^{2E} F(E') dE'$. Weyl’s law leads us to expect that the mean value of $n(\mathcal{L})$ is \mathcal{L} and likewise that of $n_f(\mathcal{L})$ is $\mathcal{L} \cdot \int f(x) dx$.

1.1 Number variance

The variance of $n_f(E, \mathcal{L})$ from its expected value is:

$$\Sigma_f^2(E, \mathcal{L}) = \langle |n_f(\mathcal{L}) - \langle n_f(\mathcal{L}) \rangle|^2 \rangle$$

It is customary to express the number variance by means of an integral kernel $K_E(\tau)$, called the “form factor”, so that as $E \rightarrow \infty$

$$\Sigma_f^2(E, \mathcal{L}) \sim \mathcal{L} \cdot \int_{-\infty}^{\infty} \widehat{f}(u)^2 K_E\left(\frac{u}{\mathcal{L}}\right) du = \int_{-\infty}^{\infty} (\mathcal{L} \widehat{f}(\mathcal{L}\tau))^2 K_E(\tau) d\tau ,$$

where $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} dx$ is the Fourier transform of f .

For “generic” surfaces, Berry [3, 4] argued that as $E \rightarrow \infty$, the behavior of $\Sigma_f^2(E, \mathcal{L})$ for \mathcal{L} in the range¹

$$1 \ll \mathcal{L} \ll \mathcal{L}_{max} = \sqrt{E} \tag{1.2}$$

is *universal*, depending only on the coarse dynamical nature of the geodesic flow on the surface, and follows that of one of a small number of random matrix ensembles: If the flow is *integrable* (as in the case of a flat torus) then $\Sigma_f^2(E, \mathcal{L}) \sim \mathcal{L} \cdot \int_{-\infty}^{\infty} f(x)^2 dx$ for $\mathcal{L} \rightarrow \infty$, as in the Poisson model of uncorrelated levels. If the flow is *chaotic* (as in the case of negative curvature) then the behavior is as in the Gaussian Orthogonal Ensemble (GOE): For the sharp window ($f = \mathbf{1}_{[-1/2, 1/2]}$), this is given by $\Sigma^2(E, \mathcal{L}) \sim \frac{2}{\pi^2} \log \mathcal{L}$ for $\mathcal{L} \rightarrow \infty$. For sufficiently smooth f , in the GOE we have $\Sigma_f^2(E, \mathcal{L}) \sim 2 \int_{-\infty}^{\infty} \widehat{f}(u)^2 |u| du$, that is the variance of sufficiently smooth statistics tends to a finite value as $\mathcal{L} \rightarrow \infty$. The form factors for the random models are $K^{pois}(\tau) \equiv 1$, and

$$K^{GOE}(\tau) = \begin{cases} 2|\tau| - |\tau| \log(1 + 2|\tau|), & |\tau| \leq 1 \\ 2 - |\tau| \log \frac{1+2|\tau|}{2|\tau|-1}, & |\tau| > 1 \end{cases} .$$

It is to be emphasized that the above behavior is only valid in the universal regime $1 \ll \mathcal{L} \ll \sqrt{E}$; for $\mathcal{L} \gg \sqrt{E}$ the integrable case is fairly well understood (at a rigorous level), see the survey [5]: The variance grows as \sqrt{E} (a classical result

¹The symbol $f(x) \ll g(x)$ means that $f(x)/g(x) \rightarrow 0$

[12] in the case of the standard flat torus). In the chaotic case it is believed [3, 4] that generically, the number variance continues to be small as in the universal regime.

1.2 Fluctuations

Our main interest here is in the value distribution of the normalized linear statistic

$$\frac{n_f(E, \mathcal{L}) - \langle n_f(\mathcal{L}) \rangle}{\sqrt{\Sigma_f^2(E, \mathcal{L})}}$$

as E varies. In all the statistical models (Poisson and GOE/GUE), it is a standard Gaussian [22, 14, 13].

In the integrable case, when $\mathcal{L} \succ \sqrt{E}$, the distribution is known ([17], [5]), and is definitely not Gaussian. Inside the universal regime (1.2), the distribution is believed to be Gaussian in both the integrable [6] and chaotic [1, 9, 25] cases. In the special case of the standard flat torus, this has been proved in a small part of the universal regime near \sqrt{E} [18].

1.3 The modular domain

We start with the upper half-plane $\mathbb{H} = \{x + \sqrt{-1}y : y > 0\}$ equipped with the hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$, which has constant curvature equal -1 . The Laplace-Beltrami operator for this metric is given by $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. The orientation-preserving isometries of the metric ds^2 are the linear fractional transformations $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\}$.

The modular domain is the Riemann surface obtained by identifying points in the upper half plane which differ by a linear fractional transformation with integer coefficients, that is by elements of the modular group

$$\Gamma := PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm I\}.$$

The resulting surface \mathbb{H}/Γ is non-compact and has cone points, but has finite hyperbolic area: $\text{area}(\mathbb{H}/\Gamma) = \pi/3$.

The spectrum of the Laplacian on the modular domain has a continuous component. Nonetheless, Selberg [24] showed that a version of Weyl's law holds for the discrete spectrum: If we write the eigenvalues of $-\Delta$ on $L^2(\mathbb{H}/\Gamma)$ in the form $E_j = 1/4 + r_j^2$, then:

$$\#\{r_j \leq T\} = \frac{\text{area}(\mathbb{H}/\Gamma)}{4\pi} T^2 - \frac{2}{\pi} T \log T + c_1 T + O\left(\frac{T}{\log T}\right) \quad (1.3)$$

where $c_1 = \frac{2 + \log \frac{2}{\pi}}{\pi}$.

