A METRIC RESULT ON THE PAIR CORRELATION OF FRACTIONAL PARTS OF SEQUENCES

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ABSTRACT. We investigate the pair correlation function of the sequence of fractional parts of $\alpha a(n)$, $n = 1, 2, \ldots, N$, where $a(n)$ is an integer valued function.

We show that if the sequence $a(n)$ is *lacunary*, in the sense that there is some $c > 1$ such that $a(n+1) \geq ca(n)$for $n \gg 1$, then for almost all α (in the sense of measure theory), the pair correlation of the fractional parts of the sequence is Poissonian, that is like that of a sequence of uncorrelated numbers. In particular this holds for the fractional parts of αg^n, $g \geq 2$ an integer.

1. INTRODUCTION

Our purpose in this note is show that the pair correlation function of several sequences of fractional parts behaves like those of random numbers. The pair correlation density for a sequence of N numbers $\theta_1, \ldots, \theta_N \in [0, 1]$ which are uniformly distributed as $N \to \infty$, measures the distribution of spacings between the numbers at distances of order of the mean spacing $1/N$. Precisely, if $\|x\| = \operatorname{distance}(x, \mathbf{Z})$ then for any interval $[-s, s]$ set

$$R_2([-s, s], N) = \frac{1}{N} \#\{1 \leq j \neq k \leq N : \|\theta_j - \theta_k\| \leq \frac{s}{N}\} \tag{1.1}$$

For random numbers θ_j chosen uniformly and independently,

$$R_2([-s, s], N) \to 2s$$

with probability tending to 1 as $N \to \infty$. In this case one says that the pair correlation function is Poissonian. A smooth form of (1.1) is to take a test function $f \in C_c^\infty(\mathbf{R})$ and set

$$R_2(f, N) := \frac{1}{N} \sum_{1 \leq j \neq k \leq N} F_N(\theta_j - \theta_k)$$

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where $F_N(y) = \sum_{m\in\mathbf{Z}} f(N(y+m))$. The Poisson case is that in the limit $N\to\infty$, $R_2(f,N)\to\int_{-\infty}^{\infty} f(x)dx$.

We will show that the pair correlation function of many sequences of fractional parts of the form $\{\alpha a(x)\}$, $x=1,\dots,N$ with $a(x)$ integers, have Poissonian pair correlation for almost all α. Our main tool is:

Theorem 1. *Let $a(x)$ be a sequence of integers so that $a(x)\neq a(y)$ if $x\neq y$ and furthermore suppose that there are at most $O(MN^{2+\epsilon})$ solutions to the equation*

$$n_1\left(a(x_1)-a(y_1)\right) = n_2\left(a(x_2)-a(y_2)\right) \tag{1.2}$$

with $1\le x_i\neq y_i\le N$, and $1\le |n_i|\le M$, $M\ll N^R$ for some $R>0$, and all $\epsilon>0$. Then for almost all α, we have

$$R_2(f,N)\to\int_{-\infty}^{\infty} f(x)dx$$

A result of this kind, was proved by Rudnick and Sarnak [2] for the spacings of αn^d, where $d\ge 2$ is an integer. Crucial use is made there of Weyl's differencing argument [1, 3] to get cancellations in sums of the exponential sums $\sum_{n\le N} e(\alpha F(n))$, where $F(n)$ is a polynomial of degree $d\ge 1$, and α is of diophantine type. No such estimate is available when we replace polynomials by functions such as the exponential function g^n (this is a key issue in the study of "normal" numbers). The idea here is to avoid this issue for *individual* α, and instead to prove this kind of result for *almost all* α - see Proposition 4.

Theorem 1 reduces the study of the generic behavior of the pair correlation of the sequence of fractional parts of $a(x)$, to estimating the number of solutions of the equation (1.2). In [2] it was shown that the number of solutions of this equation for $a(x)=x^d$, $d\ge 2$ is indeed $O(MN^{2+\epsilon})$. In section 4 we show that the same estimate holds if $a(x)$ is *lacunary*:

Proposition 2. *Let $a(x)>0$ be an increasing sequence of positive integers so that there is some $c>1$ for which*

$$a(x+1)\ge ca(x)$$

Then the equation (1.2) has at most $O(MN^2\log^2 N)$ solutions in $0<|n_i|\le M$, $1\le x_i\neq y_i\le N$, where $M\ll N^R$ for some $R>0$.

