# The Robin problem on rectangles

Cite as: J. Math. Phys. **62**, 113503 (2021); https://doi.org/10.1063/5.0061763 Submitted: 29 June 2021 • Accepted: 10 October 2021 • Published Online: 01 November 2021

🔟 Zeév Rudnick and 🔟 Igor Wigman







J. Math. Phys. **62**, 113503 (2021); https://doi.org/10.1063/5.0061763 © 2021 Author(s).

## The Robin problem on rectangles

Cite as: J. Math. Phys. 62, 113503 (2021); doi: 10.1063/5.0061763 Submitted: 29 June 2021 • Accepted: 10 October 2021 • Published Online: 1 November 2021



Zeév Rudnick<sup>1,a)</sup> (D) and Igor Wigman<sup>2,b)</sup> (D)

## AFFILIATIONS

<sup>1</sup>School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
 <sup>2</sup>Department of Mathematics, King's College London, London, United Kingdom

Note: This paper is part of the Special Collection in Honor of Freeman Dyson. <sup>a)</sup>Author to whom correspondence should be addressed: rudnick@tauex.tau.ac.il <sup>b)</sup>E-mail: igor.wigman@kcl.ac.uk

## ABSTRACT

We study the statistics and the arithmetic properties of the Robin spectrum of a rectangle. A number of results are obtained for the multiplicities in the spectrum depending on the Diophantine nature of the aspect ratio. In particular, it is shown that for the square, unlike the case of Neumann eigenvalues where there are unbounded multiplicities of arithmetic origin, there are no multiplicities in the Robin spectrum for a sufficiently small (but nonzero) Robin parameter except a systematic symmetry. In addition, uniform lower and upper bounds are established for the Robin–Neumann gaps in terms of their limiting mean spacing. Finally, the pair correlation function of the Robin spectrum on a Diophantine rectangle is shown to be Poissonian.

Published under an exclusive license by AIP Publishing. https://doi.org/10.1063/5.0061763

#### I. STATEMENT OF MAIN RESULTS

Let  $\Omega \subset \mathbb{R}^2$  be a compact planar domain with the Lipschitz boundary. The Robin eigenvalue problem on  $\Omega$  is to solve the eigenvalue equation  $-\Delta f = \lambda f$  with boundary conditions

$$\frac{\partial f}{\partial n}(x) + \sigma f(x) = 0, \quad x \in \partial \Omega,$$

where  $\frac{\partial f}{\partial n}$  is the derivative in the direction of the outward pointing normal to  $\partial \Omega$  and  $\sigma > 0$ . This boundary condition arises in the study of heat conduction; see, e.g., Chap. 1 of Ref. 12. Our goal is to study arithmetic properties and statistics of the Robin eigenvalues on a rectangle. For the results related to shape optimization for the first two eigenvalues of the Robin Laplacian on a rectangle, see Ref. 6.

Consider the case of the unit square. For  $\sigma = 0$ , the Neumann eigenvalues on the unit square are explicitly given as  $\pi^2(n^2 + m^2)$  for integer  $n, m \ge 0$ . In particular, there are multiplicities coming from the many different ways of writing some of the integers as a sum of two squares.

For  $\sigma \neq 0$ , there is no known explicit formula. The problem is, however, separable, with an orthogonal basis of eigenfunctions of the form  $u_{n,m}(x,y) = u_n(x) \cdot u_m(y)$ , where  $u_n(x)$  are the eigenfunctions of the Laplacian on the unit interval:  $-u''_n = k_n^2 u_n$ , satisfying the one-dimensional Robin boundary conditions

$$-u'(0) + \sigma u(0) = 0, \quad u'(1) + \sigma \cdot u(1) = 0.$$

The frequencies  $k_n$  are the unique solutions of the secular equation

$$\tan(k_n) = \frac{2\sigma k_n}{k_n^2 - \sigma^2} \tag{1.1}$$

in the range  $n\pi < k_n < (n+1)\pi$ ,  $n \ge 0$ ; see Lemma 2.1. The eigenfunction corresponding to  $k_n$  is  $u_n(x) = k_n \cos(k_n x) + \sigma \sin(k_n x)$ . The Robin eigenvalues on the unit square are

$$\Lambda_{n,m}=k_n^2+k_m^2,$$

with eigenfunction  $u_n(x) \cdot u_m(y)$ , and admit the symmetry  $\Lambda_{n,m} = \Lambda_{m,n}$ . For the rectangle

$$\mathcal{R}_L = [0,1] \times [0,L],$$

with the aspect ratio L > 0, the Robin energy levels of  $\mathcal{R}_L$  are all numbers

$$\Lambda_{L;n,m}(\sigma) = k_n(\sigma)^2 + \frac{1}{L^2} \cdot k_m(\sigma \cdot L)^2, \quad n, m \ge 0.$$
(1.2)

Note that if  $L \neq 1$ , there is no longer the symmetry  $(n, m) \mapsto (m, n)$ .

#### A. Multiplicities

We now consider possible multiplicities in the Robin spectrum of rectangles. Recall that for the square and, more generally, for a rectangle  $\mathcal{R}_L = [0, 1] \times [0, L]$  with  $L^2$  rational, the Neumann spectrum has large multiplicities of arithmetic nature, whereas for  $L^2$  irrational, there are no multiplicities. Our first goal is to show that for  $\sigma > 0$  sufficiently small, there are no multiplicities in the Robin spectrum of the square beyond the trivial symmetry  $\Lambda_{1;n,m} = \Lambda_{1;m,n}$ .

**Theorem 1.1.** There exists  $\sigma_0 > 0$  so that for  $0 < \sigma < \sigma_0$ , there are no spectral multiplicities other than the trivial ones  $\Lambda_{1;n,m}(\sigma) = \Lambda_{1;m,n}(\sigma)$ .

In the Proof of Theorem 1.1 (see Sec. III), we shall see that as  $\sigma$  varies, the eigenvalues  $\Lambda_{1;n,m}(\sigma)$  evolve at different rates depending on *n*, *m*. These discrepancies are sufficiently large to break the degeneracies of the Neumann case ( $\sigma = 0$ ) for  $\sigma > 0$  sufficiently small.

One should compare the statement of Theorem 1.1 asserting that, for  $\sigma > 0$  sufficiently small, the Robin spectrum of the square is non-degenerate to the recent result<sup>10</sup> asserting that the Robin spectrum of the hemisphere is non-degenerate for every  $\sigma > 0$ . On the other hand, the Robin spectrum of the square does admit nontrivial spectral degeneracies for sufficiently large  $\sigma$  (see Proposition 3.4).

Next, we consider the rectangle  $\mathcal{R}_L$  with  $L^2$  irrational. Unlike the square, here, there exist multiplicities even for small  $\sigma$ .

**Theorem 1.2.** If  $L^2$  is irrational, then there are arbitrarily small  $\sigma > 0$  for which there are multiplicities in the Robin spectrum of the rectangle  $\mathcal{R}_L$ .

The Proof of Theorem 1.2 involves some arithmetic, in particular, in showing that the set of values attained by the indefinite ternary quadratic form

$$Q(x, y, z) = L^2 x^2 + y^2 - z^2$$

at integer values of (x, y, z) intersects every neighborhood of the origin:  $-\epsilon < Q(n, m, m') < 0$  with all variables nonzero integers. This is a variation on the Oppenheim conjecture (proved by Margulis<sup>7</sup>), which turns out to admit a simple solution using only the density of the fractional parts of  $L^2n^2 \mod 1$ , due to Hardy and Littlewood.<sup>5</sup>

We next show that in some special cases, we can give an upper bound for the multiplicities and for the number of eigenvalues that are not simple. Let  $\lambda_1(\sigma) \leq \lambda_2(\sigma) \leq \cdots$  be the ordering (with multiplicities) of the Robin eigenvalues of  $\mathcal{R}_L$ . By Weyl's law, the number of eigenvalues of size at most  $\lambda$  is asymptotically

$$N(\lambda) = N_{L;\sigma}(\lambda) := \#\{\lambda_j(\sigma) \le \lambda\} \sim \frac{\operatorname{Area}(\mathcal{R}_L)}{4\pi}\lambda, \quad \lambda \to \infty.$$
(1.3)

Denote by  $N^{\text{mult}}(\lambda)$  the number of multiple eigenvalues  $\leq \lambda$  (again, counting the multiplicities in).

**Theorem 1.3.** If  $L^2$  is badly approximable, then there exists  $\sigma_0 > 0$  so that for  $\sigma < \sigma_0$ , all the multiplicities in the Robin spectrum of the rectangle  $\mathcal{R}_L$  are bounded by 3 and

$$N^{\text{mult}}(\lambda) \ll \sqrt{\lambda}. \tag{1.4}$$

*If, in addition, L is badly approximable, then the multiplicities are bounded by 2.* 

Recall that number  $\theta$  is "badly approximable" if there is some  $c = c(\theta) > 0$  so that for all integer  $p, q \in \mathbb{Z}$  with  $q \ge 1$ , we have

$$\left|\theta - \frac{p}{q}\right| \ge \frac{c}{q^2}.$$
(1.5)

For instance, quadratic irrationalities are badly approximable.

## B. Robin-Neumann gaps

We next turn to the differences between the Robin and Neumann eigenvalues, or simply RN gaps, introduced in Ref. 11. Let  $\lambda_1(\sigma) = \lambda_1^L(\sigma) \le \lambda_2(\sigma) = \lambda_2^L(\sigma) \le \cdots$  be the ordering (with multiplicities) of the Robin eigenvalues of the rectangle  $\mathcal{R}_L$ . For example, for  $\sigma = 0$ , we recover the Neumann eigenvalues and  $\lambda_j(\infty)$  are the Dirichlet eigenvalues. The RN gaps are the non-negative numbers,

$$d_j = d_j^L(\sigma) \coloneqq \lambda_j^L(\sigma) - \lambda_j^L(0).$$

For every bounded domain  $\Omega$  with a piecewise smooth boundary, it was shown in Ref. 11 (Theorem 1.1) that for  $\sigma > 0$ , there is a limiting mean RN gap, asymptotic to

$$\overline{d}(\sigma) \coloneqq \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d_j(\sigma) = \frac{2 \operatorname{length}(\partial \Omega)}{\operatorname{Area}(\Omega)} \cdot \sigma.$$
(1.6)

Concerning the individual RN gaps, the authors of Ref. 11 gave a uniform lower bound for arbitrary star-shaped domains with a smooth boundary. For an upper bound, they proved that

$$d_j(\sigma) \leq C_{\Omega} \cdot (\lambda_j(\infty))^{1/3} \cdot \sigma,$$

valid for any  $\Omega$  with a smooth boundary. This could be compared to the bound (Ref. 4, Theorem 2)

$$0 \leq \lambda_i(\infty) - \lambda_i(\sigma) \leq C \sigma^{-1/2} \lambda_i(\infty)^2$$

with C > 0 absolute, for the distance between the Robin eigenvalues and the corresponding Dirichlet eigenvalue, in the regime  $\sigma \to +\infty$ , also assuming that  $\Omega$  has a smooth boundary.