In the case of the modular domain, deviations from generic statistics were discovered [10, 7, 11]. Although the geodesic flow is chaotic, the local statistics

of the spectrum seem Poissonian, and Bogomolny, Leyvraz and Schmit [8] argued that the behavior of the form factor is given by

$$K_E(\tau) \sim \begin{cases} c_1 \frac{\exp(c_2 \sqrt{E} \tau)}{\sqrt{E}}, & \frac{1}{\sqrt{E}} \ll \tau \ll \frac{\log \sqrt{E}}{\sqrt{E}} \\ 1, & \tau \gg \frac{\log \sqrt{E}}{\sqrt{E}} \end{cases}$$

for some constants $c_1, c_2 > 0$, that is to say, in the universal regime we have

$$\Sigma_f^2(E, \mathcal{L}) \sim \begin{cases} 2c_1 \frac{\mathcal{L}}{\sqrt{E}} \int_0^\infty \widehat{f}(u)^2 \exp(c_2 \frac{\sqrt{E}u}{\mathcal{L}}) du, & \frac{\sqrt{E}}{\log E} \ll \mathcal{L} \ll \sqrt{E} \\ \mathcal{L} \cdot \int_{-\infty}^\infty \widehat{f}(u)^2 du, & 1 \ll \mathcal{L} \ll \frac{\sqrt{E}}{\log E}. \end{cases}$$

The only rigorous results known concern the closely related case where the modular group is replaced by quaternion groups: In the 1970's Selberg [15, Chapter 2.18] gave a *lower* bound for the variance $\Sigma^2(E, E)$ of the sharp counting function $n(E)$ of the form $\Sigma^2(E, E) \gg \sqrt{E}/(\log E)^2$. Luo and Sarnak [20] gave lower bounds for the averaged number variance of the sharp counting function $n(E, \mathcal{L})$:

$$\frac{1}{\mathcal{L}} \int_0^\mathcal{L} \Sigma^2(E, \mathcal{L}') d\mathcal{L}' \gg \sqrt{E}/(\log E)^2, \quad \sqrt{E}/\log E \ll \mathcal{L} \ll \sqrt{E}$$

in the case of arithmetic (co-compact) groups. In the case of the hard window $f = \mathbf{1}_{[-1/2, 1/2]}$ no *upper* bounds are currently available.

2 Formulation of results

2.1 Definition of the smooth counting functions

Let f be an even test function, whose Fourier transform $\widehat{f} \in C_c^\infty(\mathbb{R})$ is smooth and compactly supported, and normalized by requiring that

$$\int_{-\infty}^\infty f(x) dx = 1,$$

and that

$$\sup\{|\xi| : \widehat{f}(\xi) \neq 0\} = 1.$$

The eigenvalues of the Laplacian are parametrized by $E_j = 1/4 + r_j^2$. We define smooth counting functions by

$$N_{f,L}(\tau) = \sum_{j \geq 0} f(L(r_j - \tau)) + f(L(-r_j - \tau)) \tag{2.1}$$

This in essence carries the same information as (1.1). The relation between the expected number of levels \mathcal{L} and the inverse width L of the momentum window is

$$\mathcal{L} = \frac{\text{area}(\mathbb{H}/\Gamma) \sqrt{E}}{2\pi L} = \frac{\sqrt{E}}{6L}.$$

The leading order behavior of $N_{f,L}$ is given by

$$\overline{N_{f,L}}(\tau) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \{f(L(r - \tau)) + f(L(-r - \tau))\} M(r) dr$$

where

$$M(r) := \frac{\text{area}(\mathbb{H}/\Gamma)}{2} r \tanh(\pi r) - \frac{\Gamma'}{\Gamma}(1 + ir) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right).$$

(In keeping with tradition, I use the symbol Γ for both the modular group and the Gamma function.) The term $\overline{N_{f,L}}(\tau)$ is asymptotic to \mathcal{L} :

$$\overline{N_{f,L}}(\tau) \sim 2 \frac{\text{area}(\mathbb{H}/\Gamma)}{4\pi} \int_{-\infty}^{\infty} f(x) dx \frac{\tau}{L} + O\left(\frac{\log \tau}{L}\right) \sim \frac{1}{6} \frac{\tau}{L}.$$

2.2 The results

We will see that $N_{f,L} - \overline{N_{f,L}}$ has mean zero and show that the variance of $N_{f,L}$, when $\limsup \pi L / \log T < 1$, is asymptotic to

$$\sigma_L^2 := \frac{2\kappa}{\pi L} \int_0^{\infty} \widehat{f}(u)^2 e^{\pi L u} du \tag{2.2}$$

where $\kappa = \frac{1015}{864} \prod_{p \neq 2} \left(1 + \frac{p^4 - 2p^3 + 1}{(p^2 - 1)^3}\right) = 1.328 \dots$. Thus when the expected number of levels \mathcal{L} satisfies

$$\text{area}(\mathbb{H}/\Gamma) \frac{\sqrt{E}}{\log E} < \mathcal{L} \ll T = \sqrt{E},$$

the form factor $K_E(\tau)$ is given by

$$K_E(\tau) = c_1 \frac{\exp c_2 \sqrt{E} \tau}{\sqrt{E}}$$

with $c_1 = 6\kappa/\pi$, $c_2 = \pi/6$.

Our main result is that the fluctuations of $N_{f,L}$ are Gaussian:

Theorem 2.1 *Assume that $L \rightarrow \infty$ as $T \rightarrow \infty$ but $L = o(\log T)$. Then the limiting value distribution of $(N_{f,L} - \overline{N_{f,L}})/\sigma_L$ is a standard Gaussian, that is*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas}\{\tau \in [T, 2T] : \frac{N_{f,L}(\tau) - \overline{N_{f,L}}(\tau)}{\sigma_L} < x\} = \int_{-\infty}^x e^{-u^2/2} \frac{du}{\sqrt{2\pi}}.$$

The reason that we need to assume that $L = o(\log T)$ is that we prove Theorem 2.1 by the method of moments, and in computing the K th moment we find Gaussian moments for $L < c_K \log T$, where $c_K \rightarrow 0$ as K grows.

Plan of the paper: In Sections 3 and 4 we give some results on the hyperbolic conjugacy classes of the modular group. In Section 5 we use Selberg’s trace formula to express $N_{f,L}(\tau) - \overline{N_{f,L}}(\tau)$ as a sum $S_{f,L}(\tau)$ over hyperbolic conjugacy classes plus a negligible term. The variance of $S_{f,L}$ is computed in Section 6 and the higher moments in Section 7. We prove Theorem 2.1 in Section 8.

3 The modular group

3.1 Conjugacy classes

To analyze $N_{f,L}$ we use the Selberg trace formula, which for a discrete co-finite subgroup $\Gamma \subset PSL(2, \mathbb{R})$ relates a sum over the spectrum of the Laplacian on $L^2(\mathbb{H}/\Gamma)$ with a sum over the conjugacy classes of the group Γ . We review some background material on these classes for the modular group $PSL(2, \mathbb{Z})$.

