POINCARÉ SERIES

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ABSTRACT

Poincaré Series

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This thesis consists of two parts. In the first, we apply a formula of Petersson's for the Fourier coefficients of Poincaré series, to compute the traces of Hecke operators acting on spaces of cusp forms for the modular group $SL(2, \mathbf{Z})$. We compare the result with that obtained by using the Selberg Trace Formula.

In the second part, we construct a Poincaré series which reproduces the N-th symmetric power L-function for normalized Hecke eigenforms of given weight on $SL(2, \mathbf{Z})$. We show that, if $N = 7$, the initial data for this construction has the imaginary axis as its natural boundary.

Introduction

This thesis, written under the supervision of Professor Ilya Piatetski-Shapiro, investigates certain applications of Poincaré series in the theory of automorphic forms. It consists of two fairly independent chapters, and I refer the reader to the beginning of each chapter for detailed introductions. Here I will give a brief description of the content of each chapter.

In Chapter I, I show how to use a formula of Petersson's for the Fourier coefficients of Poincaré series, to compute the trace of Hecke operators acting on the space $S_k(\Gamma)$ of cusp forms of weight $k > 2$ for the full modular group $\Gamma = SL(2, \mathbf{Z})$. I also discuss the relation between this method and the Selberg Trace Formula.

In Chapter II, I construct, following **[PS]**, a Poincaré series which reproduces the N-th symmetric power L-function $L(s, f, \mathrm{sym}^N)$ for Hecke eigenforms $f \in S_k(\Gamma)$. Somewhat surprisingly, it turns out that in some instances of $N \geq 7$, the initial data for this construction cannot be meromorphically continued to the whole complex s-plane, but instead has the imaginary axis as its natural boundary. The reason for this comes from a problem in Invariant Theory: When does the Hilbert series of invariants of binary forms in N variables have zeros off the unit circle? I discuss this, in a more general setting, in the last section of Chapter II.

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Chapter I. Traces of Hecke operators via Kloosterman sums

One of the first succeses of Selberg's trace formula was the computation of the trace of the Hecke operators T_m acting on the space $S_k(\Gamma)$ of cusp forms of weight $k \geq 2$ for the modular group $\Gamma = SL_2(\mathbf{Z})$ and its congruence subgroups. Selberg reports on this in his seminal article **[Sel]** in the proceedings of the international conference on zeta functions, held at the Tata Institute in 1956.

More or less at the same time, Eichler carried out this computation by viewing the Hecke operators as *correspondences* on the modular curve X_Γ and then using a suitable version of the Lefschetz fixed-point formula. His report **[Ei 1]** also appeared in the proceedings of the Tata conference, where he treated the case of weight 2. For higher weights, he needed a suitable cohomological interpretation of $S_k(\Gamma)$, for which he invented the theory of periods of "Abelian integrals" associated to cusp forms **[Ei 2]**, later refined in **[Sh 1]**.

The formula for the trace of T_m (the "Eichler–Selberg trace formula") is given by:

$$\operatorname{tr} T_m = \delta(\sqrt{m}) m^{\frac{k}{2}-1} \frac{k-1}{12} - \frac{1}{2} \sum_{|t|<2\sqrt{m}} H(t^2-4m) \frac{\rho_t^{k-1} - \bar{\rho}_t^{k-1}}{\rho_t - \bar{\rho}_t} - {\sum_{\substack{d|m \\ d \leq \sqrt{m}}}}' d^{k-1}$$

where $H(d)$ is the class number of positive definite binary quadratic forms of discriminant d, counting forms in the class of $a(x^2+y^2)$ or $a(x^2+xy+y^2)$ with multiplicity $1/2$ (respectively $1/3$); $\sum'$ means that if m is a perfect square, we count the summand corresponding to $d = \sqrt{m}$ with multiplicity $1/2$; ρ_t is a solution of $x^2 - tx + m = 0$; and, finally,

$$\delta(x) = \begin{cases} 1, & x \in \mathbf{Z} \\ 0, & \text{otherwise.} \end{cases}$$

Formulas for the traces of Hecke operators played a crucial role in the solution of the "basis problem" for modular forms, i.e. finding a basis of $S_k(\Gamma_0(N))$ in terms of theta series **[Ei 3]**. Such formulas were also used to get various other lifting theorems for automorphic forms, e.g. Saito's work on cyclic base–change **[Sai]**. Another application is to give a unified treatment of many of the "class number relations" of classical number theory, as well as generating some new ones.

In this thesis, I will sketch yet a third approach to the proof of the Eichler–Selberg trace formula. There are three key ingredients: The first is Petersson's classical formula (slightly generalized to account for the action of the Hecke operators) for the Fourier coefficient of Poincaré series **[Pe]**:

$$\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \sum_i |a_i(n)|^2 = 1 + 2\pi i^{-k} \sum_{c>0} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c})$$

where $a_i(n)$ are the n-th Fourier coefficients of an orthonormal basis of $S_k(\Gamma)$.

Petersson's formula is then fed into the Rankin–Selberg method, to show that the trace of the Hecke operator T_m is essentialy given by the residue at $s = 1$ of the Dirichlet series:

$$D(s) = D_{k,m}(s) = \sum_{n\geq 1} \frac{1}{n^s} \sum_{c>0} \frac{T_m S(n,n;c)}{c} J_{k-1}(4\pi\sqrt{m}\frac{n}{c}).$$

We then proceed to evaluate this residue; for simplicity I take $m = 1$. Working in the region of absolute convergence of the double sum for $D(s)$, we expand the Kloosterman sums in a finite Fourier series:

$$S(n,n;c) = \sum_{t \mod c} \nu(c,t) e(\frac{n}{c}t)$$

and using Poisson summation, show that:

$$D(s) = \sum_{t\in\mathbf{Z}} F(t,s)\mathcal{L}(t,s)$$

$$\mathcal{L}(t,s) = \sum_{c>0} \frac{\nu(c,t)}{c^s}, \qquad F(t,s) = \int_0^\infty J_{k-1}(4\pi x) e(tx) \frac{dx}{x^s}.$$

The summands correspond to the *conjugacy classes* of $SL_2(\mathbf{R})$, the parameter t being the *trace* of the corresponding conjugacy class; of course the case $t = \pm 2$ is exceptional, all other values of t giving regular semi-simple classes.

$\mathcal{L}(t,s)$ is essentially the Dedekind ζ-function of the quadratic extension corresponding to the given conjugacy class, and $F(t,s)$ is some transcendental object— a Fourier–Mellin transform of the Bessel function $J_{k-1}(x)$. Remembering that one is actually working on $PSL_2(\mathbf{R})$, we lump together terms coming from conjugacy classes of $SL_2(\mathbf{R})$ which coalesce in $PSL_2(\mathbf{R})$, i.e. those with trace of fixed absolute value. We then see that all the *hyperbolic* classes give zero contribution, which one expects from experience with Selberg's trace formula. I feel that, in our case too, this is a manifestation of "Selberg's principle": Vanishing of orbital integrals for discrete series representations on non–elliptic regular semi–simple conjugacy classes. This fact was conjectured by Selberg in the same Tata paper **[Sel]** cited earlier, and proved by Harish–Chandra in 1966 **[HC]**.

Finally, we are left with a finite sum which involves the value at $s = 1$ of Dirichlet L-functions; the transition to the Eichler–Selberg formula is then completed by invoking Dirichlet's class number formula.

It is curious to note that all the ingredients of our computation were available about fifteen years prior to the Tata conference, yet this approach seems to have been overlooked by the people working on the problem of computing traces of Hecke operators. I am happy to be able to fill in this gap in the history of the subject.

A very similar approach was used by Mizumoto **[M]**,[1] who, following a question posed by Zagier, used Petersson's formula to give another proof of the entirety of the symmetric square L-function of $f \in S_k(\Gamma)$. This was first proved by Shimura **[Sh 2]**. Following Zagier **[Z]**, Mizumoto uses a Poincaré series to represent the symmetric square L-function, rather than the Rankin-Selberg construction of Shimura. I refer the reader to Chapter II of this thesis for a general construction of such Poincaré series for higher symmetric powers.

[1] I thank Bill Duke for bringing **[M]** and **[Z]** to my attention.

§1. The Rankin-Selberg Method

We start by recalling a few definitions: Let $S_k(\Gamma)$ be the space of cusp forms of weight k for the modular group $\Gamma = SL_2(\mathbf{Z})$, with k an even integer. These are holomorphic functions $f(z)$ on the upper half-plane $\mathfrak{H} = \{z : \mathrm{Im}(z) > 0\}$ which satisfy:

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and have a Fourier expansion: $f(z) = \sum_{n\geq 1} a(n)e(nz)$, where we use $e(x)$ for $e^{2\pi i x}$.

$S_k(\Gamma)$ has a hermitian structure given by the Petersson inner product:

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}y^k \frac{dx\,dy}{y^2} \qquad f, g \in S_k(\Gamma). \tag{1.1}$$

Hecke's operator $T_m : S_k(\Gamma) \longrightarrow S_k(\Gamma)$, $m \geq 1$, is given by:

$$T_m f = m^{\frac{k}{2}-1} \sum_{\alpha\in\Gamma\backslash\mathcal{M}(m)} f|_{[\alpha]_k} \tag{1.2}$$

with: $f|_{[\alpha]_k}(z) = (\det\alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha z)$ for $\alpha \in GL_2(\mathbf{Q})$, and $\mathcal{M}(m) = 2\times 2$ integral matrices of determinant m.

To compute the trace of T_m on $S_k(\Gamma)$, $k > 2$, we take an orthonormal basis of $S_k(\Gamma)$ and use:

$$\mathrm{tr}(T_m) = \sum_i \langle T_m f_i, f_i\rangle$$

We have:

$$\langle T_m f, f\rangle = \int_{\Gamma\backslash\mathfrak{H}} T_m f(z)\overline{f(z)}y^k \frac{dx\,dy}{y^2} \quad . \tag{1.3}$$

We now bring in the Rankin-Selberg method: Let

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \tag{1.4}$$

be the non-holomorphic Eisenstein series. $E(z,s)$ converges for $\mathrm{Re}(s) > 1$, has meromorphic continuation to all of the s-plane with a simple pole at $s = 1$, where it has *constant* residue:

$$\operatorname*{Res}_{s=1} E(z,s) = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})} = \frac{3}{\pi}.$$

Therefore, if $f, g \in S_k(\Gamma)$, then the *Rankin-Selberg convolution L-function* :

$$(1.5) \qquad L(f \otimes g, s) = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}E(z,s)y^k \frac{dx\,dy}{y^2}$$

has meromorphic continuation and (at most) a simple pole at $s = 1$, with:

$$\operatorname*{Res}_{s=1} L(f \otimes g, s) = \frac{1}{\operatorname{vol}(\Gamma\backslash\mathfrak{H})}\langle f, g\rangle.$$

We see then that:

$$(1.6) \qquad \frac{1}{\operatorname{vol}(\Gamma\backslash\mathfrak{H})} \operatorname{tr} T_m = \operatorname*{Res}_{s=1} \sum_i L(T_m f_i \otimes f_i, s).$$

On the other hand, if $f(z) = \sum_{n\geq 1} a(n)e(nz)$, and $g(z) = \sum_{n\geq 1} b(n)e(nz)$, then we have:

$$(1.7) \qquad L(f \otimes g, s) = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n\geq 1} \frac{a(n)\overline{b(n)}}{n^{s+k-1}}.$$

Therefore, if $f_i(z) = \sum_{n\geq 1} a_i(n)e(nz)$ are our orthonormal basis of $S_k(\Gamma)$, and:

$$T_m f_i(z) = \sum_{n\geq 1} T_m a_i(n)e(nz),$$

then:

$$(1.8) \qquad \frac{1}{\operatorname{vol}(\Gamma\backslash\mathfrak{H})} \operatorname{tr} T_m = \operatorname*{Res}_{s=1} \sum_{n\geq 1} \frac{\Gamma(s+k-1)}{(4\pi n)^{s+k-1}} \sum_i T_m a_i(n)\overline{a_i(n)}.$$

Our next step is to replace the hermitian form $\sum_i T_m a_i(n)\overline{a_i(n)}$ by a different, more explicit, quantity.