For the rectangle, the authors of Ref. 11 (Theorems 1.3 and 1.7) gave the more precise upper bound

$$d_i^L(\sigma) \le C_{L,\sigma} \tag{1.7}$$

for some constant  $C_{L,\sigma} > 0$ . Here, we give uniform upper and lower bounds for the rectangle in terms of the mean gap  $\overline{d}(\sigma)$ , in particular, refining the upper bound (1.7).

**Theorem 1.4.** There exist absolute constants C > c > 0 so that for every rectangle, for all  $\sigma > 0$  and  $j \ge 1$ ,

$$d_j(\sigma) \le C \cdot d(\sigma),\tag{1.8}$$

and for all  $\sigma \in (0, 1]$ ,

$$d_j(\sigma) \ge c \cdot \overline{d}(\sigma). \tag{1.9}$$

Note that (1.9) can only be valid for

$$\sigma \leq \frac{1}{c} \frac{\pi^2 \left( L + \frac{1}{L} \right)}{4(1+L)}.$$

Indeed,  $\lambda_1(0) \leq \lambda_1(\sigma) \leq \lambda_1(\infty)$  so that

$$c \cdot \frac{4(1+L)}{L} \cdot \sigma = c \cdot \overline{d}(\sigma) \le d_1(\sigma) \le \lambda_1(\infty) - \lambda_1(0) = \left(1 + \frac{1}{L^2}\right)\pi^2.$$

#### C. Pair correlation for the Robin energies

We next turn to study the statistics of the eigenvalues on the scale of their mean spacing. In our case, the mean spacing between the eigenvalues is constant,

$$\bar{s} := \lim_{N \to \infty} \frac{1}{N} \sum_{k \le N} (\lambda_{k+1}(\sigma) - \lambda_k(\sigma)) \sim \frac{4\pi}{\operatorname{Area} \mathcal{R}_L},$$
(1.10)

by Weyl's law (1.3). One popular local statistic is the distribution P(s) of nearest-neighbor gaps  $(\lambda_{k+1}(\sigma) - \lambda_k(\sigma))/\overline{s}$ . For the square (and more generally when  $L^2$  is rational), P(s) is a delta function at the origin.<sup>11</sup> However, we expect that if the squared aspect ratio  $L^2$  is a Diophantine irrationality, that is, there is some  $\kappa > 0$  so that  $|L^2 - p/q| > 1/q^{\kappa}$  for all integers q > 1 and p, then the nearest-neighbor gap distribution will be Poissonian:  $P(s) = e^{-s}$ , that is, as for uncorrelated levels; cf. Refs. 1, 8, and 9. However, at present, this quantity is not accessible. A more tractable statistic is the pair correlation function, defined as follows: For a test function  $f \in C_c^{\infty}(\mathbb{R})$ , we set

$$R_2^{\sigma}(f,N) = \frac{1}{N} \sum_{1 \le k \ne k' \le N} f\left(\frac{\lambda_k(\sigma) - \lambda_{k'}(\sigma)}{\bar{s}}\right).$$

The Poisson expectation is that

$$\lim_{N\to\infty}R_2^{\sigma}(f,N)=\int_{-\infty}^{\infty}f(x)dx.$$

## **Theorem 1.5.** Assume that $L^2$ is a Diophantine irrationality. Then, for every fixed $\sigma > 0$ , the pair correlation function is Poissonian.

To prove Theorem 1.5, we establish a comparison with the pair correlation of the Neumann spectrum (Proposition 6.1), which was shown to be Poissonian in the Diophantine case by Eskin, Margulis, and Mozes.<sup>3</sup> There are two key ingredients in the comparison argument: a stronger, asymptotic, form of the bound for the RN gaps of Theorem 1.4 (see Proposition 4.6) and a count of lattice points in annular regions, for which it suffices to appeal to a classical remainder term in the lattice point problem.

It is of interest to investigate analogs of our results for the case when the boundary conditions are non-constant, that is,

$$\frac{\partial f}{\partial n}(x) + \sigma(x)f(x) = 0$$

for *x* on the boundary, where  $\sigma(x) > 0$  is a continuous function on the boundary. New methods will be required since we make heavy use of the fact that  $\sigma$  is constant.

#### **II. THE ONE-DIMENSIONAL PROBLEM**

In this section, we review some classical properties of the Robin eigenvalue problem on an interval; see, e.g., Ref. 12 (Sec. 4.3).

#### A. The secular equation

Let  $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$  be the unit interval, and consider the Helmholtz equation

$$f'' + k^2 f = 0, (2.1)$$

subject to Robin boundary conditions

$$\sigma f\left(-\frac{1}{2}\right) - f'\left(-\frac{1}{2}\right) = \sigma f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) = 0.$$

We use the symmetry  $x \mapsto -x$ , which is respected by both the second derivative operator  $f \mapsto f''$  and the boundary conditions, to separate solutions into even and odd symmetry classes. The even solutions of the eigenvalue equation (2.1) are  $f(t) = \cos(kt)$ , which inserting into the boundary conditions gives

$$k \cdot \tan\left(\frac{k}{2}\right) = \sigma.$$

Likewise, the odd solutions of (2.1) are f(t) = sin(kt), and the secular equation is

$$-k \cdot \cot\left(\frac{k}{2}\right) = \sigma.$$

As we shall see below, the solutions of the even and odd secular equations interlace, and the totality of solutions  $\{k_n(\sigma) : n = 0, 1, 2, ...\}$  is the solutions of the combined secular equation

$$\left(k \, \operatorname{tan}\left(\frac{k}{2}\right) - \sigma\right) \cdot \left(k \, \operatorname{cot}\left(\frac{k}{2}\right) + \sigma\right) = 0,$$

which, after some algebra, reads

$$\tan(k) = \frac{2\sigma k}{k^2 - \sigma^2}.$$
(2.2)

We could also deduce Eq. (2.2) directly if we ignore the symmetry  $x \mapsto -x$ .

## ARTICLE

#### **B.** General intervals

Instead of the unit interval, we consider an interval of length *L*. The Laplace eigenfunctions  $f'' + k^2 f = 0$  on  $\left[-\frac{L}{2}, \frac{L}{2}\right]$  are subject to the Robin boundary conditions

$$\sigma f\left(-\frac{L}{2}\right) - f'\left(-\frac{L}{2}\right) = \sigma f\left(\frac{L}{2}\right) + f'\left(\frac{L}{2}\right) = 0.$$

We obtain solutions to the Helmholtz equation on  $\left[-\frac{L}{2}, \frac{L}{2}\right]$  by scaling the corresponding solutions on the unit interval: If g on  $\left[-\frac{1}{2}, \frac{1}{2}\right]$  solves  $g'' + k^2g = 0$  and  $\sigma g\left(-\frac{1}{2}\right) - g'\left(-\frac{1}{2}\right) = \sigma g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right) = 0$ , then  $f_L(t) = g(t/L)$  on  $\left[-\frac{L}{2}, \frac{L}{2}\right]$  satisfies

$$f'' + \left(\frac{k}{L}\right)^2 f = 0, \quad \frac{\sigma}{L} f_L\left(-\frac{L}{2}\right) - f'_L\left(-\frac{L}{2}\right) = 0 = \frac{\sigma}{L} f_L\left(\frac{L}{2}\right) + f'_L\left(\frac{L}{2}\right)$$

Hence, if we define  $k_{L;n}(\sigma) := \frac{1}{L}k_n(\sigma \cdot L)$ , then the Robin energy levels on [0, L] are

$$k_{L;n}(\sigma)^2 = \frac{1}{L^2} (k_n(\sigma \cdot L))^2, \quad n \ge 0.$$

Note that the secular equation on  $\left[-\frac{L}{2}, \frac{L}{2}\right]$  becomes

$$\tan(Lk)=\frac{2\sigma k}{k^2-\sigma^2}.$$

#### C. Properties of $k_n(\sigma)$

Lemma 2.1. For every  $n \ge 0$  and  $\sigma \ge 0$ , there is a unique solution  $k_n(\sigma)$  to the secular equation (2.2) in the range  $k_n(\sigma) \in (n\pi, (n+1)\pi)$ . The functions  $\sigma \mapsto k_n$  satisfy the following:

a. For all  $n \ge 0$ ,  $k_n(\cdot)$  are strictly increasing everywhere on  $[0, +\infty)$ , with

$$k_n(0) = n \cdot \pi, \tag{2.3}$$

and furthermore,

$$\lim k_n(\sigma) = (n+1) \cdot \pi$$

- b. For  $n \ge 1$ , the function  $\sigma \mapsto k_n(\sigma)$  is analytic everywhere. Furthermore, for  $\sigma < (n+1/2)\pi$ ,  $k_n(\sigma) \in (n\pi, (n+1/2)\pi)$ , and for  $\sigma \ge (n+1/2)\pi$ ,  $k_n(\sigma) \in ((n+1/2)\pi, (n+1)\pi)$ . Moreover,  $k_n(\sigma) = (n+1/2)\pi$  if and only if  $\sigma = (n+1/2)\pi$ .
- c. The function  $k_0(\cdot)$  is analytic everywhere except at  $(\sigma, k_0) = (0, 0)$ . Furthermore, for  $\sigma \in (0, \pi/2)$ ,  $k_0(\sigma) > \sigma$  and

$$k_0(\sigma)^2 = 2\sigma + O(\sigma^2). \tag{2.4}$$

Proof. We first consider the odd part of the spectrum: note that the function

$$S_{-}(k) \coloneqq -k \cdot \cot\left(\frac{k}{2}\right)$$

is even, and for  $k \ge 0$ , it vanishes at  $\pi$ ,  $3\pi$ , ...,  $(2n + 1)\pi$ , ...,  $n \ge 0$ ; it has singularities at

$$k = 2\pi, 4\pi, \ldots, 2n\pi, \ldots,$$

 $n \ge 1$ ; and it is increasing monotonically for  $k \ge 0$  between the singularities because it has a positive derivative there,

$$S'_{-}(k) = \frac{k - \sin k}{2\sin^2(k/2)}.$$

See Fig. 1. Thus, for  $\sigma > 0$ , there is a unique solution  $k_{2n-1}(\sigma)$  of  $S_+(k) = \sigma$  in each interval  $((2n-1)\pi, 2n\pi)$ , n = 1, 2, ... Moreover, by the analytic implicit function theorem, the solutions  $k_{2n-1}(\sigma)$  are analytic in  $\sigma$  for  $n \ge 1$ .

