# Eigenvalue statistics and lattice points

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**Abstract.** One of the more challenging problems in spectral theory and mathematical physics today is to understand the statistical distribution of eigenvalues of the Laplacian on a compact manifold. There are now several challenging conjectures about these, originating in the physics literature. In this survey, a version of a talk delivered as the Colloquio De Giorgi at the Scuola Normale Superiore of Pisa in May 2006, I will describe what is conjectured and what is known the very simple case of the flat torus, where the problems amount to counting lattice points in annuli and have a definite arithmetic flavour.

**Eigenvalue problems.** Let $M$ be a smooth, compact Riemannian manifold, $\Delta$ the Laplace-Beltrami operator associated with the metric. We consider the eigenvalue problem

$$\Delta\psi + \lambda\psi = 0, \quad \psi \in L^2(M).$$

As is well known, the spectrum is a discrete set of points, which we denote by $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$, whose only accumulation point is at infinity, and there is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions.

**Example 1.1.** The circle $M = \mathbb{R}/\mathbb{Z}$ with the standard flat metric: the Laplacian is simply $d^2/dx^2$, as a basis for the eigenfunctions we may take $\sin 2\pi kx$, $\cos 2\pi kx$ $(k > 0)$ with eigenvalues $4\pi^2k^2$ (each having multiplicity 2), together with the constant function 1 (eigenvalue zero).

**Example 1.2.** The flat torus $\mathbb{R}^2/\mathbb{Z}^2$: here the Laplacian is $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and as an orthonormal basis of eigenfunctions we may take the exponentials $\exp(2\pi i k\cdot x)$, $k \in \mathbb{Z}^2$, with eigenvalue $4\pi^2|k|^2$.

**Spectral counting functions.** We define the following spectral functions: A cumulative count

$$n(x) := \#\{m : \sqrt{\lambda_m} \le x\}$$

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and a window count

$$n(x,\delta) := \#\{m : x < \sqrt{\lambda_m} \le x+\delta\}.$$

In the case of the torus $\mathbb{R}^2/\mathbb{Z}^2$, the cumulative counting function

$$n(x) = \#\{k \in \mathbb{Z}^2 : \sqrt{4\pi^2|k|^2} \le x\}$$

is exactly the number of lattice points in a disk of radius $t = x/2\pi$. The spectral function $n(x,\delta)$ is then the number of lattice points in an annulus of width $\delta/2\pi$:

$$n(x,\delta) = \#\{k \in \mathbb{Z}^2 : T < 2\pi|k| < x+\delta\}.$$

**Weyl's law.** Weyl's law gives the asymptotics of $n(x)$ as $x \to \infty$. In our general context of a compact, $d$-dimensional manifold, Weyl's law says that

$$n(x) \sim c_d \operatorname{vol}(M) x^d, \quad x \to \infty$$

where $c_d$ is a universal constant depending only on the dimension. For the surface case, $c_2 = 1/4\pi$. Thus in the case of the torus, where the area is 1, Weyl's law says that

$$n(x) \sim \frac{1}{4\pi} x^2$$

which is the area of the circle of radius $x/2\pi$. This is consistent with the simplest intuition that the number of lattice points in such a circle is well approximated by its area.

**The remainder term.** A famous problem is to bound the remainder term for the number $N(t)$ of lattice points in a disk of radius $t$. An elementary packing argument, attributed to Gauss, gives

$$N(t) = \pi t^2 + O(t)$$

and the circle problem is to find the best exponent $\theta$ for which

$$N(t) = \pi t^2 + O(t^{\theta+\epsilon}), \quad \forall \epsilon > 0.$$

This is a problem with a long history. The first nontrivial result is due to Sierpinski (circa 1905) who showed that one can take $\theta = 2/3$. The current record is due to Huxley (2005) giving $\theta = 131/208 = 0.629\ldots$ The well-known conjecture of G. H. Hardy is that one may take $\theta = 1/2$, and it is known that one cannot do better. For instance, H. Cramér [6] showed that the second moment of the remainder is

$$\frac{1}{T}\int_0^T \left|\frac{N(t)-\pi t^2}{\sqrt{t}}\right|^2 dt = \text{const} > 0$$

and thus we certainly cannot have $\theta < 1/2$.

**The distribution of the remainder.** In the 1990's, Heath-Brown considered an interesting twist on the problem, by asking for the distribution function of the normalized remainder term

$$S(t) := \frac{N(t)-\pi t^2}{\sqrt{t}},$$

that is, to find a probability distribution $P(s)$ on the line so that the asymptotic fraction of $t$'s such that $a < S(t) < b$ is $\int_a^b P(s)ds$, precisely

$$\lim_{T\to\infty}\frac{1}{T}\,\text{meas}\{T < t \le 2T : a < S(t) < b\} = \int_a^b P(s)ds$$

where "meas" is Lebesgue measure.