For this, recall the Poincaré series

$$(1.9) \qquad P_k(z,n) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{-k} e(n\gamma z)$$

converging absolutely if $k > 2$. This is a cusp form of weight k, and represents the n-th Fourier coefficient, in the sense that:

$$(1.10) \qquad \langle f, P_k(\cdot, n)\rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a(n), \qquad f(z) = \sum_{n\geq 1} a(n)e(nz) \in S_k(\Gamma).$$

So, if we expand $P_k(z,n)$ in terms of our orthonormal basis $\{f_i\}$, we find:

$$P_k(z,n) = \sum_i \langle P_k(\cdot,n), f_i\rangle f_i(z) = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \sum_i \overline{a_i(n)} f_i(z),$$

and so:

$$T_m P_k(z,n) = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \sum_i \overline{a_i(n)} T_m f_i(z). \tag{1.11}$$

This shows that the n-th Fourier coefficient of $T_m P_k(z,n)$ is given by:

$$T_m \widehat{P_{k,n}}(n) = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \sum_i T_m a_i(n) \overline{a_i(n)} \tag{1.12}$$

–which is precisely the hermitian form appearing in the formula for $\operatorname{tr} T_m$!

We can now compute the Fourier coefficient $T_m \widehat{P_{k,n}}(n)$ from the definition of $T_m P_k(z,n)$ as the sum of an infinite series. For $m = 1$, this computation is due to Petersson [**Pe**]:

$$\widehat{P_{k,n}}(n) = 1 + 2\pi i^{-k} \sum_{c>0} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c})$$

and similarly we have:

Lemma 1.1.

$$\begin{aligned} \widehat{T_m P}_{k,n}(n') =& 2\pi i^{-k} m^{\frac{k-1}{2}} \left(\frac{n'}{n}\right)^{\frac{k-1}{2}} \sum_{c>0} \frac{T_m S(n,n';c)}{c} J_{k-1}(4\pi \sqrt{m} \frac{\sqrt{nn'}}{c}) \\ &+ \begin{cases} m^{\frac{k-1}{2}} \left(\frac{n'}{n}\right)^{\frac{k-1}{2}}, & nm = n'd^2 \textit{ for some integer } d \mid (n,m) \\ 0, & \textit{otherwise} \end{cases} \end{aligned} \tag{1.13}$$

with:

$$\delta(x) = \begin{cases} 1, & x \in \mathbf{Z} \\ 0, & \textit{otherwise} \end{cases}$$

and:

$$T_m S(n,n';c) = \sum_{\substack{a,d \bmod c \\ ad \equiv m \bmod c}} e(\frac{na + n'd}{c}) \tag{1.14}$$

is a modified Kloosterman sum.

PROOF: We have:

$$
\begin{aligned}
T_m P_{k,n}(z) &= m^{\frac{k}{2}-1} \sum_{\alpha\in\Gamma\backslash\mathcal{M}(m)} (\det\alpha)^{\frac{k-1}{2}} j(\alpha,z)^{-k} P_{k,n}(\alpha z) \\
&= m^{k-1} \sum_{\alpha\in\Gamma\backslash\mathcal{M}(m)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\alpha,z)^{-k} j(\gamma,\alpha z)^{-k} e(n\gamma\alpha z) \\
&= m^{k-1} \sum_{\Gamma_\infty\backslash\mathcal{M}(m)} j(\gamma,z)^{-k} e(n\gamma z)
\end{aligned}
$$

where we have used the cocycle relation: $j(\alpha,z)j(\gamma,\alpha z) = j(\gamma\alpha,z)$. We now follow the derivation of Petersson's formula. We divide the sum over $\Gamma_\infty\backslash\mathcal{M}(m)$ into two parts, one coming from γ in the small Bruhat cell of $GL_2(\mathbf{R})$, i.e. those γ of the form: $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)$, and the other coming from the big Bruhat cell:

$$
\widehat{T_m P_{k,n}}(n') = \int_0^1 T_m P_{k,n}(z) e(-n'z)dx = \text{small cell contribution } + \text{big cell contribution}
$$

We first compute the contribution of the small cell.

$$
\begin{aligned}
\text{small cell} &= m^{k-1} \int_0^1 \sum_{ad=m} \sum_{0\le b<d} j(\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right), z)^{-k} e(n\frac{az+b}{d}) e(-n'z)dx \\
&= m^{k-1} \sum_{ad=m} d^{-k} \sum_{0\le b<d} e(n\frac{b}{d}) \int_0^1 e(n\frac{a}{d}z - n'z)dx \\
&= m^{k-1} \sum_{ad=m} d^{-k} \cdot \begin{Bmatrix} 0, & d\nmid n \\ 1, & d\mid n \end{Bmatrix} \cdot \begin{cases} 0, & na\neq n'd \\ 1, & na = n'd \end{cases} \\
&= m^{k-1} \sum_{\substack{ad=m \\ d\mid n}} d^{1-k} \cdot \begin{cases} 0, & na\neq n'd \\ 1, & na = n'd \end{cases} \\
&= m^{k-1} \sum_{\substack{d\mid(n,m) \\ nm=n'd^2}} d^{1-k} = \begin{cases} m^{\frac{k-1}{2}} \left(\frac{n'}{n}\right)^{\frac{k-1}{2}}, & nm = n'd^2 \text{ for some } d\mid(n,m) \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}
$$

In particular, if $n = n'$, we find:

$$
\text{small cell} = \begin{cases} m^{\frac{k-1}{2}}, & m = d^2 \text{ with } d\mid n \\ 0, & \text{otherwise} \end{cases}.
$$

Now for the contribution of the big cell; for $c > 0$, we sum over all double cosets $\Gamma_\infty\backslash\mathcal{M}(m)/\Gamma_\infty$ for which the lower left-hand corner of the matrices equals c.

$$
\text{big cell} = m^{k-1} \sum_{c>0} \int_0^1 \sum_{\Gamma_\infty\backslash\mathcal{M}(m,c)} j(\gamma,z)^{-k} e(n\gamma z) e(-n'z)dx
$$

where:

$$\mathcal{M}(m,c) = \left\{ \gamma = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \mathcal{M}(m) \right\}.$$

We will need to know that as representatives of the double cosets of $\Gamma_\infty \backslash \mathcal{M}(m,c)/\Gamma_\infty$ we can take:

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \mod c, \quad ad \equiv m \mod c \right\}.$$

Now fix $c > 0$; then the sum corresponding to c above is:

$$m^{k-1} \int_{-\infty}^{\infty} \sum_{\gamma \in \Gamma_\infty \backslash \mathcal{M}(m,c)/\Gamma_\infty} j(\gamma,z)^{-k} e(n\gamma z) e(-n'z) dx$$

$$= m^{k-1} \sum_{\substack{a,d \mod c \\ ad \equiv m \mod c}} \int_{-\infty}^{\infty} (cz+d)^{-k} e(n\frac{az+b}{cz+d}) e(-n'z) dx$$

writing $(az+b)/(cz+d) = a/c - m/c(cz+d)$,

$$= m^{k-1} \sum_{\substack{a,d \mod c \\ ad \equiv m \mod c}} e(n\frac{a}{c}) \int_{-\infty}^{\infty} (cz+d)^{-k} e(\frac{-nm}{c(cz+d)} - n'z) dx$$

$$= m^{k-1} \sum_{\substack{a,d \mod c \\ ad \equiv m \mod c}} e(\frac{na+n'd}{c}) \int_{-\infty}^{\infty} (cz)^{-k} e(-\frac{nm}{c^2 z} - n'z) dx$$

$$= m^{k-1} T_m S(n,n';c) \frac{1}{c} \int_{-\infty}^{\infty} z^{-k} e(-\frac{nm/c}{z} - \frac{n'}{c} z) dx$$

$$= m^{\frac{k-1}{2}} \left(\frac{n'}{n}\right)^{\frac{k-1}{2}} \frac{T_m S(n,n';c)}{c} 2\pi i^{-k} J_{k-1}(4\pi\sqrt{m}\frac{\sqrt{nn'}}{c}).$$

This proves our formula. ∎

Lemma 1.1 shows that we have an equality:

(1.15)

$$\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \sum_i T_m a_i(n) \overline{a_i(n)} = \delta(\sqrt{m}) \delta(\frac{n}{\sqrt{m}}) m^{\frac{k-1}{2}} + 2\pi i^{-k} m^{\frac{k-1}{2}} \sum_{c>0} \frac{T_m S(n,n;c)}{c} J_{k-1}(4\pi\sqrt{m}\frac{n}{c})$$

We can now substitute this equality into our formula (1.8) for $\operatorname{tr} T_m$ to see that this trace is given by $\operatorname{vol}(\Gamma\backslash\mathfrak{H})$ times the residue at $s = 1$ of the Dirichlet series:

(1.16)

$$m^{\frac{k-1}{2}} \frac{\Gamma(s+k-1)}{(4\pi)^s \Gamma(k-1)} \sum_{n \geq 1} \frac{1}{n^s} \left\{ \delta(\sqrt{m}) \delta(\frac{n}{\sqrt{m}}) + 2\pi i^{-k} \sum_{c>0} \frac{T_m S(n,n;c)}{c} J_{k-1}(4\pi\sqrt{m}\frac{n}{c}) \right\}$$

and so we find:

$$\frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}\,\mathrm{tr}\,T_m = m^{\frac{k}{2}-1}\frac{k-1}{4\pi}\delta(\sqrt{m}) + 2\pi i^{-k} m^{\frac{k-1}{2}}\frac{k-1}{4\pi}\operatorname*{Res}_{s=1} D_{k,m}(s), \tag{1.17}$$

where we set:

$$D_{k,m}(s) = \sum_{n\geq 1}\frac{1}{n^s}\sum_{c>0}\frac{T_m S(n,n;c)}{c} J_{k-1}(4\pi\sqrt{m}\frac{n}{c}). \tag{1.18}$$

In the following sections I shall explain how to explicitly evaluate this residue.

§2. Computing $\mathrm{Res}_{s=1} D(s)$

We now shift our attention to computing the residues of the Dirichlet series appearing in §1. This task seemingly has nothing to do with the theory of automorphic forms, in the sense that it is completely elementary, requiring nothing more involved than some basic properties of Bessel functions of integral order. For simplicity, we take $m = 1$ for the rest of this section, i.e we are computing the *dimension* of $S_k(\Gamma)$. As in §1, we set:

$$D(s) = \sum_{n\geq 1} \frac{1}{n^s} \sum_{c>0} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c}). \tag{2.1}$$

We need to know the asymptotics of $J_\ell(x)$, for ℓ an odd integer. These are given by [W]:

$$J_\ell(x) \sim \frac{x^\ell}{2^\ell \Gamma(l+1)}, \qquad \text{as } x \to 0$$

$$J_\ell(x) \ll x^{-1/2}, \qquad \text{as } x \to \infty.$$

From this, it is easy to see that the sum for $D(s)$ converges absolutely for $\mathrm{Re}(s) > 2$. Recall also that if ℓ is an *odd* integer, then $J_\ell(-x) = -J_\ell(x)$.

Before proceeding, we expand the Kloosterman sums $S(n,n;c)$ in a Fourier series:

$$S(n,n;c) = \sum_{t \mod c} \nu(c,t) e(\frac{n}{c}t) \tag{2.2}$$

where:

$$\nu(c,t) = \# \left\{ a,d \mod c : \begin{matrix} ad \equiv 1 \mod c \\ a+d \equiv t \mod c \end{matrix} \right\}. \tag{2.3}$$

We also define:

$$F(t,s) = \int_0^\infty J_{k-1}(4\pi x) e(tx) \frac{dx}{x^s}. \tag{2.4}$$

Lemma 2.1. *For $1/2 < \mathrm{Re}(s) < k-1$, we have:*

$$\sum_{n=1}^\infty \frac{1}{n^s} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c}) = c^{-s} \sum_{t\in\mathbf{Z}} \nu(c,t) F(t,s)$$

Then for $\mathrm{Re}(s) > 1/2$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c}) = c^{-s} \sum_{n=1}^{\infty} \frac{S(n,n;c)}{c} f_s(\frac{n}{c})$$
$$= c^{-s} \sum_{u=1}^{c} \frac{S(u,u;c)}{c} \sum_{r=0}^{\infty} f_s(r + \frac{u}{c}).$$

We now use Poisson summation: For $1/2 < \mathrm{Re}(s) < k-1$, $0 < y \leq 1$,

$$\sum_{r=0}^{\infty} f_s(r+y) = \sum_{t \in \mathbf{Z}} \widehat{f}_s(t) e(-ty) \tag{2.5}$$

and both sides converge uniformly in y. Indeed, from the asymptotics of $J_{k-1}(4\pi x)$, we have:

$$|f_s(x)| \ll x^{-\mathrm{Re}(s) - 1/2}, \qquad \text{as } x \to \infty,$$
$$|f_s(x)| \asymp x^{k-1-\mathrm{Re}(s)}, \qquad \text{as } x \to 0,$$

and one easily sees:

$$|\widehat{f}_s(t)| \ll |t|^{-(k - \mathrm{Re}(s))}, \qquad \text{as } |t| \to \infty.$$

Therefore, the RHS of (2.5) converges uniformly for $\mathrm{Re}(s) < k-1$, while the LHS converges uniformly for $\mathrm{Re}(s) > 1/2$. Thus, for $1/2 < \mathrm{Re}(s) < k-1$, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c}) = c^{-s} \sum_{u=1}^{c} \frac{S(u,u;c)}{c} \sum_{t \in \mathbf{Z}} \widehat{f}_s(t) e(-t\frac{u}{c})$$
$$= c^{-s} \sum_{t \in \mathbf{Z}} F(t,s) \nu(c,t). \qquad \blacksquare$$

Set, for $t \in \mathbf{Z}$:

$$\mathcal{L}(t,s) = \sum_{c>0} \frac{\nu(c,t)}{c^s}.$$

Corollary 2.2. *For $2 < \mathrm{Re}(s) < k-1$, we have:*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{c>0} \frac{S(n,n;c)}{c} J_{k-1}(4\pi \frac{n}{c}) = \sum_{t \in \mathbf{Z}} F(t,s) \mathcal{L}(t,s), \tag{2.6}$$

with the RHS absolutely convergent for $\mathrm{Re}(s) > 1$.