For the even part of the spectrum, the function

$$S_+(k) \coloneqq k \cdot \tan\left(\frac{k}{2}\right)$$



is even, and for  $k \ge 0$ , it vanishes at  $0, 2\pi, ..., 2n\pi, ..., n \ge 0$ ; it has singularities at  $k = \pi, 3\pi, ..., (2n + 1)\pi, ..., n \ge 0$ ; and it is increasing monotonically for  $k \ge 0$  between the singularities because it has positive derivative there,

$$S'_+(k) = \frac{k + \sin k}{2\cos^2(k/2)}.$$

See Fig. 1. Thus, for  $\sigma > 0$ , there is a unique solution  $k_{2n}(\sigma)$  of  $S_+(k) = \sigma$  in each interval  $(2n\pi, (2n+1)\pi)$ . Moreover, by the analytic implicit function theorem, the solutions  $k_{2n}(\sigma)$  are analytic in  $\sigma$  for  $n \ge 1$ .

To see that  $k_0(\sigma) > \sigma$  for  $\sigma \in (0, \pi/2)$ , just note that  $0 < \tan(\frac{k}{2}) < 1$  for  $k \in (0, \frac{\pi}{2})$  so that  $\sigma = k \tan(\frac{k_0(\sigma)}{2}) < k_0(\sigma) \cdot 1$  in this range. Finally, to see (2.4), using  $k_0(\sigma) \to 0$  as  $\sigma \to 0$ , we expand

$$\sigma = k \tan\left(\frac{k}{2}\right) = k\left(\frac{k}{2} + O(k^3)\right) = \frac{k^2}{2} + O(k^4)$$

from which (2.4) follows.

## **D.** Auxiliary computations

*Lemma 2.2.* For  $n \ge 1$ , the functions  $k_n(\cdot)$  satisfy the following:

a.

$$k_n'(0) = \frac{2}{\pi n}.$$
 (2.5)

b.

$$(k_n(\sigma)^2)''|_{\sigma=0} = 2(k_n(\sigma) \cdot k'_n(\sigma))'|_{\sigma=0} = -\frac{8}{(\pi n)^2}.$$
(2.6)

c. Uniformly, for  $n \ge 1$ ,  $0 \le \sigma \le 1$ , one has

$$k'_{n} = \frac{2}{k_{n}} \cdot \left(1 + f_{2}(\sigma) \cdot \frac{1}{k_{n}^{2}}\right) + \mathcal{E}_{n}(\sigma)$$
(2.7)

with  $f_2(\sigma) = -\sigma(2 + \sigma)$  and

$$\mathcal{E}_n(\sigma)| = O\left(\frac{\sigma^2}{n^5}\right). \tag{2.8}$$

Proof. We treat the even secular equation, and the odd case is completely analogous. From

$$S_+(k) := k \tan\left(\frac{k}{2}\right) = \sigma,$$

J. Math. Phys. **62**, 113503 (2021); doi: 10.1063/5.0061763 Published under an exclusive license by AIP Publishing

we obtain, by implicit differentiation,  $k' = 1/S'_+(k)$ . Now,

$$S'_{+}(k) = \tan\left(\frac{k}{2}\right) + \frac{k}{2\cos^{2}\left(\frac{k}{2}\right)} = \tan\left(\frac{k}{2}\right) + \frac{k}{2}\left(1 + \tan^{2}\left(\frac{k}{2}\right)\right).$$
(2.9)

Substituting  $k_{2n}(0) = 2n\pi$ , we obtain  $S'_+(k_{2n}(0)) = n\pi = (2n\pi)/2$ , which gives (2.5) in the even case. To obtain (2.6), we use

$$(k_n^2)'' = 2(k_n')^2 + 2k_n k_n''$$
(2.10)

and  $k'_{2n} = 1/S'_+$  so that

$$k_{2n}^{\prime\prime} = \left(\frac{1}{S_{+}^{\prime}}\right)^{\prime} = -\frac{k_{2n}^{\prime}S_{+}^{\prime\prime}}{(S_{+}^{\prime})^{2}} = -(k_{2n}^{\prime})^{3}S_{+}^{\prime\prime}$$

A computation shows that

$$S_+'' = \frac{2+k\,\tan\left(\frac{k}{2}\right)}{2\cos^2\left(\frac{k}{2}\right)}.$$

Evaluating at  $\sigma = 0$ , where  $k_{2n}(0) = 2n\pi$ , we obtain  $S''_{+}(k_{2n}(0)) = 1$  and

$$k_{2n}^{\prime\prime}(0) = -k_{2n}^{\prime}(0)^{3}S_{+}^{\prime\prime}(k_{2n}(0)) = -\left(\frac{2}{2n\pi}\right)^{3}.$$

Substituting into (2.10) with  $k_{2n}(0) = 2n\pi$  and  $k'_{2n}(0) = 2/(2n\pi)$ , we deduce (2.6).

To obtain (2.7), we return to (2.9), use the secular equation to write  $tan(\frac{k}{2}) = \frac{\sigma}{k}$  for  $\sigma > 0$ , and obtain

$$S'_+(k) = \frac{\sigma}{k} + \frac{k}{2}\left(1 + \frac{\sigma^2}{k^2}\right) = \frac{k}{2} \cdot \left(1 - \frac{f_2(\sigma)}{k^2}\right).$$

Hence,

$$k' = \frac{1}{S'_{+}(k)} = \frac{2}{k} \left( 1 + \frac{f_{2}(\sigma)}{k^{2}} \right) + O\left( \frac{f_{2}(\sigma)^{2}}{k^{5}} \right),$$

which for  $\sigma \leq 1$  is (2.7).

#### III. SPECTRAL DEGENERACIES FOR THE SQUARE: PROOF OF THEOREM 1.1

## A. No multiplicities for the square near $\sigma$ = 0: Proof of Theorem 1.1

A lattice point  $(n, m) \in \mathbb{Z}_{\geq 0}^2$  gives rise to the energy [see (1.2)],

$$\Lambda_{n,m}(\sigma) = \Lambda_{1;n,m}(\sigma) \coloneqq k_n(\sigma)^2 + k_m(\sigma)^2.$$
(3.1)

In this section, we will use the shorthand  $\Lambda_{n,m}(\cdot) = \Lambda_{1;n,m}(\cdot)$ . For the Proof of Theorem 1.1, we will need the following propositions.

*Proposition 3.1.* Uniformly, for  $n, m \ge 1$  and  $0 \le \sigma \le 1$ , one has

$$\Lambda_{n,m}'(\sigma) = 4\left(2 + \frac{f_2(\sigma)}{\pi^2} \left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right) + E_{(n,m)}(\sigma)$$
(3.2)

with

$$f_2(\sigma) \coloneqq -\sigma(2+\sigma), \tag{3.3}$$

where the error term satisfies

$$|E_{(n,m)}(\sigma)| = O\left(\sigma^2 \left(\frac{1}{n^4} + \frac{1}{m^4}\right)\right).$$
(3.4)

*Proof of Proposition 3.1.* This is a direct conclusion of Lemma 2.2(d), except that we have to justify substituting, up to the admissible error term,  $\pi n$  and  $\pi m$  instead of  $k_n$  and  $k_m$ , respectively, on the rhs of (2.7). Indeed,

$$\Lambda'_{n,m}(\sigma) = 2(k'_n(\sigma)k_n(\sigma) + k'_m(\sigma)k_m(\sigma)),$$
(3.5)

and by virtue of Lemma 2.2(d), we have

$$k'_n = \frac{2}{k_n} \cdot \left(1 + f_2(\sigma) \cdot \frac{1}{k_n^2}\right) + \mathcal{E}_n(\sigma),$$

with the error term bounded by (2.8), and hence,

$$k_n \cdot k'_n = 2\left(1 + f_2(\sigma) \cdot \frac{1}{k_n^2}\right) + E_n(\sigma), \tag{3.6}$$
$$|E_n(\sigma)| = O\left(\frac{\sigma^2}{n^4}\right).$$

By the secular equation (2.2),

$$k_n(\sigma) = \pi n + O(\sigma/n),$$

and hence,

where

$$\frac{1}{k_n(\sigma)} = \frac{1}{(\pi n)(1+O(\sigma/n^2))} = \frac{1}{\pi n} + O\left(\frac{\sigma}{n^3}\right)$$

and

$$\frac{1}{k_n(\sigma)^2} = \frac{1}{\pi^2 n^2} + O\left(\frac{\sigma}{n^4}\right).$$

Substituting into (3.6) produces, after multiplication by  $f_2(\sigma)$ , an error term of  $O\left(\frac{\sigma^2}{n^4}\right)$  that can be absorbed into  $E_n(\sigma)$  so that (3.6) reads

$$k_n \cdot k'_n = 2\left(1 + f_2(\sigma) \cdot \frac{1}{(\pi n)^2}\right) + E_n(\sigma).$$

The main statement (3.2) with the prescribed error term (3.4) of Proposition 3.1 finally follows upon substituting the latter estimate corresponding to *n* and m into (3.5).