Heath-Brown [8] showed that the limiting distribution $P(s)$ exists, and remarkably it is not the Gaussian distribution. For instance it is skew (it has positive third moment) and has shorter tails than a Gaussian: as $|s| \to \infty$, $P(s)$ decays roughly as $\exp(-s^4)$.

**Other integrable systems.** The work of Heath-Brown has had a large impact in the mathematical physics community. Bleher, Dyson, Lebowitz, Sinai and others have studied the distribution of the remainder term in Weyl's law for the spectral function $n(x)$ for a number of other "integrable" systems, for instance surfaces of revolution, Liouville tori, as well as other lattice point problems. In all these integrable cases they found a similar feature: the remainder term, properly normalized, has a non-Gaussian limiting distribution. We refer to Bleher's survey [3] for more details and references.

**Lattice points in thin annuli.** One may also ask for the statistics of the window count $n(x,\delta)$. In the case of the flat torus, this is non other than the number $N(t,\rho)$ of lattice points in an annulus of width $\rho = \delta/2\pi$ and in-radius $t = x/2\pi$. Assuming that $t$ is chosen at random in the interval $[T, 2T]$, we may make the width $\rho = \rho(T)$ vary as well. Here we focus on the "mesoscopic", or "intermediate" regime, where $\rho(T) \to 0$ as $T \to \infty$, so that we have a "thin" annulus, but nonetheless we still insist that the area $2\pi t\rho + \pi\rho^2$ of the annulus (which is the expected number of lattice points) grows to infinity, that is that $T\rho(T) \to \infty$. This problem was studied by Bleher and Lebowitz, who showed [5] that the variance of $N(t,\rho)$ in this regime is

$$\frac{1}{T}\int_T^{2T} |N(t,\rho) - 2\pi t\rho|^2 \sim \text{const} \cdot T\rho|\log\rho|, \quad T \to \infty.$$

The Gaussian conjecture of Bleher and Lebowitz [4] is that the normalized remainder term

$$S(t,\rho) := \frac{N(t,\rho) - 2\pi t\rho}{\sqrt{16t\rho|\log\rho|}}$$

has a standard Gaussian value distribution, in contrast to the non-Gaussian distribution $P(s)$ for $N(t)$, that is

$$\lim_{T\to\infty} \frac{1}{T}\operatorname{meas}\{t \in [T,2T] : a < S(t,\rho) < b\} = \int_a^b e^{-s^2/2}\frac{ds}{\sqrt{2\pi}}.$$

**Results.** In 2004, Hughes and the author [9] proved this conjecture for the case that the width of the annulus shrinks "slowly" to zero, that is when $\rho(T) > 1/T^\epsilon$ for all $\epsilon > 0$ (for instance when $\rho(T) = 1/\log T$).

Subsequently, Wigman [12, 13] considered the case of elliptical annuli and showed Gaussian distribution for the case when the aspect ratio is transcendental and strongly Diophantine in a suitable sense, still in the regime where the width is slowly shrinking.

It is an open problem to establish the Gaussian law throughout the mesoscopic regime.

**The Berry-Tabor conjecture.** A different regime, sometime called the "microscopic" regime, is when the width of the annulus shrinks rapidly, while keeping the area $\lambda$ of the annulus finite (and nonzero). Let us now assume that we are dealing with a generic elliptical annulus. A special case of a conjecture of Berry and Tabor [1] is that the number $N(t,\rho)$ of lattice points in such an annulus has a Poisson distribution with parameter $\lambda$, that is for any integer $k \geq 0$,

$$\lim_{T\to\infty} \frac{1}{T}\operatorname{meas}\{t \in [T,2T] : N(t,\rho) = k\} = e^{-\lambda}\frac{\lambda^k}{k!}.$$

We cannot expect an interesting distribution for the case of *circular* annuli, as most such annuli will have no lattice points.

The Berry-Tabor conjecture is wide open. The only knowledge we have is that the second moment of $N(t,\rho)$ is as predicted. This was proved by Sarnak [11] for random ellipses, and by Eskin, Margulis and Mozes [7] for Diophantine ellipses.

**Chaotic systems.** As we saw, the question of the statistics of the spectral counting functions for the flat torus leads to interesting lattice point problems which are still largely unsolved.

The torus falls into the family of surfaces with *integrable* geodesic flow. The other extreme case is that of "chaotic" geodesic flow, for instance negatively curved surfaces. In that case, it is expected that the remainder term for the spectral function $n(x)$ has *Gaussian* fluctuations. Little is known about this, save for the special example of the modular surface [10]. In the microscopic regime, it is believed that for "generic" surfaces the statistics of the spectral counting function $n(x, \delta)$ is modelled by that of suitable random matrix ensembles [2], rather than by the Poisson model. While there have been several investigations of this by physicists and plenty of numerical evidence, virtually nothing is known here in a rigorous fashion.

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