PROOF: (2.6) follows from Lemma 2.1. From the explicit evaluation of $\nu(c,t)$ in Lemmas 3.2, 3.4, we see that if $|t| > 2$, then:

$$\nu(c,t) \ll (t^2 - 4)^{1/2} d(c), \qquad d(c) = \text{divisor function}.$$

with the RHS absolutely convergent for $\mathrm{Re}(s) > 1$.

PROOF: (2.6) follows from Lemma 2.1. From the explicit evaluation of $\nu(c,t)$ in Lemmas 3.2, 3.4, we see that if $|t| > 2$, then:

$$\nu(c,t) \ll (t^2-4)^{1/2} d(c), \qquad d(c) = \text{divisor function}.$$

Therefore, if $|t| > 2$, then for s real, $s > 1$:

$$\mathcal{L}(t,s) \ll t^{1/2} \sum_{c>0} \frac{d(c)}{c^s}.$$

Together with the estimate $|F(t,s)| \ll |t|^{\mathrm{Re}(s)-k}$ as $|t| \to \infty$, we see:

$$\sum_{|t|>2} F(t,s)\mathcal{L}(t,s) \ll \sum_{t>2} t^{1/2+\mathrm{Re}(s)-k} \sum_{c>0} \frac{d(c)}{c^s},$$

which converges for $1 < \mathrm{Re}(s) < k - 3/2$. ∎

We are now left with computing the residue at $s = 1$ of the series:

$$D(s) = \sum_{t\in\mathbf{Z}} F(t,s)\mathcal{L}(t,s).$$

The Dirichlet series $\mathcal{L}(t,s)$ have Euler products, and will be computed in §3. For now, we just list a few of them.

Example.

$$\mathcal{L}(0,s) = \frac{\zeta(s)}{\zeta(2s)} L(\chi_{-4}, s) \tag{2.7}$$

where χ_d is the quadratic character $\chi_d(p) = \left(\frac{d}{p}\right)$ given by the Legendre symbol. Similarly, we have:

$$\mathcal{L}(1,s) = \frac{\zeta(s)}{\zeta(2s)} L(\chi_{-3}, s) \tag{2.8}$$

$$\mathcal{L}(2,s) = \frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)} \tag{2.9}$$

In general (see §3), one finds that up to a finite Euler product (coming from some "bad" primes), we have:

$$\mathcal{L}(t,s) \sim \frac{\zeta(s)}{\zeta(2s)} L(\chi_{t^2-4}, s), \qquad t \neq \pm 2. \tag{2.10}$$

Note also that $\mathcal{L}(-t,s) = \mathcal{L}(t,s)$.

Next observe that from (2.10), $\mathcal{L}(t,s)$ has a simple pole at $s = 1$, while (2.9) shows that $\mathcal{L}(2,s)$ has a *double* pole at that point. This should cause some worrying, since the Rankin-Selberg method yields a *simple* pole. We are thus led into writing:

$$\operatorname*{Res}_{s=1} D(s) = \operatorname*{Res}_{s=1} \mathcal{L}(0,s)F(0,s) + \operatorname*{Res}_{s=1} \sum_{t>0} \mathcal{L}(t,s)F_{\pm}(t,s) \tag{2.11}$$

Where I set $F_{\pm}(t,s) = F(t,s) + F(-t,s), \quad t \neq 0$.

One can easily see, that in (2.11), we can take a residue term-by-term. We will now examine the various terms in (2.11). Before proceeding, recall that we have parametrized the summands in (2.11) by the parameter t; it came from taking the *trace* of an element of $SL_2(\mathbf{Z})$:

$$t \equiv a + d = \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod c.$$

Recall that conjugacy classes in $SL_2(\mathbf{R})$ are parametrized by traces; if $\gamma \neq \pm I$,then:

(1) $|\operatorname{tr}\gamma| > 2$ are *hyperbolic* classes;

(2) $|\operatorname{tr}\gamma| < 2$ are *elliptic* classes;

(3) $|\operatorname{tr}\gamma| = 2$ are *unipotent* classes.

Recall, furthermore, that orbital integrals for *discrete series* representations *vanish* on non-elliptic semi-simple conjugacy classes **[HC]**; this should intimate to us that in (2.11), the summands with $t > 2$ should vanish. This is indeed the case; to see this, it is enough to show that $F_{\pm}(t,1) = 0$ if $t > 2$, since $\mathcal{L}(t,s)$ has only a simple pole at $s = 1$.

There are (at least) two ways to see this: Since $F_{\pm}(t,s)$ is given explicitly, one should be able to compute it and check its behaviour at $s = 1$. This is indeed possible, using the Weber–Schafheitlein integral for the Mellin transform of a product of two Bessel functions; we shall come back to this later. However, from what has been said above, one should be able to come up with a "soft" proof that $F_{\pm}(t,1) = 0$ for $t > 2$. In fact, this follows from the Paley-Wiener theorem, which describes the image of the Fourier transform acting on compactly-supported functions on the real line. We have:

$$F_{\pm}(t,1) = \int_0^{\infty} J_{k-1}(4\pi x)\,[e(xt) + e(-xt)]\,\frac{dx}{x} = \int_{-\infty}^{\infty} \frac{J_{k-1}(4\pi x)}{x} e(xt)dx$$

and so $F_\pm(t,1)$ is the Fourier transform of the *entire* function $z^{-1}J_{k-1}(4\pi z)$, which is, moreover, of *exponential type*:

$$\left|\frac{J_{k-1}(4\pi z)}{z}\right| \ll e^{|z|}$$

—and so, by Paley-Wiener, is supported inside the interval $|t| \leq 2$. This shows that the residue in (2.11) is a sum of three terms, all of which can be computed explicitly.

Explicit Computations. To summarize, we have so far seen:

$$\begin{aligned}\frac{\dim S_k(\Gamma)}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})} &= \frac{k-1}{4\pi}\operatorname*{Res}_{s=1}\zeta(s)\\ &+ \frac{k-1}{2}i^{-k}\operatorname*{Res}_{s=1}\{F(0,s)\mathcal{L}(0,s)+F_\pm(1,s)\mathcal{L}(1,s)+F_\pm(2,s)\mathcal{L}(2,s)\}\end{aligned} \tag{2.12}$$

I will write this in the following suggestive manner:

$$\begin{aligned}\dim S_k(\Gamma) &= \frac{k-1}{12} && \text{the identity term}\\ &+ i^{-k}\frac{(k-1)\pi}{6}\operatorname*{Res}_{s=1}F(0,s)\mathcal{L}(0,s) && \left.\right\} \text{elliptic terms}\\ &+ i^{-k}\frac{(k-1)\pi}{6}\operatorname*{Res}_{s=1}F_\pm(1,s)\mathcal{L}(1,s) && \\ &+ i^{-k}\frac{(k-1)\pi}{6}\operatorname*{Res}_{s=1}F_\pm(2,s)\mathcal{L}(2,s) && \text{parabolic term}\end{aligned} \tag{2.13}$$

I will now briefly indicate how the explicit computation of the terms in (2.13) is carried out:

First, we have [**W**, p. 391]:

$$F(0,s) = \int_0^\infty J_{k-1}(4\pi x)\frac{dx}{x^s} = (4\pi)^{s-1}\frac{\Gamma(\frac{k-s}{2})}{2^s\Gamma(\frac{k+s}{2})}$$

and so at $s=1$ we find $F(0,1) = \frac{1}{k-1}$.

Second, we use the discontinuous integral of Weber–Schafheitlein in the special form [**W**, p. 405]:

$$\int_0^\infty J_\mu(ax)\cos(bx)\frac{dx}{x} = \begin{cases}\mu^{-1}\cos(\mu\arcsin\frac{b}{a}) & , \quad b \leq a\\ \dfrac{a^\mu\cos\frac{\mu\pi}{2}}{\mu(b+\sqrt{b^2-a^2})^\mu} & , \quad b \geq a\end{cases} \qquad (\mathrm{Re}(\mu) > -1)$$

to see that:

$$F_\pm(1,1) = \frac{2\cos\frac{(k-1)\pi}{6}}{k-1} = \begin{cases} \dfrac{\sqrt{3}}{k-1}, & k \equiv 0,2 \mod 12 \\ 0, & k \equiv 4,10 \mod 12 \\ -\dfrac{\sqrt{3}}{k-1}, & k \equiv 6,8 \mod 12. \end{cases}$$

To compute the residue at $s = 1$ of $\mathcal{L}(0,s)$ and $\mathcal{L}(1,s)$, we use (2.7), (2.8) and Dirichlet's class number formula to get:

$$\operatorname*{Res}_{s=1} \mathcal{L}(0,s) = \frac{3}{2\pi}$$
$$\operatorname*{Res}_{s=1} \mathcal{L}(1,s) = \frac{2}{\sqrt{3}\pi}$$

and so we find that the elliptic terms contribute:

$$\begin{aligned} i^{-k}\frac{(k-1)\pi}{6} \operatorname*{Res}_{s=1} F(0,s)\mathcal{L}(0,s) &= \frac{i^{-k}}{4} \\ i^{-k}\frac{(k-1)\pi}{6} \operatorname*{Res}_{s=1} F_\pm(1,s)\mathcal{L}(1,s) &= \frac{2i^{-k}\cos\frac{(k-1)\pi}{6}}{3\sqrt{3}} \end{aligned} \tag{2.14}$$

For the parabolic term in (2.13), recall (2.9):

$$\mathcal{L}(2,s) = \frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)} \tag{2.15}$$

which shows $\mathcal{L}(2,s)$ has a *double* pole at $s = 1$. We compute

$$F_\pm(2,s) = 2\int_0^\infty J_{k-1}(4\pi x)\cos(4\pi x)\frac{dx}{x^s}$$

again by using the Weber–Schafheitlein integral [W, p. 403]:

$$\int_0^\infty J_\mu(at)J_\nu(at)\frac{dt}{t^\lambda} = \frac{\left(\frac{a}{2}\right)^{\lambda-1}\Gamma(\lambda)\Gamma(\frac{\mu+\nu-\lambda+1}{2})}{2\Gamma(\frac{\lambda+\nu-\mu+1}{2})\Gamma(\frac{\lambda+\nu+\mu+1}{2})\Gamma(\frac{\lambda+\mu-\nu+1}{2})}$$
$$\operatorname{Re}(\mu+\nu+1) > \operatorname{Re}(\lambda) > 0$$

to find (remember $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$):

$$F_\pm(2,s) = \frac{2^{s-1}\pi^{s-\frac{1}{2}}\Gamma(s-\frac{1}{2})\Gamma(\frac{k-s}{2})}{\Gamma(\frac{s+1-k}{2})\Gamma(\frac{s+k-1}{2})\Gamma(\frac{k+s}{2})} \tag{2.16}$$

Notice that this has a simple zero at $s = 1$, since $\Gamma(\frac{s+1-k}{2})$ has a simple pole at $s = 1$. Using (2.15) and (2.16), it is now a simple matter to check that

$$\operatorname*{Res}_{s=1} F_{\pm}(2,s)\mathcal{L}(2,s) = -\frac{3i^k}{(k-1)\pi} \tag{2.17}$$

We can now write (2.13) using (2.14) and (2.17) as follows:

$$\begin{aligned} \dim S_k(\Gamma) = {} & \frac{k-1}{12} && \text{identity term} \\ & + \frac{i^{-k}}{4} + \frac{2i^{-k}\cos\frac{(k-1)\pi}{6}}{3\sqrt{3}} && \text{elliptic terms} \\ & - \frac{1}{2} && \text{parabolic term} \end{aligned} \tag{2.18}$$

This is the classical formula for $\dim S_k(\Gamma)$.