#### Proposition 3.2.

a. For all (n, m) and (n', m') so that  $n, m, n', m' \ge 1$  and  $n^2 + m^2 = n'^2 + m'^2$ , we have

$$\frac{1}{n^2} + \frac{1}{m^2} > \frac{1}{n'^2} + \frac{1}{m'^2} \quad \Leftrightarrow \quad nm < n'm'.$$
(3.7)

b. Uniformly, for all (n, m) and (n', m') so that  $n, m, n', m' \ge 1$  and  $n^2 + m^2 = n'^2 + m'^2$ , nm < n'm',

$$\frac{1}{n^4} + \frac{1}{m^4} + \frac{1}{n'^4} + \frac{1}{m'^4} = O\left(\left(\left(\frac{1}{n^2} + \frac{1}{m^2}\right) - \left(\frac{1}{n'^2} + \frac{1}{m'^2}\right)\right)\right).$$
(3.8)

Note that the rhs of (3.8) is positive by (3.7).

c. As  $\sigma \rightarrow 0$ , uniformly, for all (n,m) and (n',m') so that  $n,m,n',m' \geq 1$ ,  $n^2 + m^2 = n'^2 + m'^2$ , and n'm' > nm,

$$|E_{(n,m)}(\sigma)| + |E_{(n',m')}(\sigma)| = o_{\sigma \to 0} \left( f_2(\sigma) \cdot \left( \left( \frac{1}{n'^2} + \frac{1}{m'^2} \right) - \left( \frac{1}{n^2} + \frac{1}{m^2} \right) \right) \right),$$
(3.9)

with the rhs of (3.9) being positive by part (a) and (3.3).

*Proof of Proposition 3.2.* The first statement (3.7) of Proposition 3.2 is straightforward. For the second one (3.8), we denote  $K := n^2 + m^2 = n'^2 + m'^2$  and choose any parameter  $0 < \epsilon < 1$  sufficiently small, whose precise value is irrelevant, except that it will be fixed throughout this proof. We further assume w.l.o.g. that  $n \le m$  and  $n' \le m'$  (and nm < n'm'), implying, in particular, that

$$n, n' \le \sqrt{\frac{K}{2}} \text{ and } m, m' \ge \sqrt{\frac{K}{2}}$$

$$(3.10)$$

J. Math. Phys. **62**, 113503 (2021); doi: 10.1063/5.0061763 Published under an exclusive license by AIP Publishing and n < n'. We write

$$\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{n'^2} - \frac{1}{m'^2} \right) = \left(\frac{K}{n^2 m^2} - \frac{K}{n'^2 m'^2}\right)$$
  
=  $\frac{K}{n^2 m'^2 m'^2} (n'^2 m'^2 - n^2 m^2) = \frac{K(nm + n'm')}{n^2 m^2 n'^2 m'^2} (n'm' - nm).$  (3.11)

First, assume that both  $n, n' > \epsilon \sqrt{K}$ . Then, using the trivial bound  $n'm' - nm \ge 1$  in (3.11) yields

$$\left(\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{n'^2} - \frac{1}{m'^2}\right) \ge \frac{1}{n^2 n' m'} \gg \frac{\epsilon^{-2}}{n^4} \gg \epsilon^{-2} \cdot \left(\frac{1}{n^4} + \frac{1}{m^4} + \frac{1}{n'^4} + \frac{1}{m'^4}\right).$$
(3.12)

Otherwise, we assume (w.l.o.g. thanks to the above assumptions) that  $n \le \epsilon \sqrt{K}$ . In this case, we can improve upon the trivial lower bound  $n'm' - nm \ge 1$  in the following way.

Define

$$f_K(n) := n \cdot \sqrt{K - n^2} = K \cdot g(n/\sqrt{K}),$$

where  $g(y) := y \cdot \sqrt{1 - y^2}$  on  $y \in [0, 1]$  [in fact, in our context,  $y \in [0, 1/\sqrt{2}]$ ; see (3.10)], and under the assumptions mentioned above, if  $n < \epsilon \sqrt{K}$ , we have

$$f'm' - nm = f_K(n') - f_K(n) = K(g(n'/\sqrt{K}) - g(n/\sqrt{K})) \ge \frac{1}{2}\sqrt{K}(n'-n) > 0,$$

and assuming that  $\epsilon > 0$  is sufficiently small (recall that  $n \le \epsilon \sqrt{K}$ ), we claim that

r

$$g(n'/\sqrt{K}) - g(n/\sqrt{K}) \ge \frac{1}{10}\sqrt{K}(n'-n)$$
 (3.13)

so that

$$n'm' - nm \gg \sqrt{K}(n' - n) > 0 \tag{3.14}$$

improves on the trivial bound.

Indeed, by the mean value theorem, for some  $\xi \in \left(\frac{n}{\sqrt{K}}, \frac{n'}{\sqrt{K}}\right)$ ,

$$g(n'/\sqrt{K}) - g(n/\sqrt{K}) = \frac{n'-n}{\sqrt{K}}g'(\xi).$$
(3.15)

Now,  $n/\sqrt{K} < n'/\sqrt{K} < 1/\sqrt{2}$  by (3.10), and so  $\xi \in \left(0, \frac{1}{\sqrt{2}}\right)$ . In this range, the derivative

$$g'(u) = \frac{1 - 2u^2}{\sqrt{1 - u^2}}$$

is positive and decreasing until it vanishes at  $u = \frac{1}{\sqrt{2}}$ . The upshot is that so long as we stay away from this only zero, (3.15) yields a bound of the desired type (3.13) (which is why we separately treated the case  $n > \epsilon \sqrt{K}$  in the first place). To this end, we further subdivide the interval  $(0, 1/\sqrt{2})$ : first, assuming  $\frac{n}{\sqrt{K}} < \frac{n'}{\sqrt{K}} < \frac{1}{2}$  (allowed since  $n/\sqrt{K} \le \epsilon$ ), (3.15) reads

$$g(n'/\sqrt{K}) - g(n/\sqrt{K}) \ge \frac{1}{\sqrt{3}} \cdot \frac{n'-n}{\sqrt{K}}$$
(3.16)

since for  $\xi \in (0, 1/2)$ , one has  $g'(\xi) \ge g'(1/2) = \frac{1}{\sqrt{3}}$  as  $g'(\cdot)$  is decreasing. Otherwise, (3.16) holds true on the full range  $\xi \in (0, 1/\sqrt{2})$  with the constant  $1/\sqrt{3}$  replaced by a slightly smaller constant [but still bigger than  $\frac{1}{10}$  claimed in (3.13)] since g is increasing.

Inserting the nontrivial bound (3.14) into the rhs of (3.11), we have

$$\left|\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{n'^2} - \frac{1}{m'^2}\right| \gg \frac{K^{3/2}(nm + n'm')(n' - n)}{n^2m^2n'^2m'^2}$$
$$\gg \frac{(nm + n'm')(n' - n)}{n^2n'^2m} \gg \frac{(n' - n)}{n^2n'}.$$

However,

$$\frac{(n'-n)}{n^2n'} \gg \frac{1}{n^3}$$

J. Math. Phys. **62**, 113503 (2021); doi: 10.1063/5.0061763 Published under an exclusive license by AIP Publishing since the ratio of the lhs. to the rhs is

$$\frac{(n'-n)/n^2n'}{1/n^3} = \frac{n(n'-n)}{n+(n'-n)} = \frac{1}{\frac{1}{n'-n} + \frac{1}{n}} \ge \frac{1}{2}.$$

This yields

$$\left|\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{n'^2} - \frac{1}{m'^2}\right| \gg \frac{1}{n^3} \ge \frac{1}{n^4}$$

$$\gg \frac{1}{n^4} + \frac{1}{m^4} + \frac{1}{n'^4} + \frac{1}{m'^4}.$$
(3.17)

All in all, in either case, (3.12) or (3.17) yields the second statement (3.8) of Proposition 3.2. The third statement (3.9) of Proposition 3.2 follows directly from (3.8) on recalling (3.4) and (3.3).

Proposition 3.3. There exists  $\sigma_0 > 0$  so that for all  $\sigma \in (0, \sigma_0)$ , if  $n, n', m' \ge 1$ , then  $\Lambda_{n,0}(\sigma) \neq \Lambda_{n',m'}(\sigma)$ .

Proof. It follows from Proposition 3.1 that

$$\Lambda_{n',m'}(\sigma) = \pi^2 (n'^2 + m'^2) + 8\sigma - \left(\frac{1}{n'^2} + \frac{1}{m'^2}\right) \cdot (\sigma^2 + O(\sigma^3)) = \pi^2 (n'^2 + m'^2) + 8\sigma + O(\sigma^2),$$
(3.18)

where the contribution of the error term  $E_{(n',m')}$  is absorbed inside  $O(\sigma^3)$ , and

$$\Lambda_{n,0} = \pi^2 n^2 + 6\sigma + O(\sigma^2), \tag{3.19}$$

with the constant involved in the 'O'-notation in both (3.18) and (3.19) being absolute. Hence, for  $\sigma > 0$  sufficiently small, if  $\Lambda_{n,0}(\sigma) = \Lambda_{n',m'}(\sigma)$ , then necessarily  $n'^2 + m'^2 = m^2$ .

Next, if  $n'^2 + m'^2 = n^2$ , then from (3.18) and (3.19), we obtain

$$\Lambda_{n,0}(\sigma) = k_0(\sigma)^2 + k_m(\sigma)^2 < k_{n'}(\sigma)^2 + k_{m'}(\sigma)^2 = \Lambda_{n',m'}(\sigma).$$

It follows trivially that for all  $\sigma > 0$ ,  $(n, m) \neq (0, 0)$ , one has  $\Lambda_{0,0}(\sigma) < \Lambda_{n,m}(\sigma)$ .