An example of such a sequence is $a(x)=g^x$, $g\ge 2$ an integer. Thus we get:

Corollary 3. *Let $g\ge 2$ be an integer. Then for almost all α, the sequence of fractional parts of αg^n has Poisson pair correlation.*

It seems plausible that for almost all α, all correlation functions should be Poissonian in this case, and in particular the nearest neighbor spacing distribution should be exponential.

Other examples would be the sequences $a(n) = n!$ or g^{g^n} for an integer $g \geq 2$, or the integer parts $[c^n]$ where $c > 1$ is any real number.

2. A metric result for sums of exponential sums

Suppose we are given a sequence $a(x) \in \mathbf{Z}_+$, satisfying $a(x) \neq a(y)$ if $x \neq y$. Define the Weyl sum

$$S_\alpha(n, N) = \sum_{1 \leq x \leq N} e(\alpha n a(x))$$

and for each N suppose we choose $M = M(N) = N^{1+\frac{1}{100}}$, and set

$$H_N(\alpha) = \sum_{1 \leq n \leq M} |S_\alpha(n, N)|^2$$

Proposition 4. *For almost all α, we have*

$$H_N(\alpha) \ll_\alpha MN^{2-1/4}$$

Proof. Because $a(x) \neq a(y)$ if $x \neq y$, we clearly have

$$\int_0^1 |S_\alpha(n, N)|^2 d\alpha = N$$

and so

$$\int_0^1 H_N(\alpha) d\alpha = MN$$

Therefore we can estimate the measure of the set of α for which $H_N(\alpha) > MN^{2-1/4}$ by

$$\begin{aligned} \text{meas}\{\alpha : H_N(\alpha) > MN^{2-1/4}\} &\leq \frac{1}{MN^{2-1/4}} \int_{\{\alpha : H_N(\alpha) > MN^{2-1/4}\}} H_N(\alpha) d\alpha \\ &\leq \frac{1}{MN^{2-1/4}} \int_0^1 H_N(\alpha) d\alpha \\ &= \frac{1}{MN^{2-1/4}} MN = N^{-3/4} \end{aligned}$$

It follows from the Borel-Cantelli lemma that if we take a sequence of N_m's which is sparse enough so that $\sum_m N_m^{-3/4}$ converges, then along that sequence we will find that for all α in a set of full measure,

$$H_{N_m}(\alpha) \leq M_m N_m^{2-1/4}, \text{ for all } m > m_0(\alpha) \tag{2.1}$$

For simplicity, we will take $N_m = m^2$.

Now fix α for which (2.1) holds. We will show that if $N_m < N < N_{m+1}$, then

$$|H_N(\alpha) - H_{N_m}(\alpha)| \ll MN^{3/2} \tag{2.2}$$

and so we will find that together with (2.1) we have our proposition.

Note that $N - N_m < N_{m+1} - N_m = 2m + 1 \ll N^{1/2}$, and further

$$\begin{aligned} M - M_m = N^{101/100} - N_m^{101/100} &< (m+1)^{202/100} - m^{202/100} \\ &\ll m^{102/100} = N^{1/2+1/100} \end{aligned}$$

We have

$$\begin{aligned} H_N - H_{N_m} &= \sum_{n \le M} |S_\alpha(n,N)|^2 - \sum_{n \le M_m} |S_\alpha(n,N_m)|^2 \\ &= \sum_{n \le M_m} \left(|S_\alpha(n,N)|^2 - |S_\alpha(n,N_m)|^2 \right) + \sum_{M_m < n \le M} |S_\alpha(n,N)|^2 \\ &= I + II \end{aligned}$$

We use the trivial bound $|S_\alpha(n,N)|^2 \le N^2$ to estimate the term II:

$$II \ll (M - M_m)N^2 \ll N^{1/2+1/100} N^2 = MN^{3/2}$$

For the term I, note that if we square out the summands $|S_\alpha(n,N)|^2 = \sum_{x,y \le N} e(n\alpha(a(x) - a(y)))$ and likewise for $|S_\alpha(n,N_m)|^2$, we will find that I is given by

$$\begin{aligned} I = &\sum_{n \le M_m} \sum_{N_m < y \le N} e(-\alpha n a(y)) \sum_{1 \le x \le N_m} e(\alpha n a(x)) + \text{complex conjugate} \\ &+ \sum_{n \le M_m} |\sum_{N_m < x \le N} e(\alpha n a(x))|^2 \\ &= I_1 + \bar{I}_1 + I_2 \end{aligned}$$