In this section, I shall extend the methods of the previous section to compute $\operatorname{tr} T_m$ for $m > 1$. There will be one new feature – appearance of extra poles coming from the fixed-point at the cusp of the correspondence T_m.

By (1.17), we know that $\operatorname{tr} T_m$ is given by:

$$\operatorname{tr} T_m = \delta(\sqrt{m}) m^{\frac{k}{2}-1} \cdot \frac{k-1}{12} + \frac{\pi i^k m^{\frac{k-1}{2}}}{6} \operatorname*{Res}_{s=1} D_m(s), \tag{3.1}$$

where as in (1.18), $D_m(s)$ is given by:

$$D_m(s) = \sum_{n \geq 1} \frac{1}{n^s} \sum_{c>0} \frac{T_m S(n,n;c)}{c} J_{k-1}(4\pi\sqrt{m}\frac{n}{c}).$$

We now expand the Kloosterman sum $T_m S(n,n;c)$ in a Fourier series:

$$T_m S(n,n;c) = \sum_{t \bmod c} \nu_m(c,t) e(\frac{n}{c} t)$$

$$\nu_m(c,t) = \# \left\{ a, d \mod c \;\middle|\; \begin{matrix} a + d \equiv t \mod c \\ ad \equiv m \mod c \end{matrix} \right\} \tag{3.2}$$

Using Poisson summation as in the case $m = 1$, we find:

$$D_m(s) = m^{(s-1)/2} \sum_{t \in \mathbf{Z}} F(\frac{t}{\sqrt{m}}, s) \mathcal{L}_m(t,s)$$

where $F(t,s)$ is given as in (2.4), and:

$$\mathcal{L}_m(t,s) = \sum_{c>0} \frac{\nu_m(c,t)}{c^s} \tag{3.3}$$

Thus we have:

$$\begin{aligned} \operatorname*{Res}_{s=1} D_m(s) &= \operatorname*{Res}_{s=1} \sum_{t \in \mathbf{Z}} F(\frac{t}{\sqrt{m}}, s) \mathcal{L}_m(t,s) \\ &= \sum_{t \geq 0} \operatorname*{Res}_{s=1} F_{\pm}(\frac{t}{\sqrt{m}}, s) \mathcal{L}_m(t,s). \end{aligned} \tag{3.4}$$

Here, as in the case $m = 1$, we paired off the summands with fixed absolute value; as we saw in §2, $F_{\pm}(\frac{t}{\sqrt{m}}, 1) = 0$ if $|t| \geq 2\sqrt{m}$. For $m = 1$, there were no contributions to

(3.4) from the terms with $t > 2\sqrt{m} = 2$; however, if $m > 1$, we will see that there will be such a contribution from finitely many of these terms. This contribution corresponds to the cusp $i\infty$ being a fixed point for the correspondence T_m. To see all this, we have to compute $\mathcal{L}_m(t,s)$ for various values of t. For most t, we will be content to do the exact computation only for "good" primes.

Computing $\mathcal{L}_m(t,s)$. We start by observing that, for any t, $\nu_m(c,t)$ is a multiplicative function of c:

$$\nu_m(pq,t) = \nu_m(p,t)\nu_m(q,t), \qquad \text{if } \gcd(p,q) = 1.$$

This follows from the Chinese Remainder Theorem, and in fact is equivalent to "twisted multiplicativity" of the Kloosterman sums. Therefore $\mathcal{L}_m(t,s)$ has an Euler product:

$$\mathcal{L}_m(t,s) = \prod_p \mathcal{L}_m^p(t,s),$$

$$\mathcal{L}_m^p(t,s) = 1 + \sum_{k\geq 1} \frac{\nu_m(p^k,t)}{p^{ks}}.$$

Lemma 3.1. *Let $\Delta = t^2 - 4m \neq 0$, and p be a prime such that $p \nmid \Delta$. Then:*

$$\mathcal{L}_m^p(t,s) = (1-p^{-2s})(1-p^{-s})^{-1}(1-\left(\frac{\Delta}{p}\right)p^{-s})^{-1}.$$

Proof: First suppose p is odd; as a consequence of Hensel's lemma, we see that for $k \geq 2$, $\nu_m(p^k,t) = \nu_m(p,t)$. Furthermore,

$$\nu_m(p,t) = 1 + \left(\frac{\Delta}{p}\right) = \begin{cases} 2, & \Delta \text{ is a square mod } p \\ 0, & \text{otherwise} \end{cases}.$$

This shows that:

$$\begin{aligned}\mathcal{L}_m^p(t,s) &= 1 + \sum_{k\geq 1} \frac{\nu_m(p,t)}{p^{ks}} \\ &= 1 + \frac{1+\left(\frac{\Delta}{p}\right)}{1-p^{-s}}p^{-s} \\ &= \frac{1+\left(\frac{\Delta}{p}\right)p^{-s}}{1-p^{-s}} \\ &= \frac{1-p^{-2s}}{(1-p^{-s})(1-\left(\frac{\Delta}{p}\right)p^{-s})}.\end{aligned}$$

Now take $p = 2$: $2 \nmid \Delta$ iff t is odd, and so $\Delta \equiv 1 \mod 4$. Recall the Jacobi symbol is defined for $n \equiv 1 \mod 4$ by:

$$\left(\frac{n}{2}\right) = \begin{cases} 1, & n \equiv 1 \mod 8 \\ -1, & n \equiv 5 \mod 8 \end{cases}.$$

If $k \geq 1$, it is easy to see that:

$$\#\left\{x \mod 2^k \mid x^2 - tx + m \equiv 0 \mod 2^k\right\} = \frac{1}{2}\#\left\{y \mod 2^{k+2} \mid y^2 \equiv \Delta \mod 2^{k+2}\right\}.$$

However, if $k \geq 3$ and $\Delta \equiv 1 \mod 4$, then:

$$\#\left\{y^2 \equiv \Delta \mod 2^k\right\} = \begin{cases} 4, & \Delta \equiv 1 \mod 8 \\ 0, & \Delta \equiv 5 \mod 8 \end{cases} = 2\left(1 + \left(\frac{\Delta}{2}\right)\right).$$

Hence if $k \geq 1$, and t is odd, $\nu_m(2^k, t) = 1 + (\frac{\Delta}{2})$, and so:

$$\begin{aligned} \mathcal{L}_m^2(t,s) &= 1 + (1 + \left(\frac{\Delta}{2}\right)) \sum_{k \geq 1} \frac{1}{2^{ks}} \\ &= \frac{1 + (\frac{\Delta}{2})\, 2^{-s}}{1 - 2^{-s}} \\ &= \frac{1 - 2^{-2s}}{(1 - 2^{-s})(1 - (\frac{\Delta}{2})\, 2^{-s})}. \ \blacksquare \end{aligned}$$

Lemma 3.2. *Suppose p is an odd prime, $\Delta = p^\alpha \Delta'$ with $p \nmid \Delta'$. Then:*

(1) $\nu_m(p^k, t) = p^{[\frac{k}{2}]}$, *for* $1 \leq k \leq \alpha$;

(2) $\nu_m(p^k, t) = 0$, *for* $k > \alpha$, α *odd;*

(3) $\nu_m(p^k, t) = p^\beta(1 + \left(\frac{\Delta'}{p}\right))$, *for* $k > \alpha$, $\alpha = 2\beta$ *even.*

Proof: If p is odd, then:

$$\#\left\{x \mod p^k \mid x^2 - tx + m \equiv 0 \mod p^k\right\} = \#\left\{y \mod p^k \mid y^2 \equiv \Delta \mod p^k\right\}$$

If $k < \alpha$, $\Delta \equiv 0 \mod p^k$ and so

$$\nu_m(p^k, t) = \#\left\{y^2 \equiv 0 \mod p^k\right\} = p^{[\frac{k}{2}]}.$$

If α is odd, and $k > \alpha$, then Δ is not a square mod p^k, and so $\nu_m(p^k, t) = 0$, which shows (2).

Finally, if $k > \alpha = 2\beta$, then:

$$\begin{aligned}\nu_m(p^k, t) &= \#\left\{y^2 \equiv \Delta \mod p^k\right\} \\ &= p^\beta \cdot \#\left\{y^2 \equiv \Delta' \mod p^{k-\alpha}\right\} \\ &= p^\beta \cdot (1 + \left(\frac{\Delta'}{p}\right)). \ \blacksquare\end{aligned}$$

Corollary 3.3. *If p is an odd prime, and if we put $X = p^{-s}$, then:*

(1) *If $\alpha = 2\beta + 1$ is odd, then:*

$$\mathcal{L}_m^p(t, s) = (1 + X)(1 + pX^2 + \cdots + (pX^2)^\beta).$$

(2) *If $\alpha = 2\beta$ is even, then:*

$$\mathcal{L}_m^p(t, s) = \frac{1 + \left(\frac{\Delta'}{p}\right) X}{1 - X} \Lambda_p(\Delta, X)$$

$$\Lambda_p(\Delta, X) = 1 + X(pX - \left(\frac{\Delta'}{p}\right))(1 + pX^2 + \cdots + (pX^2)^{\beta - 1}).$$

Proof: (1) is immediate from the above lemma, and (2) can easily be proven from the lemma, e.g., by induction. $\blacksquare$

We now turn to the case $2 \mid \Delta$; since $\Delta = t^2 - 4m$, this implies $4 \mid \Delta$.

Lemma 3.4. *Suppose $\Delta = 2^\alpha \Delta'$, with $\alpha \geq 2$ and $2 \nmid \Delta$. Then:*

(1) $\nu_m(2^k, t) = 2^{[\frac{k}{2}]}, \quad 1 \leq k \leq \alpha - 2;$

(2) $\nu_m(2^k, t) = 0, \quad k > \alpha - 2, \quad \alpha$ *odd;*

(3) $\nu_m(2^{\alpha-1}, t) = 1, \quad \alpha$ *even;*

(4) $\nu_m(2^k, t) = 0, \quad k > \alpha - 1, \alpha$ *even and* $\Delta' \equiv 3 \mod 4;$

(5) $\nu_m(2^\alpha, t) = 2^\beta, \quad \alpha = 2\beta$ *even,* $\Delta' \equiv 1 \mod 4;$

(6) $\nu_m(2^k, t) = 2^\beta(1 + \left(\frac{\Delta'}{2}\right)), \quad \alpha = 2\beta$ *even,* $\Delta' \equiv 1 \mod 4.$

Proof: We write out the case $\alpha = 2\beta$, $k > \alpha - 2$, leaving the rest to the diligence of the reader.

Since $\Delta = t^2 - 4m$ is even iff $t = 2t'$ is even, we have:

$$\begin{aligned}\nu_m(2^k, t) &= \#\left\{(x - t')^2 - \frac{\Delta}{4} \equiv 0 \mod 2^k\right\} \\ &= 2^{\beta-1}\#\left\{z^2 \equiv \Delta' \mod 2^{k-(\alpha-2)}\right\}\end{aligned}$$

The claim follows from this, coupled with:

$$\#\left\{z^2 \equiv \Delta' \mod 2^r\right\} = \begin{cases} 1, & r = 1 \\ 0, & r \geq 2,\ \Delta' \equiv 3 \mod 4 \\ 2, & r = 2,\ \Delta' \equiv 1 \mod 4 \\ 2(1 + \left(\frac{\Delta'}{2}\right)), & r \geq 3,\ \Delta' \equiv 1 \mod 4. \end{cases} \quad \blacksquare$$

Corollary 3.5. *If $\Delta = 2^\alpha \Delta'$, with $2 \nmid \Delta'$, then putting $X = 2^{-s}$, we have:*

(1) *If $\alpha = 2\beta + 1$ is odd,*

$$\mathcal{L}_m^2(t, s) = \frac{1 - X^2}{1 - X}(1 + 2X^2 + \cdots + (2X^2)^{\beta-1})$$

(2) *If $\alpha = 2\beta$ is even, $\Delta' \equiv 3 \mod 4$,*

$$\mathcal{L}_m^2(t, s) = \frac{1 - X^2}{1 - X}(1 + 2X^2 + \cdots + (2X^2)^{\beta-1})$$

(3) *If $\alpha = 2\beta$ is even, $\Delta' \equiv 1 \mod 4$,*

$$\begin{aligned}\mathcal{L}_m^2(t, s) &= \frac{1 - X^2}{1 - X}(1 - \left(\frac{\Delta'}{2}\right) X)^{-1}\Lambda_2(\Delta, X), \\ \Lambda_2(\Delta, X) &= 1 + X(2X - \left(\frac{\Delta'}{2}\right))(1 + 2X^2 + \cdots + (2X^2)^{\beta-1}).\end{aligned}$$

The parabolic contribution. Now consider the case $\Delta \neq 0$ is a *perfect square*; this happens iff the polynomial $x^2 - tx + m = (x - a)(x - d)$ factors in the integers, and $m = ad$, $t = a + d$, $\Delta = (a - d)^2$. In this case, the results of the previous computations show that all the local factors $\mathcal{L}_m^p(t, s)$ are of the form:

$$\mathcal{L}_m^p(t, s) = \frac{1 - p^{-2s}}{(1 - p^{-s})^2}\Lambda_p(\Delta, s)$$

with $\Lambda_p(t, 1) = 1$. We thus have the global result:

Proposition 3.6. *If $t = a + d$, for positive integers $a > d$ such that $m = ad$, then:*

$$\mathcal{L}_m(t,s) = \frac{\zeta(s)^2}{\zeta(2s)}\Lambda(\Delta, s) \tag{3.5}$$

with $\Lambda(t,1) = 1$.

