Lower bounds for moments of *L*-functions

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The moments of central values of families of *L*-functions have recently attracted much attention and, with the work of Keating and Snaith [(2000) *Commun. Math. Phys.* 214, 57–89 and 91–110], there are now precise conjectures for their limiting values. We develop a simple method to establish lower bounds of the conjectured order of magnitude for several such families of *L*-functions. As an example we work out the case of the family of all Dirichlet *L*-functions to a prime modulus.

A classical question in the theory of the Riemann zeta function asks for asymptotics of the moments $\int_1^T |\zeta(\frac{1}{2}+it)|^{2k}\,dt$, where k is a positive integer. A folklore conjecture states that the $2k$th moment should be asymptotic to $C_k T(\log T)^{k^2}$ for a positive constant C_k. Only very recently with the work of Keating and Snaith (1) modeling the moments of $\zeta(s)$ by moments of characteristic polynomials of random matrices has a conjecture emerged for the value of C_k. This conjecture agrees with classical results of Hardy and Littlewood and Ingham (see ref. 2) in the cases $k = 1$ and $k = 2$, but for $k \geq 3$ very little is known. Ramachandra (3) showed that $\int_1^T |\zeta(\frac{1}{2}+it)|^{2k}\,dt \gg T(\log T)^{k^2}$ for positive integers $2k$, and Heath-Brown (4) extended this for any positive rational number k. Titchmarsh (see theorem 7.19 of ref. 2) had previously obtained a smooth version of these lower bounds for positive integers k.

Analogously, given a family of L-functions an important problem is to understand the moments of the central values of these L-functions. Modeling the family of L-functions by using the choice of random matrix ensembles suggested by Katz and Sarnak (5) in their study of low-lying zeros, Keating and Snaith (6) have advanced conjectures for such moments. We illustrate these conjectures by considering three prototypical examples. The family of Dirichlet L-functions $L(s, \chi)$ as χ varies over primitive characters (mod q) is a unitary family, and it is conjectured that

$$\sum_{\chi \ (\mathrm{mod}\ q)}^{*} |L(\tfrac{1}{2}, \chi)|^{2k} \sim C_{1,k}\, q(\log q)^{k^2}, \qquad [1]$$

where $k \in \mathbb{N}$ and $C_{1,k}$ is a specified positive constant. The family of quadratic Dirichlet L-functions $L(s, \chi_d)$, where d is a fundamental discriminant and χ_d is the associated quadratic character, is a symplectic family and it is conjectured that

$$\sum_{|d|\leq X} L(\tfrac{1}{2}, \chi_d)^k \sim C_{2,k}\, X(\log X)^{k(k+1)/2}, \qquad [2]$$

where $k \in \mathbb{N}$ and $C_{2,k}$ is a specified positive constant. The family of quadratic twists of a given newform f, $L(s, f \otimes \chi_d)$ (the L-function is normalized so that the central point is $\frac{1}{2}$); this is an orthogonal family and it is conjectured that

$$\sum_{|d|\leq X} L(\tfrac{1}{2}, f \otimes \chi_d)^k \sim C_{3,k} X(\log X)^{k(k-1)/2}, \qquad [3]$$

where $k \in \mathbb{N}$, $C_{3,k}$ is a specified constant that depends on the form f.

While asymptotics in Eqs. **1–3** are known for small values of k, for large k these conjectures appear formidable. Further, the methods used to obtain lower bounds for moments of $\zeta(s)$ do not appear to generalize to this situation. In this article we describe a simple method that furnishes lower bounds of the conjectured order of magnitude for many families of L-functions, including the three prototypical examples given above. As a rough principle, it seems that whenever one can evaluate the first moment of a family of L-functions (with a little bit to spare) then one can obtain good lower bounds for all moments. We illustrate our method by giving a lower bound (of the conjectured order of magnitude) for the family of Dirichlet characters modulo a prime.

Theorem. *Let k be a fixed natural number. Then for all large primes q*

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} |L(\tfrac{1}{2}, \chi)|^{2k} \gg_k q(\log q)^{k^2}.$$

Our methods yield corresponding lower bounds for several other families, as well as other applications, for instance to fluctuations of matrix elements of Maass wave forms in the modular domain, which are not presented here. We remark also that we may take k to be any positive rational number ≥ 1 in the theorem. If $k = r/s (\geq 1)$ is rational, then we achieve this by taking $A(\chi) = (\sum_{n\leq q^{1/(2r)}} d_{1/s}(n)\chi(n)/\sqrt{n})^s$ in the argument below.

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Proof: Let $x := q^{1/(2k)}$ be a small power of q, and set $A(\chi) = \sum_{n\leq x} \chi(n)/\sqrt{n}$. We will evaluate

$$S_1 := \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} L(\tfrac{1}{2}, \chi) A(\chi)^{k-1} \overline{A(\chi)}^{k}, \text{ and } S_2 := \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} |A(\chi)|^{2k},$$

and show that $S_2 \ll q(\log q)^{k^2} \ll S_1$. The theorem then follows from Hölder's inequality:

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} |L(\tfrac{1}{2}, \chi)|^{2k} \geq \frac{|S_1|^{2k}}{S_2^{2k-1}} \gg q(\log q)^{k^2}.$$

If $\ell \in \mathbb{N}$ then we may write $A(\chi)^\ell = \sum_{n\leq x^\ell} \chi(n) d_\ell(n; x)/\sqrt{n}$ where $d_\ell(n; x)$ denotes the number of ways of writing n as $a_1 \cdots a_\ell$ with each $a_j \leq x$. As usual $d_\ell(n)$ will denote the ℓth divisor function, and note that $d_\ell(n; x) \leq d_\ell(n)$ with equality holding when $n \leq x$.