For the term I_2 we use the trivial bound on the inner sum to get

$$I_2 \ll M_m (N - N_m)^2 \ll MN$$

For I_1 we get

$$I_1 \ll \sum_{n \le M_m} \sum_{N_m < y \le N} |S_\alpha(n,N_m)| = (N - N_m) \sum_{n \le M_m} |S_\alpha(n,N_m)|$$

By Cauchy-Schwarz we find

$$
\begin{aligned}
I_1 &\ll (N - N_m) M_m^{1/2} \left(\sum_{n \leq M_m} |S_\alpha(n, N_m)|^2 \right)^{1/2} \\
&\ll N^{1/2} M_m^{1/2} H_{N_m}(\alpha)^{1/2} \\
&\leq N^{1/2} M_m^{1/2} (M_m N^{2-1/4})^{1/2} \ll M N^{3/2-1/8} < M N^{3/2}
\end{aligned}
$$

Together with the estimates on II and I_2 we get (2.2) and so prove the proposition. □

Remark 2.1. The choice of exponents $2 - 1/2$, $1 + 1/100$ is completely arbitrary. All we needed was some improvement on the trivial bound $H_N \leq MN^2$.

3. Proof of Theorem 1

In this section we deduce Theorem 1 from Proposition 4. The argument follows closely the one given in [2].

3.1. **Bounding the variance.** Let $f \in C_c^\infty(\mathbf{R})$ be a test function and set

$$R_2(f, N) := \frac{1}{N} \sum_{1 \leq j \neq k \leq N} F_N(\theta_j - \theta_k)$$

where

$$F_N(y) = \sum_{m \in \mathbf{Z}} f(N(y + m))$$

Using the Fourier expansion of $F_N(y)$ we find

$$R_2(f, N) = \frac{1}{N^2} \sum_{n \in Z} \widehat{f}(\frac{n}{N}) \sum_{1 \leq j \neq k \leq N} e(n(\theta_j - \theta_k))$$

that is,

$$R_2(f, N)(\alpha) = \frac{1}{N^2} \sum_{n \in Z} \widehat{f}(\frac{n}{N}) s_{off}(n, N) \tag{3.1}$$

where

$$s_{off}(n, N) := \sum_{1 \leq x \neq y \leq N} e(n\alpha(a(x) - a(y)))$$

As a function of α, $R_2(f, N)(\alpha)$ is periodic and from (3.1) its Fourier expansion is

$$R_2(f, N)(\alpha) = \sum_{l \in \mathbf{Z}} b_l(N) e(l\alpha)$$

where for $l \neq 0$

$$b_l(N) = \frac{1}{N^2}\sum_{n\neq 0}\sum_{\substack{1\leq x\neq y\leq N\\ n(a(x)-a(y))=l}} \widehat{f}(\frac{n}{N}) \tag{3.2}$$

The mean of $R_2(f,N)(\alpha)$ is

$$\int_0^1 R_2(f,N)(\alpha)d\alpha = b_0(N) = \frac{1}{N^2}\sum_{1\leq x\neq y\leq N}\widehat{f}(0) = (1-\frac{1}{N})\widehat{f}(0)$$

so that

$$\int_0^1 R_2(f,N)(\alpha)d\alpha = \int_{-\infty}^{\infty} f(x)dx + O(\frac{1}{N})$$

This is the expected value for a random sequence.

We next estimate the variance of $R_2(f,N)$:

Proposition 5. *Under the assumption of Theorem 1*

$$\int_0^1 |R_2(f,N)(\alpha) - \widehat{f}(0)|^2 d\alpha \ll N^{-99/100+\epsilon}$$

for any $\epsilon > 0$, the implied constants depending on ϵ and f.

Proof. We first note that since $\widehat{f}(\frac{n}{N})$ is negligible if $|n| \gg N^{\frac{101}{100}} = M$, we can bound $b_l(N)$ by

$$\begin{aligned} b_l(N) &\ll \frac{1}{N^2}\sum_{0<|n|\ll M}\sum_{\substack{1\leq x\neq y\leq N\\ n(a(x)-a(y))=l}} \widehat{f}(\frac{n}{N}) \\ &\ll \frac{1}{N^2}\#\{0<|n|\ll M, \quad x\neq y\leq N : n(a(x)-a(y))=l\} \end{aligned}$$