Proof of Theorem 1.1. The statement of Theorem 1.1 is equivalent to having no relations,

$$\Lambda_{n,m}(\sigma) = \Lambda_{n',m'}(\sigma),$$

for  $\sigma$  sufficiently small,  $(n, m) \neq (n', m')$ , where, once again, we assume w.l.o.g. that  $n \leq m$ ,  $n' \leq m'$  [recall that for  $L = 1, \Lambda_{n,m}(\cdot) = \Lambda_{m,n}(\cdot)$ ]. By Proposition 3.3, we may further assume that  $n, m, n', m' \geq 1$ , so use Proposition 3.1 to write

$$\Lambda_{n,m}'(\sigma) = 4\left(2 + \frac{f_2(\sigma)}{\pi^2} \left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right) + E_{(n,m)}(\sigma),$$
(3.20)

with the error term given by (3.4). Using Lemma 2.2(b), we compute

$$\Lambda_{n,m}^{\prime\prime}(0) = -8\left(\frac{1}{(\pi n)^2} + \frac{1}{(\pi m)^2}\right).$$
(3.21)

Writing the analog of (3.21) for (n', m') in place of (n, m) and together with Proposition 3.2(a), we deduce that for (n, m) and (n', m') with  $n^2 + m^2 = n'^2 + m'^2$  and n'm' > nm, there exists some [a priori dependent on (n, m) and (n', m')] neighborhood of the origin so that

$$\Lambda_{n',m'}(\sigma) > \Lambda_{n,m}(\sigma).$$

To make this neighborhood absolute, we compare the expansions (3.20) of  $\Lambda'_{n,m}(\cdot)$  for (n,m) and (n',m') with  $n^2 + m^2 = n'^2 + m'^2$ . We have  $\Lambda'_{n,m}(0) = \Lambda'_{n',m'}(0)$ , and Propositions 3.2(a) and 3.2(c), bearing in mind that  $f_2(\sigma) < 0$  for all  $\sigma > 0$ , imply that there exists some absolute  $\sigma_0 > 0$  so that

$$\Lambda'_{n,m}(\sigma) < \Lambda'_{n',m'}(\sigma)$$

on  $\sigma \in (0, \sigma_0]$ , which concludes the Proof of Theorem 1.1 for (n, m) and (n', m') on the same circle.

Finally, if (n, m) and (n', m') are not on the same circle, then (3.2) shows that  $\Lambda'_{n,m}(\cdot) - \Lambda'_{n',m'}(\cdot)$  is bounded by an absolute constant around the origin (any bound B > 0 could be taken for sufficiently small neighborhood of the origin). Therefore, since

 $|\Lambda_{n,m}(\cdot) - \Lambda_{n',m'}(\cdot)| \ge 1$ 

for  $\sigma > 0$  sufficiently small,  $\Lambda_{n,m}(\cdot) - \Lambda_{n',m'}(\cdot)$  maintains its sign.

## B. Existence of spectral degeneracies

*Proposition 3.4.* There exist a number  $\sigma > 0$  so that  $\Lambda_{3,4}(\sigma) = \Lambda_{1,5}(\sigma)$ .

*Proof.* By Lemma 2.1(a) and recalling the notation (3.1), we have  $\Lambda_{3,4}(0) = 25\pi^2$  and  $\Lambda_{1,5}(0) = 26\pi^2$ , whereas  $\Lambda_{3,4}(+\infty) = 41\pi^2$  and  $\Lambda_{1,5}(+\infty) = 40\pi^2$ . Therefore, the continuous function  $\sigma \mapsto \Lambda_{3,4}(\sigma) - \Lambda_{1,5}(\sigma)$  changes sign, and so, by the intermediate value theorem, it vanishes at some  $\sigma > 0$ , i.e.,  $\Lambda_{3,4}(\sigma) = \Lambda_{1,5}(\sigma)$ , as claimed.

## IV. SPECTRAL DEGENERACIES FOR RECTANGLES: PROOF OF THEOREMS 1.2-1.3

## A. Existence of multiplicities for irrational L<sup>2</sup>

The following theorem asserts that, on recalling the notation (1.2), there exist relations of the type

$$\Lambda_{L;n,m}(\sigma) = \Lambda_{L;0,m'}(\sigma), \tag{4.1}$$

ARTICLE

with  $\sigma > 0$  arbitrarily small and  $n, m, m' \ge 1$  (depending on  $\sigma$ ). This, in particular, implies Theorem 1.2. For some L > 0, spectral degeneracies of the type (4.1) subject to  $n, m, m' \ge 1$  are the only degeneracies, at least for  $\sigma > 0$  sufficiently small; see Theorem 4.4.

**Theorem 4.1.** Let  $L^2$  be a positive irrational number. Then, there exists a sequence of Robin parameters  $\sigma_j \ge 0$  and triples of positive integers  $n, m, m' \ge 1$  (depending on  $\sigma_j$ ) so that

$$\Lambda_{L;n,m}(\sigma) = \Lambda_{L;0,m'}(\sigma). \tag{4.2}$$

The following result will be required toward giving a Proof of Theorem 4.1.

Lemma 4.2. Let  $\theta > 0$  be a positive irrational number. For every  $\epsilon > 0$ , there are positive integer solutions n, m, m' > 0 of the inequality

$$-\epsilon < n^2 \theta + m^2 - m^2 < 0. \tag{4.3}$$

*Proof.* For any irrational  $\theta$ , Hardy and Littlewood<sup>5</sup> proved in 1914 that the sequence of fractional parts { $\theta n^2 \mod 1 : n = 1, 2, ...$ } is dense in the unit interval [0, 1) (improved to uniform distribution by Weyl shortly afterward). Thus, there are  $n_1 \gg 1$  and  $j = j(n_1) \gg 1$  for which

$$-\frac{\epsilon}{4} < \theta n_1^2 - j < 0.$$

Multiplying by 4, we obtain

$$-\epsilon < \theta (2n_1)^2 - 4j < 0.$$

Let

$$n = 2n_1, \quad m = j - 1, \quad m' = j + 1$$

(which are positive). Then,  $m'^2 - m^2 = 4j$ , and we obtain

$$-\epsilon < \theta n^2 + m^2 - m'^2 < 0$$

with n, m, m' > 0, as required.

*Remark* 4.3. The Proof of Lemma 4.2 constructs infinitely many triples satisfying (4.3).

*Proof of Theorem 4.1.* Take any sequence  $\epsilon_j \rightarrow 0$ , and find a triple of positive integers (n, m, m') (depending on  $\epsilon_j$ ) as in Lemma (4.2) so that we have for  $\theta = L^2$ ,

$$-\epsilon_{j} < \Lambda_{L;n,m}(0) - \Lambda_{L;0,m'}(0) = \frac{\pi^{2}}{\theta} \cdot \left(n^{2}\theta + m^{2} - m'^{2}\right) < 0.$$

$$(4.4)$$

J. Math. Phys. **62**, 113503 (2021); doi: 10.1063/5.0061763 Published under an exclusive license by AIP Publishing

Next note that, for every L > 0 and integers  $n, m \ge 1$ , one has

$$\Lambda_{L;n,m}'(\sigma) = 4\left(\left(1 + \frac{1}{L}\right) + \frac{1}{\pi^2}\left(\frac{f_2(\sigma)}{n^2} + \frac{f_2(L \cdot \sigma)}{Lm^2}\right)\right) + E_{(n,m)}(\sigma),$$
(4.5)

where the error term is still bounded by (3.4). Indeed, in accordance with Lemma 2.2, one has

$$\left(k_{L;n}(\sigma)^{2}\right)'=\frac{1}{L}\left(k_{n}(\cdot)^{2}\right)'|_{L\sigma}=\frac{4}{L}\left(1+\frac{f_{2}(L\sigma)}{\pi^{2}n^{2}}\right)+O\left(\frac{\sigma^{4}}{n^{4}}\right),$$

and the rest follows from the definition of  $\Lambda_{L;n,m}(\cdot)$ .

Now, comparing (4.5) to (2.4), we have the following expansions:

$$\Lambda_{L;n,m}(\sigma) = \Lambda_{L;n,m}(0) + 4\sigma \cdot \left(1 + \frac{1}{L}\right) + O(\sigma^2)$$

and

$$\Lambda_{L;0,m'}(\sigma) = \Lambda_{L;0,m'}(0) + \sigma \cdot \left(2 + \frac{4}{L}\right) + O(\sigma^2).$$

$$(4.6)$$

Therefore, the difference between the Robin eigenvalues is given by

$$\Lambda_{L;n,m}(\sigma) - \Lambda_{L;0,m'}(\sigma) = \Lambda_{L;n,m}(0) - \Lambda_{L;0,m'}(0) + 2\sigma + O(\sigma^2).$$
(4.7)

In particular, if we choose  $\sigma = \epsilon_i$ , (4.4) with (4.7) together implies that for *j* sufficiently large,

$$\Lambda_{L;n,m}(\sigma) - \Lambda_{L;0,m'}(\sigma) > 0.$$

Therefore, by the intermediate value theorem, there is some  $\sigma_i \in (0, \epsilon_i)$  so that the equality (4.2) holds, which is the claimed multiplicity.

#### B. A bound on multiplicities for badly approximable $L^2$

Theorem 4.4(a) asserts that if  $\theta := L^2$  is badly approximable in the sense of (1.5), then the only possible spectral degeneracies are either the type  $\Lambda_{L;n,m}(\sigma) = \Lambda_{L;n',0}(\sigma)$  or  $\Lambda_{L;n,m}(\sigma) = \Lambda_{L;0,m'}(\sigma)$  for some  $n, m, n', m' \ge 0$ . Theorems 4.4(b) and 4.4(c) will deduce the bound for the spectral degeneracies claimed as part of Theorem 1.3.

**Theorem 4.4.** Assume that  $L^2$  is badly approximable.

- a. For  $\sigma_0 > 0$  sufficiently small, for all  $\sigma \in [0, \sigma_0]$ , there are no spectral multiplicities  $\Lambda_{L;n,m} = \Lambda_{L;n',m'}$  for  $(n,m) \neq (n',m')$  with all  $n, m, n', m' \ge 1$ .
- b. For  $\sigma \in [0, \sigma_0]$  sufficiently small, all multiplicities are bounded by 3, i.e., all eigenspaces are of dimension at most 3.
- c. If, in addition, L is badly approximable, then all multiplicities are bounded by 2.