The conjugacy classes are divided into the class which consists of the identity element, hyperbolic, elliptic and parabolic classes. The hyperbolic conjugacy classes in Γ are represented by matrices P which are diagonalizable over the reals and are conjugate to a matrix of the form $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$. The norm of P is defined as $\mathcal{N}(P) = \lambda^2$. The norm is therefore related to the trace of the corresponding group element by

$$\mathcal{N}(P)^{1/2} + \mathcal{N}(P)^{-1/2} = |\text{tr}(P)|.$$

We can write as each such P as $P = P_0^k$ where P_0 is primitive and $k \geq 1$, where an element of Γ is primitive if it cannot be written as an essential power of another element. As is well known, primitive hyperbolic conjugacy classes correspond to closed geodesics on the Riemann surface \mathbb{H}/Γ .

In the case of the modular group $\Gamma = PSL(2, \mathbb{Z})$, the traces are integers. If P is a hyperbolic class with trace $|\text{tr}(P)| = n$, $n > 2$ then its norm is

$$\mathcal{N}(n) = \left(\frac{n + \sqrt{n^2 - 4}}{2}\right)^2. \tag{3.1}$$

For the modular group, primitive hyperbolic conjugacy classes are parametrized by indefinite binary quadratic forms as follows (cf. [23]): Take a binary quadratic form $Q_{a,b,c}(x, y) := ax^2 + bxy + cy^2$, with $a, b, c \in \mathbb{Z}$. The discriminant of $Q_{a,b,c}$ is $d := b^2 - 4ac$. The form $Q_{a,b,c}$ is indefinite iff $d > 0$. We assume that d is not a perfect square. We say that $Q_{a,b,c}$ is primitive if $\text{gcd}(a, b, c) = 1$. Two binary quadratic forms Q, Q' are equivalent if $Q'(x, y) = Q(ax+by, cx+dy)$ for an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbb{Z})$; since the forms are quadratic, they are also equivalent under $-\gamma$ and hence equivalence is over $PSL(2, \mathbb{Z})$. Let $h(d)$ be the number of equivalence classes of primitive binary quadratic forms of discriminant d .

The automorphs of $Q_{a,b,c}$ are all of the form

$$\pm P(t, u) = \pm \begin{pmatrix} \frac{1}{2}(t - bu) & -cu \\ au & \frac{1}{2}(t + bu) \end{pmatrix}$$

where (t, u) solve the Pellian equation

$$t^2 - du^2 = 4 \tag{3.2}$$

If $u \neq 0$ then these are hyperbolic elements of $SL(2, \mathbb{Z})$ with norm $\mathcal{N}(P) = (t + u\sqrt{d})^2$ and trace t .

Let $\epsilon_d = \frac{1}{2}(t_d + u_d\sqrt{d})$ ($t_d, u_d > 0$) be the fundamental solution of (3.2). Then the matrix $P(t_d, u_d)$ is a primitive hyperbolic matrix P_0 of trace $\text{tr}(P_0) = t_d$ and norm $\mathcal{N}(P_0) = \epsilon_d^2$. It turns out that in this way we get a bijection between equivalence classes of primitive binary quadratic forms and conjugacy classes of primitive hyperbolic matrices in $PSL(2, \mathbb{Z})$. Thus the number of primitive hyperbolic conjugacy classes of norm ϵ_d^2 is precisely the class number $h(d)$.

3.2 The amplitude $\beta(n)$

We define, for $n > 2$,

$$\beta(n) := \frac{1}{2} \sum_{\text{tr}(P)=n} \frac{\log \mathcal{N}(P_0)}{\mathcal{N}(P)^{1/2} - \mathcal{N}(P)^{-1/2}} \tag{3.3}$$

the sum over all conjugacy classes $\{P\}$ in $PSL(2, \mathbb{Z})$ with $|\text{tr}(P)| = n$, equivalently with norm $\mathcal{N}(n)$ given by (3.1). These quantities turn out to be crucial in our analysis. The factor 1/2 in the definition is inserted among other reasons to give numbers with mean value 1 (as can be seen from the Prime Geodesic Theorem):

$$\sum_{n \leq N} \beta(n) \sim N, \text{ as } N \rightarrow \infty .$$

As representatives of the conjugacy classes of matrices with trace $n > 2$ we can take the matrices $P_0^k = P(t_d, u_d)^k = P(n, u)$ where d runs over all discriminants, $k \geq 1$ and $n^2 - du^2 = 4$, $n > 2$, $u \geq 1$. Thus we see that

$$\beta(n) = \sum_{\substack{d, u \geq 1 \\ n^2 - du^2 = 4}} \frac{h(d) \log \epsilon_d}{\sqrt{du^2}} . \tag{3.4}$$

Dirichlet’s class number formula allows us to use (3.4) to express $\beta(n)$ in terms of Dirichlet L -functions: For a discriminant d one associates the quadratic character χ_d given by $\chi_d(p) = \left(\frac{d}{p}\right)$ for p an odd prime, $\chi_d(2) = 1$ if $d \equiv 1 \pmod{8}$, $\chi_d(2) = -1$ if $d \equiv 5 \pmod{8}$ and $\chi_d(-1) = 1$. The associated L -function is $L(s, \chi_d) = \sum_{n \geq 1} \chi_d(n)n^{-s}$, $\Re(s) > 1$. Dirichlet’s class number formula is

$$h(d) \log \epsilon_d = \sqrt{d}L(1, \chi_d) .$$

Inserting the class number formula into (3.4) we find that

$$\beta(n) = \sum_{d, u \geq 1: du^2 = n^2 - 4} \frac{1}{u} L(1, \chi_d) . \tag{3.5}$$

As a consequence, one can get an upper bound of $\beta(n) = O((\log n)^2)$ by using $L(1, \chi_d) \ll \log d$. What is more useful to us is that, in the mean square, $\beta(n)$ is constant:

Lemma 3.1 (M. Peter [21])

$$\sum_{n \leq N} \beta(n)^2 \sim \kappa N, \quad N \rightarrow \infty \quad (3.6)$$

where κ is given by the product over primes

$$\kappa = \frac{1015}{864} \prod_{p \neq 2} \left(1 + \frac{p^4 - 2p^3 + 1}{(p^2 - 1)^3}\right) = 1.328 \dots \quad (3.7)$$

This (complicated) expression was derived heuristically by Bogomolny, Leyvraz and Schmit [8] and proven by Manfred Peter [21], who uses the expression (3.5) for $\beta(n)$ in terms of $L(1, \chi)$ and methods related to work on moments of class numbers [2]. For an extension to the case of congruence groups, see [19].