By Parseval

$$\begin{aligned} \int_0^1 |R_2(f,N)(\alpha) - \widehat{f}(0)|^2 d\alpha &= \left(\frac{\widehat{f}(0)}{N}\right)^2 + \sum_{l\neq 0}|b_l(N)|^2 \\ &\ll \frac{1}{N^2} + \frac{1}{N^4}A(M,N) \end{aligned}$$

where $A(M,N)$ is the number of solutions of the equation

$$n_1(a(x_1)-a(y_1)) = n_2(a(x_2)-a(y_2))$$

with $0<|n_1|,|n_2| \ll M$, and $x_1\neq y_1,\ x_2\neq y_2 \leq N$. By the assumption of the Theorem, $A(M,N) \ll MN^{2+\epsilon}$ so since $M = N^{1+1/100}$ we find

$$\int_0^1 |R_2(f,N)(\alpha) - \widehat{f}(0)|^2 d\alpha \ll MN^{-2+\epsilon} \ll N^{-1+1/100+\epsilon}$$

as required. □

3.2. **Almost everywhere convergence.** In order to prove Theorem 1 from the decay of the variance of the pair correlation (proposition 5), we first show that for each $f \in C_c^\infty(\mathbf{R})$, there is a set of full measure, depending on f, so that for all α in this set

$$R_2(f, N_m)(\alpha) \to \widehat{f}(0)$$

for a subsequence N_m which grows faster than m.

Set

$$X_N(\alpha) = R_2(f,N)(\alpha) - \widehat{f}(0)$$

By Proposition 5, $\|X_N\|_2^2 \ll_\epsilon N^{-99/100+\epsilon}$ for all $\epsilon > 0$ and so if we take

$$N_m \sim m^{101/99}$$

then

$$\int_0^1 \sum_m |X_{N_m}(\alpha)|^2 d\alpha = \sum_m \int_0^1 |X_{N_m}(\alpha)|^2 d\alpha < \infty$$

and so $\sum_m |X_{N_m}|^2 \in L^1(0,1)$. Thus the sum is finite almost everywhere, and so $X_{N_m}(\alpha) \to 0$ as $m \to \infty$ for almost all α.

We next show

Lemma 6. *If $N_m \sim m^{101/99}$, $N_m \leq N < N_{m+1}$ then for almost every α*

$$X_N(\alpha) - X_{N_m}(\alpha) \to 0$$

Since $X_{N_m}(\alpha) \to 0$ for almost all α, this Lemma will show that $R_2(f,N)(\alpha) \to \widehat{f}(0)$ for a set of full measure of α which depend on the test function f. By a diagonalization argument we can pass to a subset of full measure of α's which works for all $f \in C_c^\infty(\mathbf{R})$; for the details see [2].

3.3. **Proof of Lemma 6.** Recall that for almost all α we have, by Proposition 4

$$\sum_{1 \leq n \leq M} |S_\alpha(n,N)|^2 \ll MN^{2-1/4}$$

and applying Cauchy-Schwarz we get

$$\sum_{1 \leq n \leq M} |S_\alpha(n,N)| \ll MN^{1-1/8} \tag{3.3}$$

for all $N \gg 1$, and $M = N^{101/100}$.

We write $N = N_m + k$, with $0 \le k \ll N_m^{2/101}$. Then we claim that

$$\text{(3.4)} \quad X_{N_m+k}(\alpha) - X_{N_m}(\alpha)$$
$$= \frac{1}{N_m^2} \sum_{0<|n|\le M} \widehat{f}(\frac{n}{N_m}) \{s_{off}(n, N_m + k) - s_{off}(n, N_m)\} + O(N_m^{-\frac{1}{4}+\frac{1}{100}+\frac{2}{101}})$$

Indeed, since $\widehat{f}$ is rapidly decreasing, the trivial estimate

$$|s_{off}(n, N)| \le N + |S(n, N)|^2 \le N + N^2$$

gives

$$X_N(\alpha) = \frac{1}{N^2} \sum_{0<|n|\le M} \widehat{f}(\frac{n}{N}) s_{off}(n, N) + O(N^{-A})$$

for all $A \gg 1$. From now on we ignore this rapidly decreasing term.