Notice that this implies that $\mathcal{L}_m(t,s)$ has a double pole for $t > 2\sqrt{m}$ as above.

Proposition 3.7. *If $m = ad$, $a > d \geq 1$, and $t = a + d$ then:*

$$\operatorname*{Res}_{s=1} F_{\pm}(\frac{t}{\sqrt{m}}, s)\mathcal{L}_m(t,s) = -\frac{6m^{-(k-1)/2} i^k}{\pi(k-1)} \cdot d^{k-1} \tag{3.6}$$

Proof: Using the Weber-Schafheitlein integral [W]:

$$\int_0^\infty J_\mu(ax)J_\nu(bx)\frac{dx}{x^\lambda}$$
$$= \frac{b^\nu \Gamma(\frac{\mu+\nu-\lambda+1}{2})}{2^\lambda a^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma(\frac{\lambda+\mu-\nu+1}{2})} \cdot F(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1; \frac{b^2}{a^2})$$

for $a > b > 0$, $\mathrm{Re}(\mu+\nu+1) > \mathrm{Re}\,\lambda > -1$, we can compute $F_{\pm}(\frac{t}{\sqrt{m}}, s)$:

$$\begin{aligned}
F_{\pm}(\frac{t}{\sqrt{m}}, s) &= 2\int_0^\infty J_{k-1}(4\pi x)\cos(2\pi\frac{t}{\sqrt{m}}x)\frac{dx}{x^s} \\
&= (4\pi)^{s-1}\sqrt{\pi}\left(\frac{t}{\sqrt{m}}\right)^{\frac{1}{2}}\int_0^\infty J_{-1/2}(\frac{t}{2\sqrt{m}}x)J_{k-1}(x)\frac{dx}{x^{s-1/2}} \\
&= (4\pi)^{s-1}\sqrt{\pi}\left(\frac{t}{\sqrt{m}}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{k-s}{2})}{2^{s-1/2}\left(\frac{t}{2\sqrt{m}}\right)^{k-s+\frac{1}{2}}\Gamma(k)\Gamma(\frac{s-k+1}{2})} \\
&\quad \times F(\frac{k-s}{2}, \frac{k-s+1}{2}, k; \frac{4m}{t^2}).
\end{aligned}$$

At $s = 1$, $\Gamma(\frac{s-k+1}{2})$ has a *simple* pole, and the value of the hypergeometric function is given by:

$$\begin{aligned}
F(\frac{k-1}{2}, \frac{k}{2}, k; \frac{4m}{t^2}) &= \left(\frac{1+\sqrt{1-\frac{4m}{t^2}}}{2}\right)^{1-k} \\
&= \left(\frac{t+a-d}{2t}\right)^{1-k} \\
&= \left(\frac{t}{a}\right)^{1-k},
\end{aligned}$$

where I have used the formula:

$$F(\alpha,\alpha+1/2,2\alpha+1;z)=\left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha},\qquad |\arg(1-z)|<\pi.$$

This shows that:

$$\begin{aligned}\operatorname*{Res}_{s=1}F_\pm\left(\frac{t}{\sqrt{m}},s\right)\mathcal{L}_m(t,s)&=\zeta(2)^{-1}\sqrt{\pi}\left(\frac{t}{\sqrt{m}}\right)^{1/2}\frac{\Gamma(\frac{k-1}{2})}{2^{1/2}(\frac{t}{2\sqrt{m}})^{k-1/2}\Gamma(k)}\\&\quad\times\left(\frac{t}{a}\right)^{k-1}\operatorname*{Res}_{s=1}\frac{\zeta(s)^2}{\Gamma(\frac{s-k+1}{2})}\\&=\frac{6}{\pi^2}m^{\frac{k-1}{2}}\frac{\sqrt{\pi}\Gamma(\frac{k-1}{2})}{2^{1-k}\Gamma(k)}a^{1-k}\cdot\frac{(-1)^{\frac{k}{2}-1}\Gamma(\frac{k}{2})}{2}\\&=m^{-\frac{k-1}{2}}d^{k-1}\frac{6}{\pi^2}(-1)^{k/2-1}\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{k-1}\\&=-\frac{6i^k}{(k-1)\pi}m^{-\frac{k-1}{2}}\cdot d^{k-1}\qquad\blacksquare\end{aligned}$$

Corollary 3.8.

$$\sum_{t>2\sqrt{m}}\operatorname*{Res}_{s=1}F_\pm\left(\frac{t}{\sqrt{m}},s\right)\mathcal{L}_m(t,s)=-m^{\frac{k-1}{2}}\frac{6i^k}{(k-1)\pi}\sum_{\substack{d|m\\d<\sqrt{m}}}d^{k-1}.$$

We call the above the *parabolic* term of the trace formula; it corresponds to the "complementary term" in [D-L], where the formula is derived using the adelic version of Selberg's trace formula. In Eichler's approach, it corresponds to the contribution of the fixed point $\{i\infty\}$ of the correspondence T_m.

Suppose now that m is a *perfect square*; then we have a term in (3.4) corresponding to $t=2\sqrt{m}$. I give a computation of this term, which I include in the *parabolic* contribution.

Lemma 3.9. *If m is a perfect square, $t=2\sqrt{m}$, then:*

$$\mathcal{L}_m(t,s)=\frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)}.$$

PROOF: We can write $x^2-tx+m=(x-\sqrt{m})^2$, and so, for any prime p,

$$\nu_m(p^k,2\sqrt{m})=\#\left\{y^2\equiv 0 \mod p^k\right\}=p^{[\frac{k}{2}]}$$

which implies:

$$\mathcal{L}_m^p(2\sqrt{m},s)=\sum_{k\geq 0}\frac{p^{[\frac{k}{2}]}}{p^{ks}}=\frac{1-p^{-2s}}{(1-p^{-s})(1-p^{1-2s})}.\qquad\blacksquare$$

Corollary 3.10. *For* $t = 2\sqrt{m}$,

$$\operatorname*{Res}_{s=1} F_{\pm}(\frac{t}{\sqrt{m}}, s)\mathcal{L}_m(t,s) = -\frac{3i^k}{(k-1)\pi}$$

This was shown in (2.14).

The elliptic contribution. We now assume $|t| < 2\sqrt{m}$; then $\mathcal{L}_m(t,s)$ has a simple pole at $s = 1$, and we can compute the residue using Dirichlet's class number formula. Coupled with the computation of $F_{\pm}(\frac{t}{\sqrt{m}}, 1)$, we will get a formula for the elliptic contribution to $\operatorname{tr} T_m$.

Lemma 3.11. *For* $|t| < 2\sqrt{m}$, *let* $\rho_t = \frac{t - i\sqrt{|\Delta|}}{2}$ *be a solution of* $x^2 - tx + m = 0$. *Then:*

$$F_{\pm}(\frac{t}{\sqrt{m}}, 1) = \frac{i^{k-1} m^{-\frac{k-1}{2}}}{k-1}(\rho_t^{k-1} - \bar{\rho}_t^{k-1}).$$

Proof: Suppose $t \neq 0$; then the Weber-Schafheitlein integral:

$$\int_0^\infty J_\mu(ax)\cos(bx)\frac{dx}{x} = \mu^{-1}\cos((k-1)\arcsin\frac{b}{a}), \qquad b \le a, \ \operatorname{Re}\mu > -1,$$

shows that:

$$\begin{aligned} F_{\pm}(\frac{t}{\sqrt{m}}, 1) &= 2\int_0^\infty J_{k-1}(4\pi x)\cos(2\pi\frac{t}{\sqrt{m}}x)\frac{dx}{x} \\ &= \frac{2}{k-1}\cos((k-1)\arcsin\frac{t}{2\sqrt{m}}). \end{aligned}$$

Since:

$$e^{i\arcsin\frac{t}{2\sqrt{m}}} = \sqrt{1 - \frac{t^2}{4m}} + i\frac{t}{2\sqrt{m}} = \frac{i}{\sqrt{m}}\rho_t$$

We see that, since $k-1$ is odd,

$$\begin{aligned} F_{\pm}(\frac{t}{\sqrt{m}}, 1) &= \frac{1}{k-1}\left\{\left(\frac{i\rho_t}{\sqrt{m}}\right)^{k-1} + \overline{\left(\frac{i\rho_t}{\sqrt{m}}\right)}^{k-1}\right\} \\ &= \frac{i^{k-1}m^{-\frac{k-1}{2}}}{k-1}\left(\rho_t^{k-1} - \bar{\rho}_t^{k-1}\right). \ \blacksquare \end{aligned}$$

We can now finish evaluating the elliptic terms; the extra ingredient we need is Dirichlet's class number formula:

$$L(\chi_\Delta, 1) = \frac{2\pi h(\Delta)}{w|\Delta|^{1/2}}, \qquad \Delta < 0 \text{ a discriminant}$$

where, as usual,

$$w = \begin{cases} 4, & \Delta = -4 \\ 6, & \Delta = -3 \\ 2, & \Delta < -4 \end{cases}$$

and $h(\Delta)$ is the number of classes of positive definite, primitive binary quadratic forms of discriminant $\Delta < 0$.

Since we saw that, for $\Delta = t^2 - 4m < 0$,

$$\mathcal{L}_m(t,s) = \frac{\zeta(s)}{\zeta(2s)} L(\chi_\Delta, s)\Lambda(\Delta, s),$$

with $\Lambda(\Delta, s)$ given by Corollaries 3.3, 3.5, we have:

$$\operatorname*{Res}_{s=1} \mathcal{L}_m(t,s) = \frac{1}{\zeta(2)} L(\chi_\Delta, 1)\Lambda(\Delta, 1).$$

Noting that $\rho_t - \bar{\rho}_t = -i\sqrt{|\Delta|}$, we can combine the class number formula with lemma 3.11 above to see for $\Delta < 0$:

$$\begin{aligned} \operatorname*{Res}_{s=1} F_\pm\left(\frac{t}{\sqrt{m}}, s\right)\mathcal{L}_m(t,s) &= \frac{6}{\pi^2} \cdot \frac{2\pi}{w} h(\Delta)\Lambda(\Delta,1) \cdot \frac{i^{k-1} m^{-\frac{k-1}{2}}}{k-1} \cdot \frac{\rho_t^{k-1} - \bar{\rho}_t^{k-1}}{i(\rho_t - \bar{\rho}_t)} \\ &= -\frac{6 i^k m^{-\frac{k-1}{2}}}{\pi(k-1)} \cdot \frac{2}{w} h(\Delta)\Lambda(\Delta,1) \cdot \frac{\rho_t^{k-1} - \bar{\rho}_t^{k-1}}{\rho_t - \bar{\rho}_t}. \end{aligned}$$

One checks that:

$$\frac{2}{w} h(\Delta)\Lambda(\Delta, 1) = H(\Delta),$$

the class number of all (not necessarily primitive) positive definite forms of discriminant Δ, counted with the appropriate multiplicity. This concludes the computation of the elliptic contribution.

Putting the results of this section in (3.1), we see:

$$\begin{aligned} \operatorname{tr} T_m =& \delta(\sqrt{m}) m^{\frac{k}{2}-1} \frac{k-1}{12} \\ &- \frac{1}{2} \sum_{|t| < 2\sqrt{m}} H(t^2 - 4m) \frac{\rho_t^{k-1} - \bar{\rho}_t^{k-1}}{\rho_t - \bar{\rho}_t} \\ &- \sum_{\substack{d|m \\ d < \sqrt{m}}} d^{k-1} - \delta(\sqrt{m}) \frac{m^{\frac{k-1}{2}}}{2} \end{aligned}$$

which is exactly the Eichler-Selberg formula.

