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# Covariance of Error Terms Related to the Dirichlet Eigenvalue Problem \*

Thesis submitted in partial fulfillment of graduation requirements for a  
degree of “Master of Science” in Mathematics

by  
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# Abstract

We explore the covariance of error terms coming from Weyl's conjecture regarding the number of Dirichlet eigenvalues up to size  $X$ . We also consider this problem in short intervals, i.e. the error term of the number of eigenvalues in the window  $[X, X + S]$  for some  $S(X)$ . We look at these error terms for planar domains where the Dirichlet eigenvalues can be explicitly calculated. In these cases, the error term is closely related to the error term from the classical lattice points counting problem of expanding planar domains. We give a formula for the covariance of such error terms, for general planar domains. We also give a formula for the covariance of error terms in short intervals, for sufficiently large intervals. Going back to the Dirichlet eigenvalue problem, we give results regarding the covariance of the error terms in short intervals of 'generic' rectangles. We also explore a specific example, namely we compute the covariance between the error terms of an equilateral triangle and various rectangles.



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# 1. Introduction

## 1.1. The Dirichlet Eigenvalue Problem

### 1.1.1. Definitions

Let  $\Gamma$  be a planar domain. The Dirichlet eigenvalues of  $\Gamma$  are defined as the eigenvalues of (minus) the Laplace operator  $\Delta$  corresponding to functions on  $\Gamma$  which vanish on the boundary  $\partial\Gamma$ . In other words,  $\lambda$  is a Dirichlet eigenvalue if there is a non-zero solution  $f$  to

$$\begin{cases} -\Delta f = \lambda f & \text{in } \Gamma \\ f|_{\partial\Gamma} = 0. \end{cases}$$

It is known that there are infinitely many eigenvalues, and that these eigenvalues are positive and form a discrete set with a single accumulation point at infinity. Thus, we can denote these eigenvalues as  $0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_n \leq \dots$  where  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ .

### 1.1.2. Hearing the Shape of the Drum

In 1966 Mark Kac published an article titled "Can One Hear the Shape of a Drum?" [11]. This is a cute way of asking whether one can reconstruct the domain  $\Gamma$  given its spectrum (since one can think of the original domain  $\Gamma$  as a two dimensional membrane, and the eigenvalues are the possible frequencies of vibrations of that membrane where we hold the boundary in place). In 1992 Carolyn Gordon, David Webb and Scott Wolpert found two different domains with the same spectrum [6], which showed that the general answer to the question is negative. Nonetheless, it is still interesting to ask what information can be deduced about the original domain given its spectrum.

### 1.1.3. Weyl's Law

We denote  $n(X) = \#\{\lambda_i \leq X\}$  the cumulative counting function of the Dirichlet eigenvalues. We also define  $n(X, S) = n(X+S) - n(X)$  a window count function which counts the number of Dirichlet eigenvalues in the segment  $[X, X+S]$  where  $S(X)$  depends on  $X$ .

An important theorem regarding these counting functions is Weyl's Law [18], which for the two dimensional case states that

$$n(X) \sim \frac{1}{4\pi} \text{Area}(\Gamma) X.$$

An immediate corollary from this theorem is that the area of the original domain  $\Gamma$  can be deduced given its spectrum. Weyl also conjectured the next term in the approximation-

$$n(X) = \frac{1}{4\pi} \text{Area}(\Gamma) X - \frac{1}{4\pi} \text{Length}(\partial\Gamma) \sqrt{X} + e(X)$$

where  $e(x)$  is of smaller order. While in general this conjecture is still open, it has been proven by Victor Ivrii [10] for domains  $\Gamma$  where the set of periodic trajectories of a billiard in  $\Gamma$  has measure 0.

