A PROOF OF SIEGEL'S WEIGHT FORMULA

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1. Introduction. In this note we apply our orbit-counting method [DRS] to give a simple and conceptual proof of Siegel's weight (or "mass") formula. We begin by recalling this fundamental result. Let A and B be symmetric, nondegenerate (half) integral matrices of size $m \times m$ and $n \times n$, respectively. The formula counts the number of integral solutions to

$$ {}^{t}XAX = B \tag{1.1} $$

where X is an $m \times n$ matrix ($m \geqslant n$) in terms of p-adic solutions to (1.1) (or equivalently, solutions of (1.1) as a congruence). Let G be the orthogonal group $O(A)$. Now G acts on the variety V consisting of solutions of (1.1). $G(\mathbf{R})$ acts transitively on $V(\mathbf{R})$ and the stabilizer H_ξ of $\xi \in V$ is an orthogonal group in $m - n$ variables. $V(\mathbf{R})$ carries a normalized $G(\mathbf{R})$-invariant measure μ_∞ coming from the Hardy-Littlewood integral [ERS]

$$ \mu_\infty(E) = \int_{\eta \in \mathrm{Sym}(n)} \int_{x \in E} e(\mathrm{tr}(({}^{t}XAX - B)\eta))\, dx\, d\eta . \tag{1.2} $$

Here, $e(z) = e^{2\pi i z}$, $\mathrm{Sym}(n)$ consists of symmetric $n \times n$ matrices, and $E \subset \mathbf{R}^{m \times n}$. This measure may also be described in terms of the map $X \mapsto {}^{t}XAX$ as is done in [S1].

Let dg be a Haar measure on $G(\mathbf{R})$. In what follows we normalize Haar measure dh on H_ξ for any $\xi \in V(\mathbf{R})$ by requiring that it satisfy

$$ dg = dh\, d\mu_\infty . \tag{1.3} $$

We can now state a special case of the weight formula [S1]. Let $(r, m - r)$ be the signature of A and suppose

$$ n \leqslant \min(r, m - r), \qquad 2n + 2 < m . \tag{1.4} $$

Let $X_1, \ldots, X_\nu$ be a complete set of $G(\mathbf{Z})$-inequivalent integral solutions to (1.1) and let H_j be the stabilizer of X_j. Then

$$ \frac{1}{\mathrm{vol}(G(\mathbf{Z})\backslash G(\mathbf{R}))} \sum_{j=1}^{\nu} \mathrm{vol}(H_j(\mathbf{Z})\backslash H_j(\mathbf{R})) = \prod_p \mu_p \tag{1.5} $$

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where μ_p is the local density

$$\mu_p = \lim_{a \to \infty} p^{-ad} |\{X \bmod p^a : {}^tXAX \equiv B \bmod p^a\}| \tag{1.6}$$

and $d = mn - n(n+1)/2$ is the dimension of V.

Note that the left-hand side of (1.5) is independent of the choice of Haar measure dg. The general weight formula (that is, without the restrictions (1.4)) is similar to (1.5) except that an extra averaging over classes in the genus of A is necessary [S2]. If A is indefinite in 3 or more variables and is of square-free determinant, then by a well-known result of Meyer [Me] there is only one class in the genus of A, explaining the form of (1.5) in this case.

We give a new proof of Siegel's mass formula for indefinite quadrics ($n = 1$) in 4 or more variables and then observe that this, together with Dirichlet's class number formula, is sufficient for proving that the Tamagawa number of any special orthogonal group is 2. The general case of Siegel's formula then follows from a formal computation of adelic volumes with respect to the Tamagawa measure, for example as in Weil's paper [W1]. Our proof is by comparing two methods for counting the asymptotic number of integer points on the intersection of the quadric with a ball in $\mathbf{R}^m$.

2. Siegel's formula for quadrics. Let F be an indefinite, nondegenerate integral form in $m \geqslant 4$ variables, A the matrix of the form with respect to a suitable basis. For $k \in \mathbf{Z} - \{0\}$ consider the quadric

$$V_k = \{X \mid F(X) = k\} = \{X \mid {}^tXAX = k\}. \tag{2.1}$$

Let $N(T, V_k)$ be the number of integral points in V_k, lying in a ball B_T of radius T about the origin in $\mathbf{R}^m$. The asymptotic behaviour of $N(T, V_k)$ can be computed in two different ways.

First, by the classical Hardy-Littlewood method, which in its simplest form is valid when $m \geqslant 5$ and for $m = 4$ requires Kloosterman's method of "leveling" [E], we have

$$N(T, V_k) \sim \prod_{p < \infty} \mu_p \cdot \mu_\infty(B_T) \qquad \text{as } T \to \infty. \tag{2.2}$$

The local densities are given by (1.6), while the density "at infinity" is given by the singular integral (1.2).