Chapter II. Poincaré series for symmetric powers on $GL(2)$

Let $f \in S_k(\Gamma)$ be a normalized Hecke eigenform; its L-function then has an Euler product:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_1(n)}{n^s} = \prod_p \det(\mathrm{Id} - p^{-s} A_p)^{-1}, \qquad A_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix},$$

where I have normalized the Fourier coefficients of f relative to chapter 1 so that $a(n) = a_1(n) n^{(k-1)/2}$; in particular, $L(s,f)$ has a functional equation $s \to 1-s$, and the Ramanujan conjecture is that $A_p \in SU(2)$. The N-th symmetric power L-function for f is:

$$L(s,f,\mathrm{sym}^N) = \prod_p \det(\mathrm{Id} - p^{-s} \,\mathrm{sym}^N A_p)^{-1}$$

where $\mathrm{sym}^N : GL(2,\mathbf{C}) \to GL(N+1,\mathbf{C})$ is the representation of $GL(2,\mathbf{C})$ on $\mathrm{sym}^N(\mathbf{C}^2)$. This Euler product converges absolutely for $\mathrm{Re}(s) \gg 1$, and in a suitable right half-plane can be written as a Dirichlet series

$$L(s,f,\mathrm{sym}^N) = \sum_{n=1}^{\infty} \frac{a_N(n)}{n^s}$$

It is a major problem to show meromorphic continuation of these L-functions; in particular,

(1) $L(s,f,\mathrm{sym}^N)$ holomorphic in $\mathrm{Re}(s) > 1$ for all N is equivalent to the Ramanujan conjecture;

(2) $L(s,f,\mathrm{sym}^N)$ holomorphic in $\mathrm{Re}(s) \geq 1$ for all N, with no zeros on $\mathrm{Re}(s) = 1$ implies the Sato-Tate conjecture - equidistribution of the signs of the Fourier coefficients $a(p)$.

Langland's conjecture implies that these are L-functions for automorphic forms on $GL(N+1)$, so in particular are meromorphic and have a functional equation.

Following [PS], we will define Poincaré series $P(z,s,\mathrm{sym}^N) \in S_k(\Gamma)$ which reproduce the N-th symmetric power L-function, in the sense that its Petersson inner product with a normalized Hecke eigenform $f \in S_k(\Gamma)$ equals $L(s,f,\mathrm{sym}^N)$.

We discover a somewhat surprising phenomenon: If $N \geq 7$, then the initial data for the construction of $P(z, s, \mathrm{sym}^N)$ has a natural boundary at $\mathrm{Re}(s) = 0$ (actually, at the time of writing I can only show this for small values of $N \geq 7$). I lack an adequate understanding of why this phenomenon arises at present.

§1. Construction of $P(z, s, \mathrm{sym}^N)$

We will now find a Poincaré series $P_k(z, s, \mathrm{sym}^N) \in S_k(\Gamma)$ so that its Peterson product with all normalized Hecke eigenforms is $L(s, f, \mathrm{sym}^N)$. This was done by Piatetski-Shapiro **[PS]** in adélic language for any split reductive group; the following construction is not much more than a translation into classical language.

We try to determine $P_k(z, s, \mathrm{sym}^N)$ in the form:

$$P_k(z, s, \mathrm{sym}^N) = \sum_{n=1}^{\infty} \alpha_N(n, s) P_n^*(z)$$

where $P_n^*(z)$ are normalized Poincaré series so that

$$\langle f, P_n^* \rangle = a_1(n), \qquad f(z) = \sum_{n=1}^{\infty} a_1(n) n^{(k-1)/2} e(nz) \in S_k(\Gamma).$$

We require the coefficients $\alpha_N(n, s)$ to be *multiplicative* in the sense that

$$\alpha_N(n, s) = \prod_p \alpha_N^p(p^{\nu_p(n)}, s), \qquad n = \prod_p n^{\nu_p(n)}.$$

Thus for normalized Hecke eigenforms we get

$$\langle f, P(\cdot, \bar{s}, \mathrm{sym}^N) \rangle = \prod_p \sum_{k=0}^{\infty} \alpha_N^p(p^k, s) a_1(p^k) = L(s, f, \mathrm{sym}^N).$$

Using $a_1(p^k) = \mathrm{tr}\,\mathrm{sym}^k(A_p)$, we require that for all primes p,

$$\sum_{k=0}^{\infty} \alpha_N^p(p^k, s)\, \mathrm{tr}\,\mathrm{sym}^k(A_p) = \det(\mathrm{Id} - P^{-s}\,\mathrm{sym}^N(A_p))^{-1} = \sum_{n=o}^{\infty} \mathrm{tr}\,\mathrm{sym}^n(\mathrm{sym}^N A_p) p^{-ns}.$$

Decomposing the representation $\mathrm{sym}^n \mathrm{sym}^N(\mathbf{C}^2)$ into its irreducible constituents:

$$\mathrm{sym}^n \mathrm{sym}^N(\mathbf{C}^2) = \bigoplus_k [\mathrm{sym}^n \mathrm{sym}^N(\mathbf{C}^2) : \mathrm{sym}^k(\mathbf{C}^2)]\, \mathrm{sym}^k(\mathbf{C}^2)$$

we find that we are done if we take

$$\alpha_N^p(p^k, s) = \sum_{n=0}^{\infty} [\mathrm{sym}^n \mathrm{sym}^N(\mathbf{C}^2) : \mathrm{sym}^k(\mathbf{C}^2)] p^{-ns}.$$

This last series is known as the Poincaré-Molien series; in general, for a finite dimensional representation ρ of a compact Lie group G, one has a Poincaré-Molien series for each irreducible representation τ of G, given by:

$$F_\tau(\rho, t) = \sum_{n=0}^{\infty} [\mathrm{sym}^n \rho : \tau] t^n.$$

$F_\tau(\rho, t)$ is a rational function, which can be written in the form

$$F_\tau(\rho, t) = \frac{H_\tau(t)}{\prod_i (1 - t^{d_i})}$$

with $H_\tau(t)$ a polynomial, and the denominator independant of τ. When τ is the trivial representation of G, we get the *Hilbert function* for the graded algebra of invariants $S(\rho)^G$.

We can summarize the discussion above as follows:

Proposition 1.1. *Let $\alpha_N(n, s)$ be given by:*

$$\alpha_N(n, s) = \prod_p \alpha_N^p(p^{\nu_p(n)}, s), \qquad n = \prod_p p^{\nu_p(n)}$$

where $\alpha_N^p(p^k, s) = F_k(N, p^{-s})$ is the Poincaré-Molien series

$$F_k(N, t) = F_{\mathrm{sym}^k(\mathbf{C}^2)}(\mathrm{sym}^N(\mathbf{C}^2), t).$$

Then if $P(z, s, \mathrm{sym}^N) = \sum_n \alpha_N(n, s) P_n^*(z)$, we have:

$$\langle f, P(z, \bar{s}, \mathrm{sym}^N) \rangle = L(s, f, \mathrm{sym}^N)$$

for all normalized Hecke eigenforms $f \in S_k(\Gamma)$.

In this section, we show convergence of $P(z,s,\mathrm{sym}^N)$ for $\mathrm{Re}(s) > 1$. In order to gain some flexibility, I do not assume the Ramanujan conjecture, but rather that, if $f(z) = \sum_{n\geq 1} a_1(n)n^{(k-1)/2}e(nz)$ is a normalized Hecke eigenform, then for some $\delta \geq 0$, $|a_1(p)| \leq 2p^\delta$ for all primes p. The trivial bound is $\delta \leq 1/2$, the Ramanujan bound is $\delta = 0$, and by more elementary means one can show $\delta \leq 1/4$.

Proposition 2.1. *$\sum_{n=1}^\infty \alpha_N(n,s)P_n^*(z)$ converges absolutely for $\mathrm{Re}(s) > 1 + N\delta$, uniformly in z.*

PROOF: We first bound $P_n^*(z)$: We have a spectral expansion

$$P_n^*(z) = \sum_{i=1}^d \overline{a_{1,i}(n)} f_i(z)$$

with $f_1, \ldots, f_d$ an orthonormal basis of $S_k(\Gamma)$, $f_i(z) = \sum_n a_{1,i}(n)n^{(k-1)/2}e(nz)$.

Since $f_i(z)$ are bounded and $a_{1,i}(n) \ll n^{\delta+\epsilon}$ for all $\epsilon > 0$, we see that

$$P_n^*(z) \ll n^{\delta+\epsilon}, \qquad \text{for all } \epsilon > 0.$$

Therefore

$$|\sum_{n=1}^\infty \alpha_N(n,s)P_n^*(z)| \ll \sum_{n\geq 1} \alpha_N(n,\mathrm{Re}(s))n^{\delta+\epsilon}, \qquad \text{for all } \epsilon > 0. \tag{2.1}$$

From now on, I shall omit the dependence on ϵ, and assume s is real.

Because the RHS of (2.1) is multiplicative:

$$\sum_{n\geq 1} \alpha_N(n,s)n^\delta = \prod_p \sum_{k=0}^\infty F_k(N,p^{-s})p^{k\delta}, \tag{2.2}$$

we will need to have a suitable estimate on $F_k(N,p^{-s})$.

Lemma 2.2. *As $t \to 0$, we have:*

(1) $F_{aN}(N,t) = t^a + O(t^{a+1})$;

(2) $F_{aN+r}(N,t) = O(t^{a+1})$ *for* $1 \leq r \leq N-1$.

PROOF: $F_k(N,t) = \sum_n [\mathrm{sym}^n \rho_N : \rho_k] t^n$, where I have put $\rho_d = \mathrm{sym}^d \mathbf{C}^2$. To prove (1), we need to show

$$[\mathrm{sym}^n \rho_N : \rho_{aN}] = \begin{cases} 0, & n < a \\ 1, & n = a \end{cases}.$$

If we denote

$$\mathrm{Ten}^n \rho_N = \overbrace{\rho_N \otimes \cdots \otimes \rho_N}^{n \text{ times}}$$

then because $\mathrm{sym}^n \rho_N \subseteq \mathrm{Ten}^n \rho_N$, we have an inequality:

$$[\mathrm{sym}^n \rho_N : \rho_{aN}] \leq [\mathrm{Ten}^n \rho_N : \rho_{aN}].$$

Since the *highest* highest weight of $\mathrm{Ten}^n \rho_N$ is nN, we need that $aN \leq nN$, i.e. $a \leq n$, in order that $[\mathrm{Ten}^n \rho^N : \rho_{aN}] \neq 0$. Furthermore, ρ_{aN} appears exactly *once* in $\mathrm{Ten}^a \rho_N$; if v_N is the highest weight vector of ρ_N, then $v_N^{\otimes a} = v_N \otimes \dots v_N$ is the unique weight vector of weight aN, and is obviously symmetric, so that $\rho_{aN} \subseteq \mathrm{sym}^a \rho_N$. Thus $[\mathrm{sym}^a \rho_N : \rho_{aN}] = 1$, which shows (1).

To see (2), we again use

$$[\mathrm{sym}^n \rho_N : \rho_{aN+r}] \leq [\mathrm{Ten}^n \rho_N : \rho_{aN+r}]$$

so we need to see $[\mathrm{Ten}^n \rho_N : \rho_{aN+r}] = 0$ for $n < a+1$. This follows from the previous discussion: The highest weight of any representation appearing in $\mathrm{Ten}^n \rho_N$ is nN, so $[\mathrm{Ten}^n \rho_N : \rho_{aN+r}] = 0$ if $nN < aN + r$, i.e. if $n < a+1$. Thus $[\mathrm{sym}^n \rho_N : \rho_{aN+r}] = 0$ for $n < a+1$, so that

$$F_{aN+r}(N,t) = \sum_{n=1}^{\infty} [\mathrm{sym}^n \rho_N : \rho_{aN+r}] t^n = O(t^{a+1}). \qquad \blacksquare$$

Corollary 2.3. *The product* $\alpha_N(n,s) = \prod_p \alpha_N^p(p^{\nu_p(n)}, s)$ *converges for* $\mathrm{Re}(s) > 1$.

We can now prove Proposition 2.1. By Lemma 2.2, we have (setting $t = p^{-s}$, $|t| < 1$):

$$\begin{aligned} \sum_{k=0}^{\infty} F_k(N,t) &= \sum_{r=0}^{N-1} \sum_{a=0}^{\infty} F_{aN+r}(N,t) p^{(aN+r)\delta} \\ &= \sum_{a=0}^{\infty} \{t^a + O(t^{a+1})\} p^{aN\delta} + \sum_{r=1}^{n-1} O(\sum_{a=0}^{\infty} t^{a+1} p^{(aN+r)\delta}) \\ &= \sum_{a=0}^{\infty} t^a p^{N\delta} \cdot (1 + O(t)) \\ &= \frac{1}{1 - p^{N\delta} t} \cdot (1 + O(t)) \end{aligned}$$

We thus see that the product (2.2) converges for $\mathrm{Re}(s) > 1 + N\delta$. ∎

Remark. If N is odd then $F(N,t) = F_0(N,t)$ is a function of t^2 rather than t, so $\prod_p F(N, p^{-s})$ converges for $\mathrm{Re}(s) > 1/2$. This follows from the fact that if both N and n are *odd*, the center of $SU(2)$ acts nontrivially on $\mathrm{Ten}^n \rho_N$, which therfore does not contain the trivial representation ρ_0.