*Proof of Theorem 4.4(a).* We will show that, under the hypotheses of Theorem 4.4, the sign of

$$\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma),$$

which does not vanish at the origin, will be maintained in a neighborhood of the origin, which is independent of n, m, n', m'. At this point, we will assume for simplicity that  $n \neq n'$ . We will further assume w.l.o.g. that  $\Lambda_{L;n,m}(0) > \Lambda_{L;n',m'}(0)$ . Abbreviating

$$\theta \coloneqq L^2$$
,

then necessarily

$$\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0) = \pi \theta \left( (n^2 \cdot \theta + m^2) - (n'^2 \cdot \theta + m'^2) \right) = \pi \theta \left( (n^2 - n'^2) \cdot \theta + (m^2 - m'^2) \right) \gg (n^2 - n'^2)^{-1}$$
(4.8)

since  $\theta = L^2$  is badly approximable. On the other hand, (4.5) implies that

$$\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma) = \Lambda_{L;n',m'}(0) - \Lambda_{L;n',m'}(0) + O\left(\sigma^2 \cdot \left(\left|\frac{1}{n^2} + \frac{1}{n'^2} + \frac{1}{m^2} + \frac{1}{m'^2}\right|\right)\right),$$
(4.9)

J. Math. Phys. **62**, 113503 (2021); doi: 10.1063/5.0061763 Published under an exclusive license by AIP Publishing We claim that

so that, bearing in mind (4.8), the main term on the rhs of (4.9) is at least of the same order of magnitude as the error term on the rhs of (4.9)not factoring in the factor  $\sigma^2$ , implying no multiplicities for  $\sigma$  sufficiently small. First, clearly,

also following from a more direct argument, i.e., a truncated version of the expansion (2.7), where the error term is of smaller order of magnitude compared to the secondary term in (2.7). Note that the statement of Theorem 4.4(a) is trivial, unless the lhs of (4.8) is <1, which

 $|n^2 - n'^2| \ll |m^2 - m'^2|.$ 

 $\frac{1}{|n^2 - n'^2|} \gg \left| \frac{1}{n^2} + \frac{1}{n'^2} + \frac{1}{m^2} + \frac{1}{m'^2} \right|$ 

so we are left to deal with bounding

To this end, we use (4.10) to obtain

*Proof of Theorem 4.4(b) and 4.4(c).* Thanks to Theorem 4.4(a), for  $\sigma$  sufficiently small, a spectral multiplicity is of the type

 $\frac{1}{|n^2 - n'^2|} \gg \frac{1}{|m^2 - m'^2|} \gg \frac{1}{m^2} + \frac{1}{m'^2},$ 

$$k_n(\sigma)^2 + \frac{1}{\theta} \cdot k_m(\sigma \cdot L)^2 = k_0(\sigma)^2 + \frac{1}{\theta} \cdot k_{m'}(\sigma \cdot L)^2$$
(4.12)

or

$$k_n(\sigma)^2 + \frac{1}{\theta} \cdot k_m(\sigma \cdot L)^2 = k_{n'}(\sigma)^2 + \frac{1}{\theta} \cdot k_0(\sigma \cdot L)^2$$
(4.13)

or

$$k_n(\sigma)^2 + \frac{1}{\theta} \cdot k_0(\sigma \cdot L)^2 = k_0(\sigma)^2 + \frac{1}{\theta} \cdot k_{m'}(\sigma \cdot L)^2$$
(4.14)

for some  $n, m, n', m' \in \mathbb{Z}_{\geq 1}$  (taking into account that  $k_0(\sigma)^2 + \frac{1}{\theta} \cdot k_0(\sigma \cdot L)^2$  is arbitrarily small for  $\sigma$  sufficiently small). Given  $(n, m) \in \mathbb{Z}_{\geq 1}^2$ or  $m' \in \mathbb{Z}_{\geq 1}$ , at most one lattice point (n', 0) can possibly satisfy either (4.13) or (4.14) [respectively, (4.12) or (4.14)], and the same holds analogously for (0, m'). It follows that the multiplicities are bounded by 3, concluding Theorem 4.4(b).

The above also shows that if multiplicity 3 actually occurs with  $\sigma$  arbitrarily small, then

$$k_n(\sigma)^2 + \frac{1}{\theta} \cdot k_m(\sigma \cdot L)^2 = k_0(\sigma)^2 + \frac{1}{\theta} \cdot k_{m'}(\sigma \cdot L)^2 = k_{n'}(\sigma)^2 + \frac{1}{\theta} \cdot k_0(\sigma \cdot L)^2$$

$$(4.15)$$

is satisfied. Using only the latter of the two equalities of (4.15), together with (2.7) and (2.4), we obtain

$$0 = \left(k_0(\sigma)^2 + \frac{1}{\theta} \cdot k_{m'}(\sigma \cdot L)^2\right) - \left(k_{n'}(\sigma)^2 + \frac{1}{\theta} \cdot k_0(\sigma \cdot L)^2\right) = \frac{\pi^2}{\theta}(m'^2 - n'^2\theta) + O(\sigma).$$
(4.16)

In particular, if, as we assumed, (4.15) occurs with  $\sigma$  arbitrarily small [i.e., (4.16) holds for a sequence  $\sigma_i \rightarrow 0$ ], then the quadratic form

$$n^{\prime 2}\theta - m^{\prime 2} \tag{4.17}$$

attains arbitrarily small values. However,

$$n'^{2}\theta - m'^{2} = (n'L - m') \cdot (n'L + m') \gg \frac{1}{n'} \cdot (n'L + m') \gg 1$$

by the assumption on L being badly approximable, contradicting our conclusion on the quadratic form (4.17) attaining arbitrarily small values. 

(4.10)

$$\frac{1}{|n^2 - n'^2|} \gg \frac{1}{n^2} + \frac{1}{n'^2},$$

$$\frac{1}{|n^2 - n'^2|} \gg \frac{1}{m^2} + \frac{1}{m'^2}.$$
(4.11)

#### C. A bound on the number of degenerate eigenvalues

Recall Weyl's law (1.3) for the Robin spectrum of  $\mathcal{R}_L$  and the function  $N^{\text{mult}}(\lambda)$  counting the number of multiple eigenvalues  $\leq \lambda$ , including multiplicities. As a corollary of the arguments above, we deduce the bound (1.4) for  $N^{\text{mult}}(\lambda)$ , thus concluding the Proof of Theorem 1.3.

*Corollary* 4.5. *If*  $L^2$  *is badly approximable, then* 

$$N^{\mathrm{mult}}(\lambda) \ll \sqrt{\lambda}.$$

*Proof.* Indeed, we saw that the only possible source of multiplicities is when  $\Lambda_{L;n,m} = \Lambda_{L;n',0}$  or  $\Lambda_{L;n,m} = \Lambda_{L;0,m'}$  and that each source, e.g.,  $\Lambda_{L;0,m'}$ , coming from one of the axes can at most contribute a threefold degeneracy because we cannot have  $\Lambda_{L;0,m'} = \Lambda_{L;0,m''}$  with  $m' \neq m''$ . Moreover, the number of eigenvalues

$$\Lambda_{L;0,m'} = k_0(\sigma)^2 + \left(\frac{1}{L}k_{m'}(L\sigma)\right)^2 \leq \lambda$$

is at most the number of  $m' \ge 0$  with  $k_{m'}(L\sigma) \le L\sqrt{\lambda}$ , which is  $O(\sqrt{\lambda})$  since  $k_{m'} = n\pi + O(1)$ .

Λ

#### D. An auxiliary result

For future reference, we record the following result concerning the asymptotic behavior of the RN gaps, similar to one used previously, but simpler in that it has no control over the error term as  $\sigma > 0$  is varying.

*Proposition 4.6.* For  $n, m \ge 0$ ,

$$L_{I;n,m}(\sigma) - \Lambda_{L;n,m}(0) = \left(1 + \frac{1}{L}\right) 4\sigma + O_{L,\sigma} \left(\frac{1}{1+n^2} + \frac{1}{1+m^2}\right).$$

Note that the "main term" is

 $\left(1+\frac{1}{L}\right)4\sigma = 2\frac{\operatorname{length}\partial\mathcal{R}_L}{\operatorname{Area}\mathcal{R}_L}\sigma,\tag{4.18}$ 

ARTICLE

which is the mean value of the RN gaps, by our general theory.<sup>11</sup> For *n* fixed,  $m \to \infty$ , the corresponding sequence of RN gaps is

$$\begin{split} \Lambda_{L;n,m}(\sigma) &- \Lambda_{L;n,m}(0) = k_n(\sigma)^2 - k_n(0)^2 + \frac{1}{L^2} \left( k_m(\sigma L)^2 - k_m(0)^2 \right) \to \\ k_n(\sigma)^2 &- k_n(0)^2 + \frac{1}{L^2} 4\sigma L = k_n(\sigma)^2 - k_n(0)^2 + 4\frac{\sigma}{L} \end{split}$$

by Lemma 2.2(d), and we observe that

$$k_n(\sigma)^2 - k_n(0)^2 + 4\frac{\sigma}{L} < 4\sigma(1+1/L)$$

by Lemma 2.2(b), at least, for  $\sigma$  sufficiently small. That is, the RN gaps for this infinite (though rare) sequence of energies are strictly less than the mean (4.18). In particular, for n = 0, we obtain, recalling Lemma 2.1(b),

$$k_0(\sigma)^2 - k_n(0)^2 + 4\frac{\sigma}{L} < 2\sigma + 4\frac{\sigma}{L} = 4\sigma(1/2 + 1/L),$$

again at least for  $\sigma$  sufficiently small. Likewise, one may obtain infinite sequences of RN gaps that are asymptotic to a value strictly less than (4.18) by fixing m and taking  $n \to \infty$ .

Figure 2 illustrates 2000 RN gaps for the square,  $\sigma = 1$ . Here,  $k_0(\sigma)^2 - k_n(0)^2 + 4 = 5.707...$  corresponds to the bottom trend line in the picture and  $k_1(\sigma)^2 - k_n(1)^2 + 4 = 7.62275...$  corresponds to the second to bottom trend line.

*Proof.* The statement of Proposition 4.6 follows directly from (4.5) for  $n, m \ge 1$  and from (4.6) for  $n = 0, m \ge 1$  [and the trivial bound  $\Lambda_{L;0,0} = O(1)$ ].

#### V. BOUNDEDNESS OF ROBIN-NEUMANN GAPS: PROOF OF THEOREM 1.4

*Lemma* 5.1. *There exists an absolute constant*  $C_0 > 0$  *so that for all*  $n \ge 0$  *and*  $\sigma > 0$ *,* 

$$k_n(\sigma)^2 - k_n(0)^2 \le C_0 \cdot \sigma.$$
(5.1)

ARTICLE



**FIG. 2**. 2000 RN gaps for the square,  $\sigma = 1$ . The bulk of the RN gaps tends to mean 8. The secondary curves correspond to lattice points whose minimal coordinate is small, in particular, lattice points lying on the axes, whose RN gaps are less than 6. Red: mean 8; green: moving average.