Further, we have from Proposition 4 and $|s_{off}(n, N)| \le N + |S(n, N)|^2$ that

$$\sum_{0<|n|\le M} |s_{off}(n, N_m + k)| \le M(N_m + k) + \sum_{0<|n|\le M} |S(n, N_m + k)|^2$$
$$\ll M(N_m + k) + M(N_m + k)^{2-1/4} \ll M N_m^{2-1/4}$$

Next we claim that

$$\text{(3.5)} \quad \frac{1}{(N_m + k)^2} \sum_{0\ne|n|\le M} \widehat{f}(\frac{n}{N_m + k}) s_{off}(n, N_m + k) =$$
$$\frac{1}{N_m^2} \sum_{0\ne|n|\le M} \widehat{f}(\frac{n}{N_m}) s_{off}(n, N_m + k) + O(N_m^{-\frac{1}{4}+\frac{1}{100}+\frac{2}{101}})$$

This will immediately give (3.4). Indeed, write

$$\frac{1}{(N_m + k)^2} = \frac{1}{N_m^2} + O(\frac{k}{N_m^3}) = \frac{1}{N_m^2} + O(N_m^{-3+2/101})$$

and

$$\frac{n}{N_m + k} = \frac{n}{N_m} + O(\frac{nk}{N_m^2})$$
$$= \frac{n}{N_m} + O(\frac{M}{N_m^{2-2/101}}) = \frac{n}{N_m} + O(N_m^{-1+\frac{1}{100}+\frac{2}{101}})$$

so that for $|n| \le M \sim N_m^{101/100}$, $k < N_m^{2/101}$,

$$\widehat{f}(\frac{n}{N_m + k}) = \widehat{f}(\frac{n}{N_m}) + O(\frac{M}{N_m^{2-2/101}}) = \widehat{f}(\frac{n}{N_m}) + O(N_m^{-1+\frac{1}{100}+\frac{2}{101}})$$

Therefore

$$
\begin{aligned}
&\frac{1}{(N_m+k)^2}\sum_{0\neq|n|\leq M}\widehat{f}(\frac{n}{N_m+k})s_{off}(n,N_m+k)\\
&\qquad-\frac{1}{N_m^2}\sum_{0\neq|n|\leq M}\widehat{f}(\frac{n}{N_m})s_{off}(n,N_m+k)\\
&=\left(\frac{1}{N_m^2}+O(\frac{1}{N^{3-\frac{2}{101}}})\right)\sum_{0\neq|n|\leq M}\left(\widehat{f}(\frac{n}{N_m})+O(N_m^{-1+\frac{1}{100}+\frac{2}{101}})\right)s_{off}(n,N_m+k)\\
&\qquad-\frac{1}{N_m^2}\sum_{0\neq|n|\leq M}\widehat{f}(\frac{n}{N_m})s_{off}(n,N_m+k)\\
&\ll N_m^{-3+2/101}\sum_{0\neq|n|\leq M}|s_{off}(n,N_m+k)|\\
&\ll N_m^{-3+2/101}\cdot MN_m^{2-1/4}\ll N_m^{-1/4+1/100+2/101}\qquad\text{by (3.3)}
\end{aligned}
$$

as required. This proves (3.5) and so (3.4).

As our last step we express the difference $s_{off}(n,N_m+k)-s_{off}(n,N_m)$ in the form

$$
\begin{aligned}
&s_{off}(n,N_m+k)-s_{off}(n,N_m)\\
&=2\operatorname{Re}\sum_{y=N_m+1}^{N_m+k}e(-n\alpha a(y))\sum_{1\leq x\leq N_m}e(n\alpha a(x))+\sum_{N_m+1\leq x\neq y\leq N_m+k}e(n\alpha(a(x)-a(y)))
\end{aligned}
$$

We estimate the second term trivially by $k^2\ll N_m^{4/101}$:

$$|s_{off}(n,N_m+k)-s_{off}(n,N_m)|\leq k|S(n,N_m+k)|+k^2$$

Then inserting this into (3.4) and using (3.3) we get

$$
\begin{aligned}
X_{N_m+k}-X_{N_m}&\ll\frac{1}{N_m^2}\sum_{0<|n|\leq M}\left(k|S(n,N_m+k)|+k^2\right)+N_m^{-\frac{1}{4}+\frac{1}{100}+\frac{2}{101}}\\
&\ll\frac{k}{N_m^2}\sum_{0<|n|\leq M}|S(n,N)|+\frac{Mk^2}{N_m^2}+N_m^{-\frac{1}{4}+\frac{1}{100}+\frac{2}{101}}\\
&\ll\frac{k}{N_m^2}MN_m^{\frac{7}{8}}+\frac{Mk^2}{N_m^2}+N_m^{-\frac{1}{4}+\frac{1}{100}+\frac{2}{101}}\qquad\text{by (3.3)}\\
&\ll N_m^{-\frac{1}{8}+\frac{2}{101}+\frac{1}{100}}+N_m^{-1+\frac{1}{100}+\frac{2}{101}}+N_m^{-\frac{1}{4}+\frac{1}{100}+\frac{2}{101}}\\
&\ll N_m^{-\frac{1}{8}+\frac{2}{101}+\frac{1}{100}}
\end{aligned}
$$