4 The length spectrum

We will need to study alternating sums of the form

$$\sum_{j=1}^K \pm \log \mathcal{N}(n_j).$$

The first question is to when these alternating sums vanish.

We say that a relation

$$\sum_{j=1}^K \eta_j \log \mathcal{N}(n_j) = 0, \quad \eta_j = \pm 1 \quad (4.1)$$

is *non-degenerate* if no sub-sum vanishes, that is if there is no proper subset $S \subset \{1, \dots, K\}$ for which $\sum_{j \in S} \eta_j \log \mathcal{N}(n_j) = 0$. The existence of non-degenerate relations (4.1) forces severe constraints. To explain these, recall that $\mathcal{N}(n)$ is a unit in the real quadratic field $\mathbb{Q}(\sqrt{n^2 - 4})$. We claim that such relations can occur only if all these units lie in the same quadratic field.

Lemma 4.1 *Let $\sum_{j=1}^K \pm \log \mathcal{N}(n_j) = 0$ be a non-degenerate relation. Then all the norms $\mathcal{N}(n_i)$ lie in the same quadratic field, that is for some common d we have $n_i^2 - 4 = df_i^2$ for all i .*

Proof. We can write each norm as a power of the fundamental unit of the quadratic field in which it lies. Thus it will suffice to show that if F_1, \dots, F_K be distinct real

quadratic fields, then the fundamental units ϵ_i of F_i are multiplicatively independent.

Let $E = F_1 \vee \dots \vee F_K$ be the compositum of the fields F_i . This is a Galois extension of the rationals with Galois group $G = Gal(E/\mathbb{Q})$ an elementary Abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^s$, for some $s \leq K$. If we denote by U_E the unit group of E , then G acts on U_E and hence we get a linear representation on the vector space $\mathbb{Q} \otimes U_E$.

We claim that the ϵ_i are eigenvectors of G , that is $\sigma(\epsilon_i) = \epsilon_i^{\chi_i(\sigma)}$ for all $\sigma \in G$, where $\chi_i : G \rightarrow \{\pm 1\}$ are *distinct* characters. This forces them to be multiplicatively independent.

Indeed, since we have an Abelian extension, all subfields are Galois and in particular F_i are preserved by G . Since the unit group is also preserved this means that under the action of any element $\sigma \in G$, ϵ_i is taken to a unit of F_i which is necessarily $\epsilon_i^{\pm 1}$. That is we have a character χ_i of G with $\sigma(\epsilon_i) = \epsilon_i^{\chi_i(\sigma)}$. The characters χ_i are distinct since the kernel of χ_i is precisely $Gal(E/F_i)$. \square

We next get a lower bound for $\sum_{j=1}^K \pm \log \mathcal{N}(n_j)$ in the case it is non-zero.

Lemma 4.2 i) *If $m \neq n$ then*

$$|\log \mathcal{N}(m) - \log \mathcal{N}(n)| \gg \frac{1}{\min(m, n)}.$$

ii) *Suppose $\sum_{j=1}^K \pm \log \mathcal{N}(n_j)$ is nonzero. Then*

$$\left| \sum_{j=1}^K \pm \log \mathcal{N}(n_j) \right| \gg \frac{1}{\left(\prod_{j=1}^K \mathcal{N}(n_j) \right)^{2^{K-1}-1/2}}$$

Proof. i) Indeed, since $\log \mathcal{N}(n) = 2 \log n + O(1/n^2)$, if $m \neq n$, say $m > n$, then

$$\log \mathcal{N}(m) - \log \mathcal{N}(n) = 2 \log \frac{m}{n} + O\left(\frac{1}{n^2}\right).$$

Since $\log \frac{m}{n} \geq \log \frac{n+1}{n} \gg \frac{1}{n}$, we find

$$\log \mathcal{N}(m) - \log \mathcal{N}(n) \gg \frac{1}{n} = \frac{1}{\min(m, n)}.$$

ii) Let $\lambda_j = \frac{n_j + \sqrt{n_j^2 - 4}}{2}$ so that $\mathcal{N}(n_j) = \lambda_j^2$, and set $\alpha = \prod_{j=1}^K \lambda_j^{\pm 1}$. If $|\alpha - 1| \leq 1/2$ then

$$\left| \sum_{j=1}^K \pm \log \mathcal{N}(n_j) \right| = 2 |\log \alpha| \gg |\alpha - 1|.$$

So it suffices to give a lower bound for $|\alpha - 1|$, assuming $\alpha \neq 1$. This follows from Liouville's theorem on Diophantine approximation of algebraic numbers by

rational; we give an explicit proof as follows: Let $E = \mathbb{Q}(\sqrt{n_1}, \dots, \sqrt{n_K})$, which is a Galois extension of the rationals with Galois group $G = Gal(E/\mathbb{Q})$ which is an elementary abelian 2-group of order 2^s , for some $s \leq K$. Moreover $\alpha \in E$ is an algebraic integer, and hence the norm $\mathcal{N}_{E/\mathbb{Q}}(\alpha - 1)$ is a nonzero rational integer, hence has absolute value at least 1. Thus

$$|\mathcal{N}_{E/\mathbb{Q}}(\alpha - 1)| = |\alpha - 1| \prod_{id \neq \sigma \in G} |\alpha^\sigma - 1| \geq 1.$$

Since $\lambda_j^\sigma = \lambda_j^{\pm 1}$ for all $\sigma \in G$, we have

$$|\alpha^\sigma - 1| \leq \prod_{j=1}^K \lambda_j + 1.$$

Thus

$$|\alpha - 1| \geq \frac{1}{(\prod_{j=1}^K \lambda_j + 1)^{|G|-1}} \gg \frac{1}{(\prod_{j=1}^K \mathcal{N}(n_j))^{(2^K-1)/2}}. \quad \square$$

5 An expansion for $N_{f,L}$

5.1 The Selberg trace formula

We will transform $N_{f,L}$ by using the Selberg trace formula [24]: Let $g \in C_c^\infty(\mathbb{R})$ be an even, smooth and compactly supported function, and let

$$h(r) = \int_{-\infty}^\infty g(u)e^{iru} du$$

so that

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^\infty h(r)e^{-iru} dr.$$

The Selberg trace formula for a discrete co-compact sub-group $\Gamma \subset PSL(2, \mathbb{R})$ with no elliptic elements is the identity [24]

$$\begin{aligned} \sum_{j \geq 0} h(r_j) &= \frac{\text{area}(\mathbb{H}/\Gamma)}{4\pi} \int_{-\infty}^\infty h(r)r \tanh(\pi r) dr \\ &+ \sum_{\{P\} \text{ hyperbolic}} \frac{\log \mathcal{N}(P_0)}{\mathcal{N}(P)^{1/2} - \mathcal{N}(P)^{-1/2}} g(\log \mathcal{N}(P)) \end{aligned} \tag{5.1}$$

where the sum is over all hyperbolic conjugacy classes of Γ .