*Proof.* For  $\sigma > c_0 \cdot (n+1)$  with a  $c_0$  sufficiently small parameter to be chosen later, we use the trivial bound

$$k_n(\sigma)^2 - k_n(0)^2 = (k_n(\sigma) - k_n(0)) \cdot (k_n(\sigma) + k_n(0)) \leq \pi \cdot 2(n+1)\pi \leq \frac{2\pi^2}{c_0} \cdot \sigma.$$

Otherwise, for  $\sigma \le c_0 \cdot (n+1)$  with  $c_0$  sufficiently small, assume that  $n \ge 1$ , and we will take care of n = 0 separately below. Recall the secular equation

$$\tan(k) = \frac{2\sigma k}{k^2 - \sigma^2}.$$
(5.2)

In this case, the denominator on the rhs of (5.2) is bounded away from 0 so that the rhs of (5.2) is  $\leq 4\frac{\sigma}{k} < 4c_0 \frac{n+1}{n\pi}$  arbitrarily small by appropriately choosing  $c_0$ , and then, since  $\arctan(x) \leq x$  for x > 0,

$$k_n(\sigma) - k_n(0) = k_n(\sigma) - n\pi \le 4 \frac{\sigma}{k_n(\sigma)}$$

We then have

$$k_n(\sigma)^2 - k_n(0)^2 = (k_n(\sigma) - k_n(0)) \cdot (k_n(\sigma) + k_n(0)) \le 4 \frac{\sigma}{k_n(\sigma)} \cdot 2(n+1)\pi \le 8\pi \frac{n+1}{n\pi} \cdot \sigma$$
$$= 8 \frac{n+1}{n} \cdot \sigma \le 16\sigma.$$

Finally, we take care of the remaining case of n = 0 under the assumption  $\sigma \le c_0(n + 1) = c_0$ . Recall that [Lemma 2.1(b)] here  $k_0(\sigma) > \sigma$ . Denote  $k = k_0(\sigma) < \frac{\pi}{2}$  for  $\sigma < \frac{\pi}{2}$  [Proposition 2.1(c)]. Therefore, we can use  $k < \frac{\pi}{2}$  so that  $k < \tan(k)$ , and (2.2) reads

$$k < \tan k = \frac{2\sigma k}{k^2 - \sigma^2}$$

and we manipulate with that to write (recall that the denominator is positive)

$$k^2-\sigma^2<2\sigma,$$

and then,

$$k^2 < 2\sigma + \sigma^2 < 3\sigma,$$

valid for  $\sigma < c_0$ , provided that  $c_0$  is sufficiently small.

*Lemma 5.2.* There exists an absolute constant  $c_0 > 0$  so that for all  $n \ge 0$  and  $\sigma \in [0, 1]$ ,

$$k_n(\sigma)^2 - k_n(0)^2 \ge c_0 \cdot \sigma.$$
(5.3)

J. Math. Phys. **62**, 113503 (2021); doi: 10.1063/5.0061763 Published under an exclusive license by AIP Publishing

*Proof.* The main argument behind the Proof of Lemma 5.2 is similar to that of Lemma 5.1. First, assume that  $n \ge 1$  so that here  $k_n(\sigma) \ge \pi$ , and the denominator of the rhs of (2.2) is bounded away from 0. It then follows that  $\tan k > \frac{2\sigma}{k}$ , and then,

$$k_n(\sigma) - k_n(0) = k_n(\sigma) - n\pi > c_1 \frac{\sigma}{k_n(\sigma)}$$

with  $c_1$  sufficiently small. We then have for  $n \ge 1$ ,

$$k_n(\sigma)^2 - k_n(0)^2 = (k_n(\sigma) - k_n(0)) \cdot (k_n(\sigma) + k_n(0)) > c_1 \frac{\sigma}{k_n(\sigma)} \cdot k_n(\sigma) = c_1 \cdot \sigma,$$

which is (5.3) with  $c_1 > 0$  in place of  $c_0$ . That (5.3) holds with n = 0 follows directly from (2.4) for all  $\sigma \in [0, 1]$ , at the expense of further decreasing the constant to some  $c_0$ .

Proof of Theorem 1.4. Recall that the energies  $\{\lambda_{L}^{l}(\sigma)\}_{j\geq 1}$  are the sorted list of  $\{\Lambda_{L;n,m}(\sigma)\}_{n,m\geq 0}$ . The main obstacle in inferring the upper and the lower bounds (1.8) and (1.9) of Theorem 1.4 directly from the corresponding bounds in lemmas 5.1 and 5.2 is that the numbers  $\Lambda_{L;n,m}(\sigma)$  can mix so that the gaps  $d_j$  will not, in general, be equal to  $\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n,m}(0)$ . We will overcome this obstacle by appealing to an argument inspired by an idea behind the proof of Ref. 11 (Theorem 1.7) for both (1.8) and (1.9). Recall the spectral function  $N(\lambda) = N_{L;\sigma}(\lambda)$ as in (1.3). Set

$$a_L := 1 + \frac{1}{L} = \frac{1}{2} \frac{\text{length}(\mathcal{R}_L)}{\text{Area}(\mathcal{R}_L)}.$$
(5.4)

ARTICLE

First, we prove (1.8). Lemma 5.1 yields a number  $C_0 > 0$  so that if  $t := \lambda_j(0)$ , then for every  $\sigma > 0$  and  $n, m \ge 0$  so that  $\Lambda_{L;n,m}(0) = t$ , one has

$$\Lambda_{L;n,m}(\sigma) \leq t + C_0 \cdot \sigma + C_0 \cdot \frac{1}{L^2} \sigma L = t + C_0 \left(1 + \frac{1}{L}\right) \sigma.$$

We deduce that  $N_{\sigma}(t + C_0a_L \cdot \sigma) \ge j$  with  $a_L$  as in (5.4), and hence,  $\lambda_j(\sigma) \le t + C_0a_L \cdot \sigma$ . Finally, we infer  $d_j(\sigma) = \lambda_j(\sigma) - t \le C_0a_L \cdot \sigma$ , which, thanks to (5.4) and (1.6), is identified as (1.8).

Next, we show (1.9). Using the same idea as above, Lemma 5.2 gives an absolute  $c_0 > 0$  so that if  $t := \lambda_j(0)$ , then for every  $\sigma \in [0, 1]$  and  $n, m \ge 0$  with  $\Lambda_{L;n,m}(0) = t$ , one has

$$\Lambda_{L;n,m}(\sigma) \geq t + c_0 a_L \cdot \sigma.$$

Therefore, for every  $t' < t + c_0 \cdot \sigma$ ,  $N_{\sigma}(t') < j$ , and thus,

$$\lambda_j(\sigma) \geq t + c_0 a_L \cdot \sigma.$$

Finally, we obtain

$$d_j(\sigma) = \lambda_j(\sigma) - t \ge c_0 a_L \sigma,$$

which is (1.9) on recalling (5.4) and (1.6) again.

#### VI. PAIR CORRELATION: PROOF OF THEOREM 1.5

Fix  $f \in C_c(\mathbb{R})$  even. The associated pair correlation function is

$$\mathsf{R}_{2}^{\sigma}(f,N) \coloneqq \frac{1}{N} \sum_{1 \le j \ne k \le N} f\left(\frac{\lambda_{j}(\sigma) - \lambda_{k}(\sigma)}{\bar{s}}\right)$$

where the mean spacing  $\overline{s}$  is given by (1.10).

*Proposition 6.1.* For any rectangle  $\mathcal{R}_L$  and any fixed  $\sigma > 0$ ,

$$|R_2^{\sigma}(f,N) - R_2^0(f,N)| \ll N^{-1/10} \to 0.$$

*Proof.* For notational convenience, in what follows, we will neglect the asymptotically constant mean spacing (1.10) being equal to  $\frac{4\pi}{\operatorname{Area}\mathcal{R}_L}$  and proceed as if  $\{\lambda_j\}$  had mean spacing asymptotic to unity. Note that a feature of the pair correlation function is that, by its definition, the ordering of the eigenvalues is irrelevant. Therefore, we can compute it by taking for some large  $N \gg 1$ ,

## $\widetilde{N}(\sigma) = \#\{k: \lambda_k(\sigma) \le N\} = \#\{n, m \ge 0: \Lambda_{L;n,m}(\sigma) \le N\},\$

which, by Weyl's law, is asymptotically  $\widetilde{N}(\sigma) \approx N$ , and then,

$$R_{2}^{\sigma}(f,\widetilde{N}) = \frac{1}{\widetilde{N}(\sigma)} \sum_{\substack{\Lambda_{L;n,m}(\sigma),\Lambda_{L;n',m'}(\sigma) \leq N \\ (n,m) \neq (n',m')}} f(\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma)).$$

Since  $\Lambda_{L;n,m}(\sigma) = \Lambda_{L;n,m}(0) + O_{\sigma}(1)$ , we also know that  $\widetilde{N}(\sigma) \sim \widetilde{N}(0)$ .

Therefore, we can bound the difference between the Neumann and Robin pair correlations as

$$|R_{2}^{\sigma}(f,\widetilde{N}) - R_{2}^{0}(f,\widetilde{N})| \ll \frac{1}{N} \sum_{\substack{\Lambda_{L;n,m}(0),\Lambda_{L;n',m'}(0) \leq N\\(n,m) \neq (n',m')}} |f(\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma)) - f(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0))|.$$
(6.1)

Set

$$d_{n,m}(\sigma) \coloneqq \Lambda_{L;n,m}(\sigma) - \Lambda_{L;n,m}(0)$$

[not to be confused with the actual RN gaps  $\lambda_k(\sigma) - \lambda_k(0)$ ]. These are bounded, say  $d_{n,m}(\sigma) \leq C$  (which depends on *L* and  $\sigma$ ); moreover, by Proposition 4.6, there is  $C_1 > 0$  so that

$$\left| d_{n,m}(\sigma) - \left( 1 + \frac{1}{L} \right) 4\sigma \right| = \left| \Lambda_{L;n,m}(\sigma) - \Lambda_{L;n,m}(0) - \left( 1 + \frac{1}{L} \right) 4\sigma \right|$$

$$\leq C_1 \left( \frac{1}{1+n^2} + \frac{1}{1+m^2} \right).$$
(6.2)

Assume that *f* is supported in  $[-\rho, \rho]$ . Take a function  $g \in C_c^{\infty}(\mathbb{R})$  that is non-negative:  $g \ge 0$ , and so,

$$g \equiv \max |f'|$$
 on  $[-2(\rho + 2C_1), 2(\rho + 2C_1)],$  (6.3)

where  $C_1$  is as in (6.2). In particular,  $g \ge |f'|$ , as depicted in Fig. 3.