We start with S_2. Note that $A(\chi_0) \ll \sqrt{x}$ and so

$$\sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} |A(\chi)|^{2k} = \sum_{\chi \ (\mathrm{mod}\ q)} |A(\chi)|^{2k} + O(x^k)$$

$$= \sum_{m,n\leq x^k} \frac{d_k(m, x) d_k(n, x)}{\sqrt{mn}} \sum_{\chi \ (\mathrm{mod}\ q)} \chi(m)\bar{\chi}(n) + O(x^k).$$

Since $x^k = \sqrt{q} < q$ the orthogonality relation for characters (mod q) gives that only the diagonal terms $m = n$ survive. Thus

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$$S_2 = \phi(q) \sum_{n \le x^k} \frac{d_k(n,x)^2}{n} + O(\sqrt{q}).$$

Since $d_k(n, x) \le d_k(n)$ and $\Sigma_{n\le y}\, d_k(n)^2/n \sim c_k(\log y)^{k^2}$ for a positive constant c_k, we find that $S_2 \ll q(\log q)^{k^2}$, as claimed.

We now turn to S_1. If $\mathrm{Re}(s) > 1$ then integration by parts gives

$$L(s,\chi) = \sum_{n\le X} \frac{\chi(n)}{n^s} + \int_X^\infty \frac{1}{y^s}\, d\Bigg(\sum_{X<n\le y} \chi(n)\Bigg)$$

$$= \sum_{n\le X} \frac{\chi(n)}{n^s} + s\int_X^\infty \frac{\sum_{X<n\le y}\chi(n)}{y^{s+1}}\, dy.$$

Since the numerator of the integrand above is $\ll \sqrt{q}\log q$ by the Pólya-Vinogradov inequality (see chapter 23 of ref. 7) the above expression furnishes an analytic continuation of $L(s,\chi)$ to $\mathrm{Re}(s) > 0$. Moreover we obtain

$$L(\tfrac{1}{2},\chi) = \sum_{n\le X} \frac{\chi(n)}{\sqrt{n}} + O\Bigg(\frac{\sqrt{q}\log q}{\sqrt{X}}\Bigg).$$

We choose here $X = q\log^4 q$ and obtain

$$S_1 = \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \sum_{n\le X} \frac{\chi(n)}{\sqrt{n}} A(\chi)^{k-1}\, \overline{A(\chi)}^k$$

$$+ O\Bigg(\frac{1}{\log q} \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} |A(\chi)|^{2k-1}\Bigg).$$

Since $|A(\chi)|^{2k-1} \le 1 + |A(\chi)|^{2k}$ the error term above is $\ll (q + S_2)/\log q$. The main term is

$$\sum_{\chi\,(\mathrm{mod}\, q)} \sum_{n\le X} \frac{\chi(n)}{\sqrt{n}} A(\chi)^{k-1}\, \overline{A(\chi)}^k + O(\sqrt{X}x^{k-\frac{1}{2}}).$$

Recalling that $x = q^{1/(2k)}$ and using the orthogonality relation for characters we conclude

$$S_1 = \phi(q) \sum_{a\le x^{k-1}} \sum_{b\le x^k} \sum_{\substack{n\le X\\ an\equiv b\,(\mathrm{mod}\, q)}} \frac{d_{k-1}(a;x)d_k(b;x)}{\sqrt{abn}}$$

$$+ O\Bigg(\frac{S_2}{\log q}\Bigg).$$

The main term above will arise from the diagonal terms $an = b$. Let us first estimate the contribution of the off-diagonal terms. Here we may write $an = b + q\ell$, where $1 \le \ell \le Xx^{k-1}/q = x^{k-1}(\log q)^4$. The contribution of these off-diagonal terms is

$$\ll q \sum_{b\le x^k} \frac{d_k(b;x)}{\sqrt{b}} \sum_{\ell \le x^{k-1}(\log q)^4} \frac{1}{\sqrt{q\ell}} \sum_{an=b+q\ell} d_{k-1}(a;x)$$

$$\ll q^{\frac{1}{2}+\varepsilon} x^{\frac{k-1}{2}+\frac{k}{2}} \ll \frac{q}{\log q}$$

since $\Sigma_{an=b+q\ell}\, d_{k-1}(a;x) \le d_k(b + q\ell) \ll (q\ell)^\varepsilon$. Therefore,

$$S_1 = \phi(q) \sum_{b\le x^k} \frac{d_k(b;x)}{b} \sum_{\substack{a\le x^{k-1}, n\le X\\ an=b}} d_{k-1}(a;x) + O\Bigg(\frac{S_2}{\log q}\Bigg).$$

Since

$$\sum_{\substack{a\le x^{k-1}, n\le X\\ an=b}} d_{k-1}(a;x) \le \sum_{\substack{a\le x^{k-1}, n\le x\\ an=b}} d_{k-1}(a;x) = d_k(b;x),$$

and $d_k(b; x) = d_k(b)$ for $b \le x$, we deduce that

$$S_1 \ge \phi(q) \sum_{b\le x^k} \frac{d_k(b;x)^2}{b} + O\Bigg(\frac{S_2}{\log q}\Bigg)$$

$$\ge \phi(q) \sum_{b\le x} \frac{d_k(b)^2}{b} + O\Bigg(\frac{S_2}{\log q}\Bigg) \gg q(\log q)^{k^2}.$$

This proves the theorem.

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