§3. Computing $F_k(N,t)$

As our next step, we need to compute the Poincaré-Molien series $F_k(N,t)$. This is trivial for $N=1$, since $\mathrm{sym}^1(\mathbf{C}^2)=\mathbf{C}^2$, so:

$$[\mathrm{sym}^n\,\mathrm{sym}^1\,\mathbf{C}^2:\mathrm{sym}^k\,\mathbf{C}^2]=[\mathrm{sym}^n\,\mathbf{C}^2:\mathrm{sym}^k\,\mathbf{C}^2]=\begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}.$$

Thus $F_k(1,t)=t^k$.

For $N=2$, we have the decomposition:

$$\mathrm{sym}^n\,\mathrm{sym}^2\,\mathbf{C}^2=\sum_{\substack{k\leq n\\ k\equiv n \mod 2}}\mathrm{sym}^{2k}\,\mathbf{C}^2$$

which is at the heart of the classical theory of spherical harmonics on $SO(3)$. Therefore, we have:

$$F_k(2,t)=\begin{cases}0, & k \text{ odd}\\ \frac{t^{k/2}}{1-t^2}, & k \text{ even}\end{cases}.$$

This is well known,and is equivalent to the expansion:

$$L(s,f,\mathrm{sym}^2)=\zeta(2s)\sum_{n=1}^{\infty}\frac{a(n^2)}{n^s}.$$

The corresponding Poincaré series was used by Zagier [**Z**] and Mizumoto [**M**] give new proofs of Shimura's theorem [**Sh 2**] on entirety of $L(s,f,\mathrm{sym}^2)$.

For $N\geq 3$, all this is more complicated, and as we shall see, so is the answer. We use an algorithm of T. Springer's [**Sp 2**]: He starts from the formula

$$F_k(N,t)=\sum_{n=0}^{\infty}[\mathrm{sym}^n\,\mathrm{sym}^N:\mathrm{sym}^k]t^n=\int_{SU(2)}\det(\mathrm{Id}-t\,\mathrm{sym}^N(g))^{-1}\,\mathrm{tr}\,\mathrm{sym}^k(g)dg$$

where the Haar measure on $SU(2)$ is normalized to have volume one.

We have:

$$\mathrm{tr}\,\mathrm{sym}^k(t(\phi))=\frac{\sin(k+1)\phi}{\sin\phi},\qquad t(\phi)=\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}$$

and using Weyl's integration formula, one sees that

$$F_k(N,t)=\frac{1}{\pi}\int_0^{2\pi}\sin(k+1)\phi\sin\phi\det(\mathrm{Id}-t\,\mathrm{sym}^N(t(\phi)))^{-1}d\phi$$

The crux of Springer's method is a partial fraction decomposition for

$$\det(\mathrm{Id} - t\,\mathrm{sym}^N(t(\phi)))^{-1} = \prod_{j=0}^{N}(1 - te^{i(N-2j)\phi})^{-1},$$

together with the elementary formula:

$$\frac{1}{\pi}\int_0^{2\pi}(1-te^{i\phi})^{-1}\sin(k+1)\phi\sin\phi d\phi = \frac{1}{2}(1+\delta_{k,0})t^k - \frac{1}{2}t^{k+2}, \qquad |t|<1.$$

From these, one gets an "explicit" formula

$$F_k(N,t) = \sum_{0\le j<\frac{N}{2}} \phi_{N-2j}((t^k - t^{k+2})\gamma_{N,j}) \tag{3.1}$$

where $\phi_n : \mathbf{C}(t) \to \mathbf{C}(t)$ is the linear map such that

$$\phi_n f(t^n) = \frac{1}{n}\sum_{j \bmod n} f(\zeta_n^j t), \qquad \zeta_n = e^{\frac{2\pi i}{n}}$$

and

$$\gamma_{N,k}(t) = \frac{(-1)^k t^{k(k+1)}}{\prod_{j=1}^{k}(1-t^{2j})\prod_{j=1}^{N-k}(1-t^{2j})}.$$

To compute $\phi_n f$ for $f \in \mathbf{C}(t)$, one uses the following:

(1)

$$\phi_n(t^k)(t) = \begin{cases} t^{k/n}, & k \equiv 0 \mod n \\ 0, & \text{otherwise.}\end{cases}$$

(2)

$$\phi_n(g_n f)(t) = g(t)\phi_n f(t), \qquad \text{if } g_n(t) = g(t^n).$$

For example, if $f(t) = (1-t^n)^{-1}h(t)$, then $\phi_n f(t) = (1-t)^{-1}\phi_n h(t)$.

(3) If $q(t) = (1-t^q)^{-1}t^k$, $q \not\equiv 0 \mod n$, then rewriting $q(t)$ in the form:

$$q(t) = \frac{t^k(1 + t^q + \cdots + t^{q(n-1)})}{1-t^{qn}}$$

we find:

$$\phi_n q(t) = \frac{\phi_n(\sum_{j=0}^{n-1} t^{k+jq})}{1-t^q} = \frac{\sum_{j=0}^{n-1}\phi_n(t^{k+jq})(t)}{1-t^q}.$$

One uses the same procedure for dealing with expressions of the form

$$f(t) = \frac{t^k}{\prod_j(1-t^{m_j})}.$$

In (3.3), (3.4) I shall give the results of applying this algorithm to compute $F_k(N,t)$, for $N=3,4$. As an example, I shall detail the computation of $F_k(3,t)$. By (3.1),

$$F_k(3,t) = \phi_3((t^k - t^{k+2})\gamma_{3,0}) + \phi_1((t^k - t^{k+2})\gamma_{3,1}) \tag{3.2}$$

Since $\phi_1 = \mathrm{Id}$, the second term on the RHS of (3.2) is given by

$$(t^k - t^{k+2})\gamma_{3,1}(t) = -\frac{t^{k+2}}{(1-t^2)(1-t^4)}.$$

For the first term on the RHS of (3.2), we apply ϕ_3 to:

$$f(t) = (t^k - t^{k+2})\gamma_{3,0}(t) = \frac{t^k}{(1-t^4)(1-t^6)} = \frac{t^k(1+t^4+t^8)}{(1-t^6)(1-t^{12})}$$

to find

$$\phi_3 f(t) = \frac{\phi_3(t^k + t^{k+4} + t^{k+8})}{(1-t^2)(1-t^4)} = \begin{cases} \frac{t^{k/3}}{(1-t^2)(1-t^4)}, & k \equiv 0 \mod 3 \\ \frac{t^{(k+4)/3}}{(1-t^2)(1-t^4)}, & k \equiv 2 \mod 3 \\ \frac{t^{(k+8)/3}}{(1-t^2)(1-t^4)}, & k \equiv 1 \mod 3 \end{cases}$$

Therefore, we have:

$$\begin{aligned} F_{3k}(3,t) &= \frac{t^k(1+t^2+\cdots+t^{2k})}{1-t^4} \\ F_{3k+1}(3,t) &= \begin{cases} 0, & k=0 \\ \frac{t^{k+3}(1+t^2+\cdots+t^{2(k-1)})}{1-t^4}, & k \geq 1 \end{cases} \\ F_{3k+2}(3,t) &= \frac{t^{k+2}(1+t^2+\cdots+t^{2k})}{1-t^4}. \end{aligned} \tag{3.3}$$

Likewise, a similar computation shows:

$$\begin{aligned} F_{4k}(4,t) &= \frac{t^k(1+t+\cdots+t^k)}{(1-t^2)(1-t^3)} \\ F_{4k+2}(4,t) &= \frac{t^{k+2}(1-t^k)}{(1-t)(1-t^2)(1-t^3)} \\ F_{2k+1}(4,t) &= 0 \end{aligned} \tag{3.4}$$

The corresponding expression for $N = 5, 6$ are more complicated; for example, for $N = 5$, we need 15 formulas to describe $F_k(5,t)$, depending on the residue class of k modulo 15. We omit these formulas.

We have seen that we have a representation

$$P_k(z,s,\mathrm{sym}^N) = \sum_{n=1}^{\infty} \alpha_N(n,s) P_n^*(z), \tag{4.1}$$

absolutely convergent for $\mathrm{Re}(s) > 1$, and that in that range we have a spectral expansion:

$$P_k(z,s,\mathrm{sym}^N) = \sum_{i=1}^{d(k)} L(s,f_i,\mathrm{sym}^N) \frac{f_i(z)}{\langle f_i, f_i \rangle} \tag{4.2}$$

where $f_1, \ldots, f_{d(k)}$ is a basis of $S_k(\Gamma)$ consisting of normalized Hecke eigenforms.

It is widely believed that $L(s,f_i,\mathrm{sym}^N)$ have meromorphic continuation to all of $\mathbf{C}$ with functional equation - for instance, it would follow from Langlands' Functoriality Conjecture. By (4.2), the same should be true of $P_k(z,s,\mathrm{sym}^N)$. It is therefore surprising to see that if $N \geq 7$, the coefficients $\alpha_N(n,s)$ in (4.1) have *natural boundary* at $\mathrm{Re}(s) = 0$. Of course, this does not prevent $P_k(z,s,\mathrm{sym}^N)$ from being meromorphic in $\mathbf{C}$. Nonetheless, it is interesting to note that, to date, meromorphic continuation of $L(s,f_i,\mathrm{sym}^N)$ is unknown for $N \geq 6$.

Theorem 4.1. *Let $\alpha_N(n,s)$ be defined as in Proposition 1.1 for $N \geq 1$, $\mathrm{Re}(s) > 1$.*

(1) *For all N, $\alpha_N(n,s)$ has meromorphic continuation to $\mathrm{Re}(s) > 0$;*

(2) *If $N \leq 6$, $\alpha_N(n,s)$ extends to a meromorphic function in all of $\mathbf{C}$;*

(3) *If $N = 7$ then $\alpha_N(n,s)$ has the imaginary axis $\mathrm{Re}(s) = 0$ as its natural boundary; more precisely, every point on $\mathrm{Re}(s) = 0$ is a limit point of zeros of $\alpha_N(n,s)$.*

PROOF: To see all this, we recall that for $p \nmid n$, the p-th component of $\alpha_N(n,s)$ is the Hilbert function $F(N,p^{-s})$ for the invariants of a binary N-ic:

$$\alpha_N^p(n,s) = F(N,p^{-s}) = \frac{H_N(p^{-s})}{\prod_i (1-p^{-d_i s})}$$

with $H_N(t) \in \mathbf{Z}[t]$, $H_N(0) = 1$. Thus

$$\alpha_N(n,s) = g_N(n,s) \prod_i \zeta(d_i s) \prod_{p \nmid n} H_N(p^{-s}),$$

with $g_N(n,s)$ meromorphic in $\mathbf{C}$. In particular,

$$\alpha_N(1,s) = \prod_i \zeta(d_i s) \prod_p H_N(p^{-s}).$$

We now use a theorem of Esterman's [**Est**], who considered Euler products of the form:

$$L(H,s) = \prod_p H(p^{-s})^{-1}$$

with $H(t) = 1 + a_1 t + \cdots + a_d t^d \in \mathbf{Z}[t]$ a polynomial with integer coefficients. $L(H,s)$ converges for $\mathrm{Re}(s) > 1$, has meromorphic continuation to $\mathrm{Re}(s) > 0$, and if we factor $H(t)$ as:

$$H(t) = \prod_{i=1}^{d} (1 - \beta_i t),$$

then there are two alternatives:

(1) If $|\beta_i| = 1$ for all $i = 1, \ldots, d$ then $L(H,s)$ has meromorphic continuation to all of $\mathbf{C}$; we say then that $H(t)$ is *unitary.*

(2) If $|\beta_i| \neq 1$ for some i, then every point of $\mathrm{Re}(s) = 0$ is a limit point of poles of $L(H,s)$ in the half-plane $\mathrm{Re}(s) > 0$, and so the imaginary axis is the natural boundary of $L(H,s)$.