We first show that

$$\begin{aligned} \left| R_2^{\sigma}(f,\widetilde{N}) - R_2^0(f,\widetilde{N}) \right| \ll \\ \frac{1}{N} \sum g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)) \cdot \left( \frac{1}{1+n^2} + \frac{1}{1+m^2} + \frac{1}{1+n'^2} + \frac{1}{1+m'^2} \right). \end{aligned}$$
(6.4)

Indeed, to contribute to  $R_2^{\sigma}(f, \widetilde{N}) - R_2^0(f, \widetilde{N})$  in (6.1), it is forced that at least one of the two eigenvalue differences

$$\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma), \ \Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)$$
(6.5)



**FIG. 3.** Sketch of |f'| and g.

is in supp  $f \subseteq [-\rho, \rho]$ . Since the difference between these two expressions (6.5) is

$$d_{n,m}(\sigma) - d_{n',m'}(\sigma) \in \left[-2C_1, 2C_1\right]$$

by (6.2), if one of the expressions (6.5) is in  $[-\rho, \rho]$ , then both

$$\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma), \ \Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0) \in [-(\rho + 2C_1), \rho + 2C_1].$$
(6.6)

For such a pair, we have by the mean value theorem,

$$f(\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma)) - f(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)) = (d_{n,m}(\sigma) - d_{n',m'}(\sigma))f'(\xi(n,m,n',m'))$$
(6.7)

for some  $\xi(n, m, n', m')$  between  $\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma)$  and  $\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)$ . Proposition 4.6 implies that

$$d_{n,m}(\sigma) - d_{n',m'}(\sigma) \le C_1 \left( \frac{1}{1+n^2} + \frac{1}{1+m^2} + \frac{1}{1+n'^2} + \frac{1}{1+m'^2} \right).$$
(6.8)

In addition, if some summand

$$f(\Lambda_{L;n,m}(\sigma) - \Lambda_{L;n',m'}(\sigma)) - f(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0))|$$

on the rhs of (6.1) does not vanish, then

$$\left|f'(\xi(n,m,n',m'))\right| \le \max |f'| = g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0))$$
(6.9)

by (6.3) and (6.6). The claimed inequality (6.4) follows upon substituting (6.9) and (6.8) into (6.7) and finally into (6.1). Next, we claim that the rhs of (6.4) satisfies the inequality

$$\frac{1}{N}\sum g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)) \cdot \left(\frac{1}{1+n^2} + \frac{1}{1+m^2} + \frac{1}{1+n'^2} + \frac{1}{1+m'^2}\right) \ll N^{-\frac{1}{10}},$$
(6.10)

where the sum is over all pairs with  $(n, m) \neq (n', m')$ . To see this, we take a large parameter M > 0 to be chosen later and divide the summands into two categories: (1) those with min(n, m, n', m') > M and (2) the rest. An individual summand with min(n, m, n', m') > M is bounded,

$$\frac{1}{1+n^2} + \frac{1}{1+m^2} + \frac{1}{1+n'^2} + \frac{1}{1+m'^2} \ll \frac{1}{M^2},$$

so that the total contribution of the summands of the first category is bounded by

$$\frac{1}{N}\sum g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0))\frac{1}{M^2} = \frac{1}{M^2}R_2^0(g,N)$$

by forgetting the restriction  $\min(n, m, n', m') > M$ . Since the pair correlation function for any rectangle is bounded (Ref. 2, Lemma 3.1) by

$$R_2^0(g,N) \ll_g N^{\varepsilon}$$
,

it follows that the contribution to the sum (6.10) of summands of the first category is dominated by

$$\ll \frac{N^{\epsilon}}{M^2}.$$
(6.11)

We next treat the contribution to the sum (6.10) of second category summands, those with at least one of the coordinates small  $\leq M$ , say  $n \leq M$ , where we use the trivial bound

$$\frac{1}{1+n^2} + \frac{1}{1+m^2} + \frac{1}{1+n'^2} + \frac{1}{1+m'^2} \ll 1.$$

Hence, the contribution to the sum (6.10) of the second category summands is bounded by

$$\frac{1}{N} \sum_{0 \le n \le M} \sum_{0 \le m \ll \sqrt{N}} \sum_{0 \le n', m' \ll \sqrt{N}} g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)) \\
\ll \frac{1}{N} \sum_{0 \le n \le M} \sum_{0 \le m \ll \sqrt{N}} \# \left\{ (n', m') \in [0, \sqrt{N}]^2 : |\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)| \le C_2 \right\}$$
(6.12)

since supp  $g \subseteq [-C_2, C_2]$  with  $C_2 = \rho + 2C_1$ . Given (n, m), the term

$$\#\left\{\left(n',m'\right)\in\left[0,\sqrt{N}\right]^{2}:\left|\Lambda_{L;n,m}(0)-\Lambda_{L;n',m'}(0)\right|\leq C_{2}\right\}$$

is the number of lattice points in a quarter of an elliptic annulus of width  $\ll \frac{1}{\sqrt{\Lambda_{L_{n,m}}(0)}}$  and constant area (both depending on  $C_2$ ).

While we expect the number of points in such a narrow annulus to be very small, say  $\ll N^{e}$ , we are unable to show this for irrational  $L^{2}$ . Instead, we give a crude bound of  $\ll N^{1/3}$ : We use the classical bound on the number of lattice point in a dilated ellipse (for the circle, this is due to Sierpinski in 1906),

$$\#\{(n',m')\in\mathbb{Z}^2:\Lambda_{L;n',m'}(0)\leq x\}=Ax+O(x^{1/3}),$$

where A is the area of the ellipse. Therefore, a crude bound for the number of points in the annulus is

$$\#\{(n',m')\in\mathbb{Z}_{\geq 0}^{2}:|\Lambda_{L;n,m}(0)-\Lambda_{L;n',m'}(0)|\leq C_{2}\} \\ =A(\Lambda_{L;n,m}(0)+C_{2})-A(\Lambda_{L;n,m}(0)-C_{2})+O(\Lambda_{L;n,m}(0)^{1/3})\ll\Lambda_{L;n,m}(0)^{1/3}.$$

Since  $\Lambda_{L;n,m}(0) \ll N$ , we obtain

$$\#\left\{ (n',m') \in \mathbb{Z}_{\geq 0}^2 : |\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)| \le C_2 \right\} \ll N^{1/3}.$$
(6.13)

Summing the inequality (6.13) [whose l.h.s. is clearly greater than or equal to the summands on the rhs of (6.12)] over  $n \le M$  and  $m \ll \sqrt{N}$  and substituting into (6.12) yield the bound

$$\frac{1}{N} \sum_{0 \le n \le M} \sum_{0 \le m \ll \sqrt{N}} \sum_{0 \le n', m' \ll \sqrt{N}} g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0))$$

$$\ll \frac{1}{N} \sum_{1 \le n \le M} \sum_{0 \le m \ll \sqrt{N}} N^{1/3} \ll M N^{-1/6}$$
(6.14)

for the contribution of the second category summands. Consolidating the contributions (6.11) and (6.14) of the first and the second categories summands, respectively, we finally obtain a bound for the sum in (6.10),

$$\frac{1}{N} \sum g(\Lambda_{L;n,m}(0) - \Lambda_{L;n',m'}(0)) \cdot \left(\frac{1}{1+n^2} + \frac{1}{1+m^2} + \frac{1}{1+n'^2} + \frac{1}{1+m'^2}\right) \\ \ll \frac{N^{\epsilon}}{M^2} + MN^{-1/6} \ll N^{-1/10}$$

on taking  $M = N^{1/18 + \epsilon/3}$ .

#### ACKNOWLEDGMENTS

This research was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 786758) and by the Israel Science Foundation (Grant No. 1881/20). We wish to thank Nadav Yesha for many stimulating conversations and a referee for a simplification of the arguments in Sec. II.

## AUTHOR DECLARATIONS

#### **Conflict of Interest**

The authors have no conflicts to disclose..

#### DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

#### REFERENCES

<sup>1</sup>Berry, M. V. and Tabor, M., "Level clustering in the regular spectrum," Proc. R. Soc. A 356, 375–394 (1977).

<sup>2</sup>Bleher, P. and Lebowitz, J., "Variance of number of lattice points in random narrow elliptic strip," Ann. Inst. H. Poincaré: Probab. Stat. 31(1), 27–58 (1995).

<sup>3</sup>Eskin, A., Margulis, G. A., and Mozes, S., "Quadratic forms of signature (2, 2) and eigenvalue spacings on rectangular 2-tori," Ann. Math. 161(2), 679–725 (2005).

<sup>4</sup>Filinovskiy, A., "On the eigenvalues of a Robin problem with a large parameter," Math. Bohemica 139(2), 341–352 (2014).

<sup>5</sup>Hardy, G. H. and Littlewood, J. E., "Some problems of diophantine approximation," Acta Math. 37(1), 155–191 (1914).

<sup>6</sup>Laugesen, R. S., "The Robin Laplacian—Spectral conjectures, rectangular theorems," J. Math. Phys. **60**(12), 121507 (2019).

<sup>7</sup>Margulis, G. A., "Formes quadratriques indéfinies et flots unipotents sur les espaces homogènes," C. R. Acad. Sci. Paris Sér. I: Math. **304**(10), 249–253 (1987).

<sup>8</sup>Marklof, J., "The Berry-Tabor conjecture," in Proceedings of the 3rd European Congress of Mathematics, Barcelona 2000, Progress in Mathematics Vol. 202 (Birkhäuser, Basel, 2001), pp. 421-427.

<sup>9</sup>Rudnick, Z., "What is quantum chaos," Not. AMS 55(1), 32–34 (2008).

<sup>10</sup>Rudnick, Z. and Wigman, I., On the Robin Spectrum for the Hemisphere (Annales Mathématiques du Québec, 2021), pp. 1-17.

<sup>11</sup>Rudnick, Z., Wigman, I., and Yesha, N., "Differences between Robin and Neumann eigenvalues," Comm. Math. Phys. (to be published).

<sup>12</sup>Strauss, W. A., Partial Differential Equations, An Introduction, 2nd ed. (John Wiley & Sons, Chichester, 2008).