In the case of the modular group, the hyperbolic terms can be written as

$$2 \cdot \sum_{n > 2} \beta(n) g(\log \mathcal{N}(n)) \tag{5.2}$$

where the amplitude $\beta(n)$ is given by (3.3).

For groups with elliptic elements, there is an extra contribution to the RHS of (5.1) which is a sum over the finitely many conjugacy classes of elements E of finite order $m \geq 2$:

$$\sum_{\{E\}} \sum_{k=1}^m \frac{1}{m \sin(\pi k/m)} \int_{-\infty}^{\infty} h(r) \frac{e^{-2\pi k r/m}}{1 + e^{-2\pi r}} dr. \tag{5.3}$$

For discrete groups whose fundamental domain is non-compact but of finite volume, that is with cusps, there are extra terms coming from the contribution of the continuous spectrum and parabolic elements. For $\Gamma = PSL(2, \mathbb{Z})$, these terms are given explicitly by [16]:

$$g(0) \log \frac{\pi}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \left(\frac{\Gamma'}{\Gamma}(1 + ir) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right) \right) dr + 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} g(2 \log n) \tag{5.4}$$

where $\Lambda(n)$ is the von Mangold function.

5.2 Transforming $N_{f,L}$

We now apply the trace formula to derive an alternative expression for $N_{f,L}$. Taking

$$h(r) = f(L(r - \tau)) + f(L(-r - \tau))$$

so that

$$g(u) = \frac{1}{2\pi L} \widehat{f}\left(\frac{u}{2\pi L}\right) (e^{-i\tau u} + e^{i\tau u})$$

we find that

$$N_{f,L}(\tau) = \overline{N_{f,L}}(\tau) + S_{f,L}(\tau) + \mathcal{E} \tag{5.5}$$

where:

- The term $\overline{N_{f,L}}$ is given by the contribution of the identity class to (5.1) and part of the parabolic terms in (5.4):

$$\overline{N_{f,L}}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{f(L(r - \tau)) + f(L(-r - \tau))\} M(r) dr$$

where

$$M(r) = \frac{\text{area}(\mathbb{H}/\Gamma)}{2} r \tanh(\pi r) - \frac{\Gamma'}{\Gamma}(1 + ir) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right).$$

By Stirling's formula, we have

$$\overline{N_{f,L}}(\tau) = \frac{1}{6} \int_{-\infty}^{\infty} f(x) dx \frac{\tau}{L} + O\left(\frac{\log \tau}{L}\right).$$

- The term $S_{f,L}(\tau)$ is the contribution of the hyperbolic classes (5.2):

$$S_{f,L}(\tau) = \frac{1}{\pi L} \sum_{n>2} \beta(n) \widehat{f}\left(\frac{\log \mathcal{N}(n)}{2\pi L}\right) \left(e^{-i\tau \log \mathcal{N}(n)} + e^{i\tau \log \mathcal{N}(n)} \right). \tag{5.6}$$

The sum (5.6) contains only terms with $\log \mathcal{N}(n) \leq 2\pi L$, that is $n < e^{\pi L}$.

As we will see below, it is the term $S_{f,L}(\tau)$ which is responsible for the fluctuations of $N_{f,L}(\tau)$, and its variance is asymptotic to σ_L^2 . As we can see from the formula (5.6), since \widehat{f} has compact support we have $S_{f,L}(\tau) \equiv 0$ for $L \ll 1$.

- \mathcal{E} is the contribution of the elliptic classes (5.3) and the remaining part of the parabolic contribution (5.4), namely

$$\frac{1}{\pi L} \widehat{f}(0) \log \frac{\pi}{2} + \frac{1}{\pi L} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \widehat{f}\left(\frac{\log n}{\pi L}\right) 2 \cos(2\tau \log n). \tag{5.7}$$

\mathcal{E} is easily seen to be negligible, that is $\mathcal{E} = o(\sigma_L)$. Indeed, the contribution of the elliptic elements is easily seen to be $O(e^{-\text{const} \cdot \tau/L})$. As for (5.7), this is bounded as $L \rightarrow \infty$ by (say) Mertens' theorem. Moreover the mean value of (5.7) clearly vanishes as $T \rightarrow \infty$.

We thus see that the difference between the centered counting function $N_{f,L}(\tau) - \overline{N_{f,L}}$ and the sum $S_{f,L}(\tau)$ over hyperbolic conjugacy classes is negligible relative to the standard deviation σ_L of $S_{f,L}(\tau)$, and thus for our purposes we need only investigate the statistics of $S_{f,L}(\tau)$.

6 The mean and variance of $S_{f,L}$

6.1 The averaging procedure

We define an averaging procedure by taking a non-negative weight function $w \geq 0$, which is smooth and compactly supported in $(0, \infty)$, with $\int_{-\infty}^{\infty} w(x) dx = 1$. We then get an averaging operator:

$$\langle F \rangle_{w,T} := \frac{1}{T} \int_{-\infty}^{\infty} F(\tau) w\left(\frac{\tau}{T}\right) d\tau .$$

Let $\mathbb{P}_{w,T}$ be the associated probability measure:

$$\mathbb{P}_{w,T}(f \in \mathcal{A}) = \frac{1}{T} \int_{-\infty}^{\infty} \mathbb{1}_{\mathcal{A}}(f(t)) w\left(\frac{t}{T}\right) dt .$$

Note that the requirement $w \in C_c^\infty(0, \infty)$ implies that the Fourier transform of w decays rapidly:

$$\widehat{w}(x) \ll |x|^{-A}, \text{ as } |x| \rightarrow \infty$$

for all $A > 1$. In the concluding Section 8 we will relax the restrictions on w to allow other averages, e.g., $w = \mathbf{1}_{[1,2]}$ so that we take t uniformly distributed in $[T, 2T]$, or $w(t) = 2t \mathbf{1}_{[1,\sqrt{2}]}$ when we take the eigenvalue $\lambda = 1/4 + t^2$ uniformly distributed in $[E, 2E]$, $E = 1/4 + T^2$.