In view of Esterman's theorem, our theorem will follow if we show that for the Hilbert function

$$F(N,t) = \frac{H_N(t)}{\prod_i (1 - t^{d_i})}$$

the numerator $H_N(t)$ is unitary for $N \leq 6$, and nonunitary otherwise. Note that the numerator $H_N(t)$ is not unique, but it is well defined up to unitary factors. In fact, $H_N(t)$ is unitary if and only if the prime factors of $H_N(t)$ over $\mathbf{Z}$ are all cyclotomic polynomials. Using the list in the Appendix, we see $H_1(t) = H_2(t) = H_3(t) = H_4(t) = 1$, $H_5(t) = 1 + t^{18}$, $H_6(t) = 1 + t^{15}$ - which are unitary. For $N = 7$ the Hilbert function is given by:

$$F(7,t) = \frac{H_7(t)}{(1-t^4)(1-t^8)(1-t^{12})^2(1-t^{20})},$$

with the prime factorization of $H_7(t)$ being:

$$H_7(t) = (t^2+1)^2(t^4-t^2+1)(t^8-t^6+t^4-t^2+1)$$
$$\cdot (t^{32} - t^{26} + 2t^{24} - t^{22} + 5t^{20} + 2t^{18} + 6t^{16} + 2t^{14} + 5t^{12} - t^{10} + 2t^8 - t^6 + 1)$$

from which one sees $H_7(t)$ is not unitary.

Likewise, we have the Hilbert function for the invariants of binary octavics:

$$F(8,t) = \frac{1+t^8+t^9+t^{10}+t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)}$$

with $H_8(t) = 1+t^8+t^9+t^{10}+t^{18}$ having prime factorization

$$(4.3)\quad H_8(t) = (1-t+t^2)(1+t-t^3-t^4+t^6+t^7+t^8+t^9+t^{10}-t^{12}-t^{13}+t^{15}+t^{16}).$$

Inspection shows that the last factor in (4.3) is not cyclotomic. It seems likely that none of the numerators $H_N(t)$ of the Hilbert functions are unitary if $N \geq 7$. Using the computer-generated tables in **[Sal]**, I have checked this for $N \leq 12$. ∎

We digress to discuss a question in Invariant Theory. Let G be a complex connected semi-simple group, and V a finite dimensional representation of G (we may also consider compact Lie groups, or Chevalley groups over a finite field). We form the Poincaré-Molien series for the algebra of invariants for G acting on the symmetric algebra $S(V)$:

$$F(V,t) = \sum_{n=0}^{\infty} \dim S(V)_n^G t^n,$$

where $S(V)_n^G$ are the homogeneous invariants of degree n. Then we can write $F(V,t)$ in the form:

$$F(V,t) = \frac{H(t)}{\prod_i (1-t^{d_i})}, \qquad H(t) = 1 + a_1 t + \cdots + a_e t^e \in \mathbf{Z}[t]. \tag{5.1}$$

If V contains no invariant vectors, then we can take all $d_i \geq 2$. We say (G,V) is *unitary* if the roots of the numerator $H(t)$ in (5.1) are all of modulus 1 (in which case they will all be roots of unity). This notion is independent of the choice of $H(t)$ in (5.1).

I would like to pose the following:

Problem. *Classify all unitary pairs (G,V).*

There is an obvious class of examples of unitary pairs (G,V) - those pairs such that $S(V)^G$ is a *polynomial algebra*, i.e. admits a finite set of homogeneous generators $p_1,\ldots,p_r$ which are algebraically independent. In this case, if the degrees of the generators are $d_i = \deg p_i$, then:

$$F(V,t) = \frac{1}{\prod_{i=1}^{r}(1-t^{d_i})}.$$

Such pairs have been classified - see **[K-P-V]** for G connected, simple, acting irreducibly on V. For example, one knows $S(V)^G$ is a polynomial ring in the following cases:

(1) G has a Zariski dense orbit in the projective space $P(V)$, in which case $S(V)^G = \mathbf{C}$; for example, take $SL(n,\mathbf{C})$ acting on $\mathbf{C}^n$.

(2) The adjoint representation of G on its Lie algebra $\mathfrak{g}$: By Chevalley's theorem, $S(\mathfrak{g})^G = S(\mathfrak{h})^W$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, and W is the corresponding Weyl group. $(W,\mathfrak{h})$ is a finite reflection group, so $S(\mathfrak{h})^W$ is a polynomial ring.

(3) $G = SL(2, \mathbf{C})$, $V_d = \mathrm{sym}^d\,\mathbf{C}^2$ is the space of binary forms of degree d, then $S(V_d)^G$ is a polynomial ring if and only if $d \leq 4$ [**K-P-V**].

There are some less trivial examples of unitary pairs (G, V); as we have seen, $SL(2, \mathbf{C})$ acting on V_d for $d = 5, 6$ is unitary, but the invariants are not a polynomial ring. These are examples of (G, V) such that $S(V)^G$ is a *complete intersection.* We recall the relevant definitions (see [**St**]): Let B be a finitely generated graded algebra over $\mathbf{C}$, with a minimal set of homogeneous generators $p_1, \ldots, p_n$ of positive degree. Then, if $A = \mathbf{C}[x_1, \ldots, x_n]$ is the polynomial algebra with grading given by $\deg x_i = \deg p_i$, we can write $B = A/I$ for some homogeneous ideal I. If I can be generated by $h = n - \dim B$ homogeneous elements, we say that B is a complete intersection. Usually, we need more than $n - \dim B$ generators; in our case, $B = S(V)^G$ is Cohen-Macauley and so $\mathrm{hd}\, B = n - \dim B$ is the homological dimension of B. Let $\mathbf{y} = \{y_1, \ldots, y_h\}$ be a minimal set of homogeneous generators of the ideal I. Then if B is a complete intersection, we get a free resolution of B (as an A-module) from the *Koszul complex* $K_\bullet(\mathbf{y})$ with respect to the elements $\mathbf{y}$. Thus we have the exact sequence of graded A-modules:

$$0 \longrightarrow K_h(\mathbf{y}) \xrightarrow{d} \ldots \xrightarrow{d} K_1(\mathbf{y}) \xrightarrow{d} K_0(\mathbf{y}) \xrightarrow{\epsilon} B \to 0$$

where the $K_p(\mathbf{y})$ are free A-modules of rank $\binom{h}{p}$ with basis:

$$e_{i_1} \wedge \cdots \wedge e_{i_p}, \qquad 1 \leq i_1 < \cdots < i_p \leq h.$$

The boundary maps $d : K_p(\mathbf{y}) \to K_{p-1}(\mathbf{y})$ are

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} y_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$$

and $\epsilon : K_0(\mathbf{y}) \simeq A \to B$ is the canonical projection. From this one sees [**St**] that the Hilbert series of B is given by:

$$F(B, t) = \sum_{p=0}^{h} (-1)^p F(K_p(\mathbf{y}), t) = \frac{\prod_{i=1}^{h} (1 - t^{e_i})}{\prod_{j=1}^{n} (1 - t^{d_j})}$$

where $d_j = \deg p_j$, $e_i = \deg y_i$.

Thus, if $S(V)^G$ is a complete intersection then (G, V) is unitary. N.D. Beklemishev has determined all the cases where the invariants of n-ary forms of degree r are a complete intersection [**B**]; in particular, for $SL(2,\mathbf{C})$ this happens only for V_d with $d = 3,4,5,6$. I think it likely that if $d \geq 7$, then V_d is not unitary[2] . More generally, it is natural to conjecture that, given G, there are only finitely many representations V (up to addition of trivial summands) for which (G, V) is unitary. A related finiteness result (which strongly influenced [**B**]) is due to V.L. Popov [**Po**], who showed that for each G there are only finitely many V such that the homological dimension of $S(V)^G$ is less than any given constant. In the case of $SL(2,\mathbf{C})$, hd $S(V_d)^G = 0$ for $d \leq 4$, hd $S(V_d)^G = 1$ for $d = 5,6$, and is at least 3 if $d \geq 7$ [**Po**].

In the case of finite Chevalley groups, the problem is likely to be at least as complicated. There certainly are cases of non-unitary invariants. For example,[3] take $G = PSL(2, \mathbb{F}_7)$, the projective special linear group over the field of 7 elements, which is a simple group of order 168. Let $V = \mathrm{St}$ be the *Steinberg representation*, which can be realised as the space of functions on the projective line $P(\mathbb{F}_7)$, whose mean is zero. For any finite group G and a representation $\rho : G \to GL(V)$, we can use the formula:

$$\begin{aligned} F(V,t) &= \frac{1}{|G|}\sum_{g\in G}\det(1-t\rho(g))^{-1} \\ &= \sum_{\{g\}}\frac{1}{C_g}\det(1-t\rho(g))^{-1}, \end{aligned}$$

where the second sum runs over the conjugacy classes $\{g\}$ of G, and C_g is the order of the centralizer of the conjugacy class of g. One can see that:

$$\begin{aligned} F(\mathrm{St},t) &= \frac{1}{168}\frac{1}{(1-t)^7} + \frac{1}{8}\frac{1}{(1+t)^4(1-t)^3} \\ &+ \frac{1}{3}\frac{1}{(1-t)(1-t^3)^2} + \frac{1}{4}\frac{1}{(1+t)(1+t^2)(1-t^4)} \\ &+ 2\frac{1}{7}\frac{1}{1-t^7} \\ &= \frac{H(t)}{(1-t^2)^2(1-t^3)^2(1-t^4)^2(1-t^7)}, \end{aligned} \tag{5.2}$$

[2] I have verified this for $d \leq 12$

[3] I thank Walter Feit for suggesting this example.

$$H(t) = 1 - t^2 - t^3 + t^4 + 2t^5 + 4t^6 + 2t^7 + 3t^8 + 2t^9 + 3t^{10} + 2t^{11} + 4t^{12}$$
$$+ 2t^{13} + t^{14} - t^{15} - t^{16} + t^{18}$$

(In (5.2), the factor 2 in the last summand comes from the two unipotent classes of G). This shows that (G, St) is non-unitary.

For the reader's convenience, I present a list of the Hilbert functions $F(N,t)$ for invariants of binary forms of degree $N \leq 10$. This is very classical, and can be found, for instance, in [**Syl**]. For computer-generated tables of these Hilbert functions for $N \leq 24$, $N \neq 21, 23$, see [**Sal**]. For additional information and background, see [**H**], [**Sp 1**].

(A1)
$$F(1,t) = 1$$

(A2)
$$F(2,t) = \frac{1}{1-t^2}$$

(A3)
$$F(3,t) = \frac{1}{1-t^4}$$

(A4)
$$F(4,t) = \frac{1}{(1-t^2)(1-t^3)}$$

(A5)
$$F(5,t) = \frac{1+t^8}{(1-t^4)(1-t^8)(1-t^{12})}$$

(A6)
$$F(6,t) = \frac{1+t^{15}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}$$

(A7)
$$F(7,t) = \frac{H_7(t)}{(1-t^4)(1-t^8)(1-t^{12})^2(1-t^{20})}$$

$$H_7(t) = 1 + 2t^8 + 4t^{12} + 4t^{14} + 5t^{16} + 9t^{18} + 6t^{20} + 9t^{22} + 8t^{24} + 9t^{26} + 6t^{28} + 9t^{30} + 5t^{32} + 4t^{34} + 4t^{36} + 2t^{40} + t^{48}$$

(A8)
$$F(8,t) = \frac{1+t^8+t^9+t^{10}+t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)}$$

(A9)
$$F(9,t) = \frac{H_9(t)}{(1-t^4)(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})(1-t^{14})(1-t^{16})}$$

$$H_9(t) = t^{60} + t^{56} - t^{54} + 5t^{52} + 3t^{50} + 18t^{48} + 15t^{46} + 44t^{44} + 43t^{42}$$
$$+ 82t^{40} + 76t^{38} + 122t^{36} + 107t^{34} + 147t^{32} + 119t^{30} + 147t^{28}$$
$$+ 107t^{26} + 122t^{24} + 76t^{22} + 82t^{20} + 43t^{18} + 44t^{16} + 15t^{14}$$
$$+ 18t^{12} + 3t^{10} + 5t^{8} - t^{6} + t^{4} + 1$$

(A10) $$F(10,t) = \frac{H_{10}(t)}{(1-t^2)(1-t^4)(1-t^5)(1-t^6)^2(1-t^7)(1-t^8)(1-t^9)}$$

$$H_{10}(t) = 1 - t^5 + 2t^6 - t^7 + 4t^8 + 4t^9 + 8t^{10} + 6t^{11} + 16t^{12}$$
$$+ 9t^{13} + 17t^{14} + 15t^{15} + 19t^{16} + 12t^{17} + 23^{18} + 12t^{19}$$
$$+ 19t^{20} + 15t^{21} + 17t^{22} + 9t^{23} + 16t^{24} + 6t^{25} + 8t^{26}$$
$$+ 4t^{27} + 4t^{28} - t^{29} + 2t^{30} - t^{31} + t^{36}$$

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