Second, we can count $N(T, V_k)$ using the orbit counting method of [DRS], which counts asymptotically the contributions of each $G(\mathbf{Z})$-orbit in $V_k(\mathbf{Z})$. This method basically uses nonabelian harmonic analysis on $G(\mathbf{Z})\backslash G(\mathbf{R})$; see [EM] for another proof using ergodic theory. If $X_1, \ldots, X_\nu$ are the representatives of the $G(\mathbf{Z})$-orbits in $V_k(\mathbf{Z})$ and $H_1, \ldots, H_\nu$ are their stabilisers in G (for $k \neq 0$ these are orthogonal groups in $m - 1$ variables), then with the normalizations in Section 1 we have for each orbit $x_j G(\mathbf{Z})$

$$\sum_{\substack{\gamma \in G(\mathbf{Z}) \\ x_j\gamma \in B_T}} 1 \sim \frac{\operatorname{vol}(H_j(\mathbf{Z})\backslash H_j(\mathbf{R}))}{\operatorname{vol}(G(\mathbf{Z})\backslash G(\mathbf{R}))}\mu_\infty(B_T).$$

Hence, summing over the representatives, we find

$$N(T, V_k) \sim \sum_{i=1}^{\nu} \frac{\operatorname{vol}(H_i(\mathbf{Z})\backslash H_i(\mathbf{R}))}{\operatorname{vol}(G(\mathbf{Z})\backslash G(\mathbf{R}))} \cdot \mu_\infty(B_T). \tag{2.3}$$

Comparing (2.2) and (2.3), we conclude

$$\frac{1}{\operatorname{vol}(G(\mathbf{Z})\backslash G(\mathbf{R}))} \sum_{i=1}^{\nu} \operatorname{vol}(H_i(\mathbf{Z})\backslash H_i(\mathbf{R})) = \prod_{p<\infty} \mu_p \tag{2.4}$$

which is exactly Siegel's formula (1.5) for these quadrics.

Siegel's formula for forms in 2 variables is equivalent to Dirichlet's class number formula, and for forms in 3 variables which are isotropic over $\mathbf{Q}$ it is again equivalent to Dirichlet's class number formula, since any such form is rationally equivalent to γF, where $\gamma \in \mathbf{Q}^*$ and F is the determinant form

$$F(a, b, c) = \det\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}. \tag{2.5}$$

3. The Tamagawa number of the special orthogonal group. Tamagawa discovered that Siegel's formula for representing a form by itself is equivalent to knowing that the volume of the adelic homogeneous space $G(\mathbf{Q})\backslash G(\mathbf{A})$ with repect to the "Tamagawa measure" τ equals 2: $\tau(G(\mathbf{Q})\backslash G(\mathbf{A})) = 2$, where G is now the special orthogonal group. Kneser [K] and Weil [W1] then showed that $\tau(G) = 2$ implies the general case of Siegel's mass formula. We will now show that the mass formula (2.4) for indefinite quadrics, derived in the previous section, suffices to show that $\tau(G) = 2$ for all orthogonal groups and hence implies the general mass formula. We will assume the reader is familiar with the contents of [W1].

The proof is by induction on the number of variables m. We first deal with $\mathbf{Q}$-isotropic forms. It is enough to deal with square-free discriminants since orthogonal groups of forms whose discriminants differ by a rational square are isomorphic over $\mathbf{Q}$. The case $m = 2$ requires special consideration and is equivalent to Dirichlet's class number formula (which we assume—see [S2]).

Now assume $m \geqslant 3$; pick $k \in \mathbf{Z} - \{0\}$ and a vector $X \in \mathbf{Z}^m$ for which $F(X) = k$, so that F restricted to the orthocomplement of X is still isotropic over $\mathbf{Q}$. The stabiliser H of X is then the special orthogonal group of an isotropic form in $m - 1$ variables, and so by induction $\tau(H) = 2$.

Proceeding now as in [W1], but without assuming the value of $\tau(G)$, we obtain the mass formula (2.4) save for a factor of $\tau(G)/2$ on the left-hand side. Since we have already established (2.4) and both sides are nonzero, we conclude that $\tau(G) = 2$.

To deal with anisotropic forms over $\mathbf{Q}$ (for $m \geqslant 5$ this is equivalent to being definite), take $c \in \mathbf{Z} - \{0\}$ square-free, odd, represented by F, and relatively prime to disc(F). (Such a c is easily seen to exist.) Consider the form

$$F^*(x_1, \ldots, x_{m+1}) = F(x_1, \ldots, x_m) - cx_{m+1}^2 . \tag{3.1}$$

Then disc(F^*) $= -c$ disc(F) and F^* is now isotropic and still has one class in its genus by Meyer's theorem.

Consider now the mass formula (2.4) for the quadric $F^*(x_1, \ldots, x_{m+1}) = -c$; in this formula all stabilisers appearing are $\mathbf{Q}$-equivalent (e.g., by Witt's theorem) and so have the same Tamagawa number $\tau(SO(F)) = 2$. The formal computation of volumes in [W1] gives the mass formula (2.4) upon using $\tau(SO(F^*)) = 2$, save for a factor of $2/\tau(SO(F))$. Therefore, $\tau(SO(F)) = 2$.

Remarks. The above proof that $\tau(G) = 2$ follows the standard inductive procedure as in [M] or [W2] though the beginning of the induction above uses Dirichlet's class number formula rather than resorting to accidental isomorphisms. The new feature is that, by using [DRS] and the classical Hardy-Littlewood method [E], we avoid the use of Poisson summation or, equivalently, the zeta-functions that are introduced when summing over all values of k. In sticking to one quadric V_k, our proof is along the lines of Dirichlet's proof of his class number formula. Siegel's proof in [S1] also has this feature, but it relies heavily on theta-functions and Siegel modular forms.

Our result (2.3) is valid in much greater generality, with the quadric V_k being replaced by any affine symmetric variety. In [ERS] we investigate the extent to which the general homogeneous variety is Hardy-Littlewood in the sense that the asymptotic count (2.2) is (or is not) valid.

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