6.2 The expected value of $S_{f,L}$

We will first show that the mean value $\langle S_{f,L} \rangle_{w,T}$ tends to zero as $T \rightarrow \infty$ provided $L = O(\log T)$: Averaging (5.6) we find

$$\langle S_{f,L} \rangle_{w,T} = \frac{1}{\pi L} \sum_{2 < n < e^{\pi L}} \beta(n) \widehat{f}\left(\frac{\log \mathcal{N}(n)}{2\pi L}\right) 2\Re e \widehat{w}\left(\frac{T}{2\pi} \log \mathcal{N}(n)\right).$$

Note that since $\log \mathcal{N}(n) \sim 2 \log n$ and $\text{supp } \widehat{f} \subseteq [-1, 1]$, the sum is over $n \leq e^{\pi L}$. Using $\widehat{w}(x) \ll x^{-A}$ as $x \rightarrow \infty$, we have

$$\langle S_{f,L} \rangle_{w,T} \ll \frac{1}{L} \sum_{n < e^{\pi L}} \beta(n) \frac{1}{(T \log n)^A}.$$

Since $\sum_{n \leq x} \beta(n)^2 \ll x$ by Lemma 3.1, we have by the Cauchy-Schwartz inequality that

$$\langle S_{f,L} \rangle_{w,T} \ll \frac{e^{\pi L}}{T^A L^{A+1}}$$

which goes to zero since we assume $L = O(\log T)$.

Note that this argument also works when we allow straight averages (such as $w = \mathbf{1}_{[1,2]}$) as long as $L < \frac{1}{\pi} \log T$.

6.3 The variance of $S_{f,L}$

Proposition 6.1 *If $\limsup \pi L / \log T < 1$ then as $T \rightarrow \infty$:*

$$\langle (S_{f,L})^2 \rangle_{w,T} \sim \sigma_L^2 =: \frac{2\kappa}{\pi L} \int_0^\infty \widehat{f}(u)^2 e^{\pi L u} du \tag{6.1}$$

where κ is given by (3.7).

Note that we have

$$e^{(1-\epsilon)\pi L/2} \ll \sigma_L \ll \frac{e^{\pi L/2}}{L}$$

for all $\epsilon > 0$.

Proof. To compute $\langle (S_{f,L})^2 \rangle_{w,T}$, use (5.6) to get

$$\begin{aligned} \langle (S_{f,L})^2 \rangle_{w,T} &= \frac{1}{(\pi L)^2} \sum_{m,n < e^{\pi L}} \beta(m)\beta(n) \widehat{f}\left(\frac{\log \mathcal{N}(m)}{2\pi L}\right) \widehat{f}\left(\frac{\log \mathcal{N}(n)}{2\pi L}\right) \\ &\quad \times \sum_{\epsilon_1, \epsilon_2 = \pm 1} \widehat{w}\left(\frac{T}{2\pi} (\epsilon_1 \log \mathcal{N}(m) + \epsilon_2 \log \mathcal{N}(n))\right). \end{aligned}$$

We now deduce that as $T \rightarrow \infty$, the only non-vanishing contribution is from the “diagonal terms” where $\epsilon_1 = -\epsilon_2$ and

$$\mathcal{N}(m) = \mathcal{N}(n)$$

that is $m = n$.

If $m \neq n$ we may use Lemma 4.2 to get a lower bound

$$|\log \mathcal{N}(m) \pm \log \mathcal{N}(n)| \gg \frac{1}{\min(m, n)}. \tag{6.2}$$

To be included in the sum, we need $\mathcal{N}(m), \mathcal{N}(n) \leq e^{2\pi L}$, that is $m, n \leq e^{\pi L}$, and so

$$\widehat{w}\left(\frac{T}{2\pi}(\epsilon_1 \log \mathcal{N}(m) + \epsilon_2 \log \mathcal{N}(n))\right) \ll \left(\frac{\min(m, n)}{T}\right)^A \ll \left(\frac{e^{\pi L}}{T}\right)^A.$$

Moreover, from $\sum_{n < x} \beta(n)^2 \ll x$ we get by Cauchy-Schwartz that the off-diagonal contribution is dominated by

$$\frac{1}{L^2} \left(\frac{e^{\pi L}}{T}\right)^A \sum_{m, n < e^{\pi L}} \beta(m)\beta(n) \ll \frac{e^{A\pi L}}{L^2 T^A} e^{2\pi L}$$

for all $A > 1$. This goes to zero if $\pi L \leq (1 - \delta)(\log T)$ for some $\delta > 0$, which we assume.

The diagonal terms $m = n$ give

$$\frac{1}{(\pi L)^2} \sum_{n > 2} \beta(n)^2 \widehat{f}\left(\frac{\log \mathcal{N}(n)}{2\pi L}\right)^2$$

(where we used $\widehat{w}(0) = 1$). Since there are two such terms (corresponding to $\epsilon_1 = -\epsilon_2 = +1$ or -1), we have the total diagonal contribution being

$$2 \frac{1}{(\pi L)^2} \sum_{n > 2} \beta(n)^2 \widehat{f}\left(\frac{\log \mathcal{N}(n)}{2\pi L}\right)^2.$$

This can be evaluated asymptotically as $L \rightarrow \infty$ using Peter’s formula (Lemma 3.1) to give

$$\frac{2\kappa}{\pi L} \int_0^\infty \widehat{f}(u)^2 e^{\pi L u} du =: \sigma_L^2.$$

Thus we find

$$\langle (S_{f,L})^2 \rangle_{w,T} \sim \sigma_L^2$$

if $\limsup \pi L / \log T < 1$. □

7 Higher moments

We can now show that $S_{f,L}(\tau)$ has Gaussian moments:

Theorem 7.1 *For $K \geq 3$ the K th moment of $S_{f,L}/\sigma_L$ converges to that of a normal Gaussian provided that $L \rightarrow \infty$ with T but that $L = o(\log T)$:*

$$\lim_{T \rightarrow \infty} \left\langle \left(\frac{S_{f,L}(\tau)}{\sigma_L} \right)^K \right\rangle_{w,T} = \begin{cases} \frac{(2k)!}{k!2^k}, & K = 2k \text{ even} \\ 0, & K \text{ odd.} \end{cases}$$