This proves our Lemma. □

4. Proof of Proposition 2

We assume that $a(x) > 0$ is an increasing sequence of positive integers so that there is some $c > 1$ for which

$$a(x+1) \geq c a(x) \tag{4.1}$$

and will show that the equation

$$n_1(a(x_1) - a(y_1)) = n_2(a(x_2) - a(y_2)) \tag{4.2}$$

has at most $O(MN^2 \log^2 N)$ solutions in $0 < |n_i| \leq M$, $1 \leq x_i \neq y_i \leq N$, where $M \ll N^R$ for some $R > 0$.

By changing the sign of n_i and exchanging the roles of x_1 and y_1 and of x_2 for y_2 as needed, we may assume that

$$x_1 > y_1, \qquad x_2 > y_2, \qquad n_1, n_2 > 0 \tag{4.3}$$

Moreover by changing the roles of the right- and left-hand sides of (4.2), we may further assume

$$x_1 \geq x_2 \tag{4.4}$$

We begin by observing that for solution of (4.2) satifying the above normalization conditions (4.3),(4.4), we must have

$$x_1 - x_2 \ll \log_c M \tag{4.5}$$

Indeed, the LHS of (4.2) is by (4.1) at least

(4.6)

$$n_1(a(x_1) - a(y_1)) \geq 1 \cdot (a(x_1) - a(y_1)) \geq a(x_1) - a(x_1 - 1) \geq a(x_1)(1 - \frac{1}{c})$$

The RHS of (4.2) is at most

$$n_2(a(x_2) - a(y_2)) \leq M a(x_2)$$

From (4.1) we have

$$a(x_1) \geq c^{x_1 - x_2} a(x_2)$$

so that the RHS of (4.2) is at most

$$RHS \leq \frac{M a(x_1)}{c^{x_1 - x_2}} \tag{4.7}$$

Combining (4.6) and (4.7) gives

$$a(x_1)(1 - \frac{1}{c}) \leq \frac{M a(x_1)}{c^{x_1 - x_2}}$$

so that

$$x_1 - x_2 \leq \log_c M$$

Now fix n_1, x_1, y_1. We need to show that the number of triples (n_2, x_2, y_2) solving (4.2) and the normalization conditions (4.3),(4.4) is at most $O(\log^2 M)$. Since $x_1 - x_2 \leq \log_c M$ we may also fix x_2 and show that the number of pairs (n_2, y_2) solving (4.2) and the normalization conditions (4.3),(4.4) is at most $O(\log M)$. Since y_2 will now determine n_2, it suffices to determine y_2. For this, it suffices to show that there is at most **one** solution with $x_2 - y_2 > 2\log_c M$.

Indeed, if (n_2, y_2) is a solution with $x_2 - y_2 > 2\log_c M$ then

$$a(y_2) \leq \frac{a(x_2)}{c^{x_2 - y_2}} < \frac{a(x_2)}{M^2}$$

Thus the LHS of (4.2) equals

$$n_2(a(x_2) - a(y_2)) = n_2 a(x_2)\left(1 - \frac{a(y_2)}{a(x_2)}\right) = n_2 a(x_2)(1 + O(\frac{1}{M^2}))$$

If (n_2', y_2') is another such solution then

$$n_2(a(x_2) - a(y_2)) = n_2'(a(x_2) - a(y_2'))$$

so that we find

$$\frac{n_2'}{n_2} = \frac{1 + O(\frac{1}{M^2})}{1 + O(\frac{1}{M^2})} = 1 + O(\frac{1}{M^2})$$

However, since $n_2, n_2' \leq M$ this forces $n_2 = n_2'$. Thus there are at most $1 + 2\log_c M$ solutions of (4.2) with n_1, x_1, y_1, x_2 fixed (and satisfying the normalization conditions). This shows that the total number of solutions of (4.2) is $O(MN^2 \log^2 N)$.

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