7.1 Reduction to the pre-diagonal

By (5.6) the K th moments of $S_{f,L}$ is given by

$$\begin{aligned} \langle (S_{f,L})^K \rangle_{w,T} &= \frac{1}{(\pi L)^K} \sum_{n_1, \dots, n_K < e^{\pi L}} \prod_{j=1}^K \beta(n_j) \hat{f}\left(\frac{\log \mathcal{N}(n_j)}{2\pi L}\right) \\ &\quad \times \sum_{\eta_j = \pm 1} \hat{w}\left(\frac{T}{2\pi} \left(\sum_j \eta_j \log \mathcal{N}(n_j)\right)\right). \end{aligned} \tag{7.1}$$

We now show that as $T \rightarrow \infty$, the only (possibly) non-vanishing contribution to (7.1) is for terms satisfying:

$$\sum_{j=1}^K \eta_j \log \mathcal{N}(n_j) = 0$$

that is we have

$$\begin{aligned} \langle (S_{f,L})^K \rangle_{w,T} &= \frac{1}{(\pi L)^K} \sum_{\eta_j = \pm 1} \sum_{\sum_j \eta_j \log \mathcal{N}(n_j) = 0} \prod_{j=1}^K \beta(n_j) \hat{f}\left(\frac{\log \mathcal{N}(n_j)}{2\pi L}\right) \\ &\quad + O\left(\frac{e^{\alpha_K L}}{T^{\gamma_K}}\right) \end{aligned} \tag{7.2}$$

for some $\alpha_K, \gamma_K > 0$. Since $L = o(\log T)$ the remainder term vanishes as $T \rightarrow \infty$.

To prove this, recall that by Lemma 4.2, if $\sum_{j=1}^K \eta_j \log \mathcal{N}(n_j) \neq 0$ then for some $\delta_K > 0$

$$\left| \sum_{j=1}^K \eta_j \log \mathcal{N}(n_j) \right| \gg_K \left(\prod_{j=1}^K \mathcal{N}(n_j) \right)^{-2^{K-1} + 1/2} \gg e^{-\pi \delta_K L}$$

since only terms with $\mathcal{N}(n_j) < e^{2\pi L}$ appear in (7.1). Thus for these terms we have

$$\widehat{w}\left(\frac{T}{2\pi}\left(\sum_j \eta_j \log \mathcal{N}(n_j)\right)\right) \ll \left(\frac{e^{\pi L \cdot \delta_K}}{T}\right)^A.$$

Replacing $\beta(n)$ by $\log^2 n \ll L^2$ in (7.1) gives that the contribution of the terms with $\sum_{j=1}^K \eta_j \log \mathcal{N}(n_j) \neq 0$ is dominated by

$$\frac{1}{L^K} \sum_{n_1, \dots, n_K < e^{\pi L}} L^{2K} \left(\frac{e^{\pi L \cdot \delta_K}}{T}\right)^A \ll L^K \frac{e^{\pi L(K+A\delta_K)}}{T^A}.$$

Since $L = o(\log T)$, this vanishes as $T \rightarrow \infty$ (in fact we need only assume that $L < c_K \log T$ for this, if c_K is sufficiently small). This proves (7.2).

7.2 Off-diagonal terms

In (7.2) we consider the sum of non-diagonal terms, that is terms for which there is at least one index j such that $n_j \neq n_i$ for all $i \neq j$. To handle these, we use Lemma 4.1 which forces the relation

$$\prod_{j=1}^K \mathcal{N}(n_j)^{\eta_j} = 1$$

to decompose into a union of such relations. Thus there is a decomposition

$$\{1, 2, \dots, K\} = \coprod S_j$$

so that in each subset S_j we have

$$\prod_{i \in S_j} \mathcal{N}(n_i)^{\eta_i} = 1 \tag{7.3}$$

and the norms $\mathcal{N}(n_i) = (n_i + \sqrt{n_i^2 - 4})/2$ lie in the same real quadratic field $\mathbb{Q}(\sqrt{d_j})$ for all $i \in S_j$. In the diagonal case there are $K/2$ such sets, e.g., $S_1 = \{1, K/2 + 1\}$, $S_2 = \{2, K/2 + 2\}, \dots$ and the identities are of the form $\mathcal{N}(n_j) \mathcal{N}(n_{K/2+j})^{-1} = 1$, $j = 1, \dots, K/2$.

In the off-diagonal case we assume that there is a subset S_j contains at least 3 elements. The number r of subsets is then at most $(K - 1)/2$, since

$$K = \sum_{j=1}^r \#S_j \geq 3 + 2(r - 1).$$

To count such tuples of n_i , we denote for each subset S_j by d_j the common value of the square-free kernel of $n_i^2 - 4$, $i \in S_j$ and then write

$$n_i^2 - 4 = d_j f_i^2, \quad i \in S_j.$$

Let $\epsilon(d_j)$ be the fundamental unit of the field $\mathbb{Q}(\sqrt{d_j})$ and write $\mathcal{N}(n_i) = \epsilon(d_j)^{2k_i}$, $i \in S_j$. Since $\log \mathcal{N}(n_i) \ll L$ we have $k_i \ll L/\log \epsilon(d_j)$, $i \in S_j$ and the relation (7.3) implies $\sum_{i \in S_j} \pm k_i = 0$. Thus for each subset S_j there are at most $O((L/\log \epsilon(d_j))^{\#S_j-1})$ solutions of (7.3) with $\log \mathcal{N}(n_i) \ll L$.

Recall that we are summing over $\log \mathcal{N}(n) \leq 2\pi L$. Using $\beta(n) \ll (\log n)^2 \ll L^2$ we find that the off-diagonal contribution is bounded by the sum over all partitions $\{1, \dots, K\} = \prod_{j=1}^r S_j$ of

$$L^2 \prod_{j=1}^r \sum_{\epsilon(d_j) \leq e^{\pi L}} \left(\frac{L}{\log \epsilon(d_j)}\right)^{(\#S_j-1)/2} \ll L^K (\#\{d \text{ fundamental} : \epsilon(d) \leq e^{\pi L}\})^r \tag{7.4}$$

where $r \leq (K - 1)/2$ is the total number of subsets S_j in our partition.

Lemma 7.2 *The number of fundamental discriminants $d > 0$ for which $\epsilon(d) < X$ is $O(X^{1+\delta})$ for all $\delta > 0$.*

Proof. We need to bound the number of fundamental discriminants d for which the fundamental solution $\epsilon(d) = (x_d + \sqrt{d}y_d)/2$ of $x^2 - dy^2 = 4$ is at most X . Since $\epsilon(d) \sim x_d$, this is equivalent to bounding the number of fundamental d 's for which $x_d \ll X$. In turn, this number is majored by the number $\nu(X)$ of all triples (d, x, y) of positive integers, with $d \equiv 0, 1 \pmod 4$, for which $x^2 - dy^2 = 4$ and $x < X$, which is the sum

$$\nu(X) = \sum_{x < X} \#\{d, y \geq 1, d \equiv 0, 1 \pmod 4 : dy^2 = x^2 - 4\}.$$

Since for $x \neq 2$ the number of pairs (d, y) with $dy^2 = x^2 - 4$ is at most the number of divisors $\tau(x^2 - 4)$ of $x^2 - 4$, we find that

$$\nu(X) \leq \sum_{2 < x < X} \tau(x^2 - 4) \ll \sum_{2 < x < X} x^\delta \ll X^{1+\delta}$$

for all $\delta > 0$, by virtue of the bound $\tau(n) \ll n^\delta$ for all $\delta > 0$.

Note: A more refined argument [23, Lemma 4.2] shows that $\nu(X)$ is asymptotic to $\frac{35}{16}X$, so that one can replace the bound $O(X^{1+\delta})$ by $O(X)$. \square

Thus we find that (7.4) is bounded by

$$L^K e^{(1+\delta)\pi Lr} \ll L^K e^{(1+\delta)\pi L(K-1)/2}$$

for all $\delta > 0$. Since $\sigma_L \gg e^{(1-\epsilon)\pi L/2}$ for all $\epsilon > 0$, this shows that the sum of the off-diagonal terms is $O(\sigma_L^{K-1+\epsilon})$, for all $\epsilon > 0$. To prove Theorem 7.1 it thus suffices to evaluate the diagonal contributions.

7.3 The diagonal contribution

Assume now that there is the same number of + signs as there are − signs. That is $K = 2k$ is even, and there are $\binom{2k}{k}$ such choices of signs. For simplicity assume the first k are + and the last k are −. Thus we have to evaluate the sum

$$\frac{1}{(\pi L)^{2k}} \sum_{\prod_{j=1}^k \mathcal{N}(n_j) = \prod_{j=k+1}^{2k} \mathcal{N}(n_j)} \prod_{j=1}^{2k} \beta(n_j) \widehat{f}\left(\frac{\log \mathcal{N}(n_j)}{2\pi L}\right). \tag{7.5}$$

There are $k!$ ways to pair off variables from the first k and the last k , such as the pairing $n_j = n_{k+j}$, $1 \leq j \leq k$. Each such pairing contributes a term

$$\frac{1}{(\pi L)^{2k}} \left(\sum_{n>2} \beta(n)^2 \widehat{f}\left(\frac{\log \mathcal{N}(n)}{2\pi L}\right)^2 \right)^k \sim \left(\frac{\sigma_L^2}{2}\right)^k.$$

There are overlaps between the different ways of pairing off variables, which correspond to intersection of diagonals such as $n_1 = n_2 = n_3 = n_4$. The contribution of these was already estimated in the study of the non-diagonal terms, as they correspond to relations (7.3) where some subset has more than two elements.

Thus the total contribution of diagonal terms to $\langle (S_{f,L})^{2k} \rangle_{w,T}$ is asymptotically

$$\binom{2k}{k} \cdot k! \cdot \left(\frac{\sigma_L^2}{2}\right)^k = \frac{(2k)!}{k!2^k} \sigma_L^{2k}.$$

This proves Theorem 7.1. □

8 Conclusion

Since the Gaussian distribution is determined by its moments, Theorem 7.1 implies

Theorem 8.1 *Assume that $L \rightarrow \infty$ as $T \rightarrow \infty$ but $L = o(\log T)$. Then*

$$\lim_{T \rightarrow \infty} \mathbb{P}_{w,T} \left(\frac{N_{f,L}(\tau) - \overline{N_{f,L}}(\tau)}{\sigma_L} < x \right) = \int_{-\infty}^x e^{-u^2/2} \frac{du}{\sqrt{2\pi}},$$

So far we have assume that the weight function w defining the averages is in $C_c^\infty(0, \infty)$. To deduce the results for the standard averages ($w = \mathbf{1}_{[1,2]}$) as in Theorem 2.1, one proceeds by approximating $\mathbf{1}_{[1,2]}$ by “admissible” w ’s in a standard fashion, see, e.g., [18]. The details are as follows: Fix $\epsilon > 0$, and approximate the indicator function $\mathbf{1}_{[1,2]}$ above and below by smooth functions $\chi_\pm \geq 0$ so that $\chi_- \leq \mathbf{1}_{[1,2]} \leq \chi_+$, where both χ_\pm and their Fourier transforms are smooth and of rapid decay, and so that their total masses are within ϵ of unity: $|\int \chi_\pm(x) dx - 1| < \epsilon$. Now set $\omega_\pm := \chi_\pm / \int \chi_\pm$. Then ω_\pm are “admissible” and for all t ,

$$(1 - \epsilon)\omega_-(t) \leq \mathbf{1}_{[1,2]}(t) \leq (1 + \epsilon)\omega_+(t). \tag{8.1}$$

Now

$$\text{meas} \left\{ t \in [T, 2T] : \frac{S_{f,L}(\tau)}{\sigma_L} \in \mathcal{A} \right\} = \int_{-\infty}^{\infty} \mathbb{1}_{\mathcal{A}} \left(\frac{S_{f,L}(\tau)}{\sigma_L} \right) \mathbb{1}_{[1,2]} \left(\frac{t}{T} \right) dt$$

and since (8.1) holds, we find

$$\begin{aligned} (1 - \epsilon) \mathbb{P}_{\omega_-, T} \left\{ \frac{S_{f,L}(\tau)}{\sigma_L} \in \mathcal{A} \right\} &\leq \frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{S_{f,L}(\tau)}{\sigma_L} \in \mathcal{A} \right\} \\ &\leq (1 + \epsilon) \mathbb{P}_{\omega_+, T} \left\{ \frac{S_{f,L}(\tau)}{\sigma_L} \in \mathcal{A} \right\}. \end{aligned}$$

By Theorem 8.1 we find that

$$(1 - \epsilon) \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-x^2/2} dx \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{S_{f,L}(\tau)}{\sigma_L} \in \mathcal{A} \right\}$$

with a similar statement for lim sup; since $\epsilon > 0$ is arbitrary this shows that the limit exists and equals

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{S_{f,L}(\tau)}{\sigma_L} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-x^2/2} dx$$

which proves Theorem 2.1.

The same consideration applies to other positive statistics, such as the number variance.

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