#### 1.1.4. The Error Term and the Window Count Error Term

There has been a lot of work dedicated to the study of  $e(X)$  and other statistical properties of the spectrum. For example, another important function is the error term for the window counting function, which can be expressed as  $e(X, S) = e(X + S) - e(X)$ . The statistics of  $e(X, S)$  behave differently depending on the growth rate of  $S(X)$  as  $X \rightarrow \infty$ . The case where  $S(X)$  is constant is especially important due to the Berry-Tabor conjecture (their original paper at [1], and a survey can be found in [15]). This is a major conjecture from the field of quantum chaos, which suggests a connection between the statistical behaviour of  $e(X, S)$  and whether the billiard flow for  $\Gamma$  is chaotic or integrable.

## 1.2. Lattice Point Counting Problem

### 1.2.1. Definitions

We now consider another classical problem, that of estimating the number of lattice points inside an expanding planar domain. Let  $\Omega$  be a convex shape in the plane, with smooth boundary  $\gamma = \partial\Omega$  and positive curvature. We denote

$$N(R) = \#\{\mathbb{Z}^2 \cap R\Omega\}.$$

It is known that  $N(R)$  is approximated by the area of  $R\Omega$  which is  $R^2 \text{Area}(\Omega)$ . And so, we define the error function

$$E(R) = N(R) - \text{Area}(\Omega) R^2$$

### 1.2.2. Bounding the Lattice Point Error Function

There has been a lot of research dedicated to try and bound this error term  $E(R)$ . One can easily show that  $E(R) = \mathcal{O}(R)$ . However, it is believed that  $E(R) = \mathcal{O}(R^{1/2+\epsilon})$  for all  $\epsilon > 0$ . A special case of this problem is when  $\Omega$  is the unit disk. In this case showing that  $E(R) = \mathcal{O}(R^{1/2+\epsilon})$  for all  $\epsilon > 0$  is known as the Gauss circle problem.

While we still don't know to show that  $E(R) = \mathcal{O}(R^{1/2+\epsilon})$ , there have been significant improvements in this area. The bound  $E(R) = \mathcal{O}(R^{2/3})$  was obtained by Sierpiński [16] at the start of the twentieth century. Since then, various improvements to this bound were obtained. The current best known bound for the case of the circle is  $E(R) = \mathcal{O}(R^{517/824+\epsilon})$  due

to Bourgain and Watt [5]. The lower bound  $\limsup_{R \rightarrow \infty} E(R)/R^{1/2} > 0$  was shown independently by Hardy and Landau in 1915 [7] [13].

### 1.2.3. The Asymptotic Distribution of the Lattice Point Error Function

While proving the bound  $E(R) = \mathcal{O}(R^{1/2+\epsilon})$  is still outside our reach, in the 1940's Wintner considered a related problem [19]. Consider the normalized error function

$$F(R) = E(R)/R^{1/2}. \quad (1.1)$$

We can ask whether this function has an asymptotic density. That is, we want to know if there exists a density function  $p(t)$ , such that for every segment  $[a, b] \subset \mathbb{R}$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mu(\{R \in [0, T] : F(R) \in [a, b]\}) = \int_a^b p(t) dt$$

where  $\mu$  is the standard Lebesgue measure.

Wintner and Heath-Brown showed that for the case of the circle such a density function does exist [19] [8]. This was later generalized by Bleher for general convex domains with a smooth boundary and positive curvature [3].

## 1.3. The Connection Between Counting Dirichlet Eigenvalues and Counting Lattice Points

### 1.3.1. Domains with Explicit Eigenvalues Formulas

There is a connection between the Dirichlet eigenvalue counting problem and the lattice point counting problem. This connection becomes apparent once we consider the Dirichlet eigenvalue problem for domains where the eigenvalues can be explicitly calculated. Very few such domains are known, and these include rectangles, ellipses, and a few select triangles such as the equilateral triangle.

Consider for example the case of a rectangle  $\Gamma = [0, a] \times [0, b]$ . The Dirichlet eigenfunctions in this case are given by

$$f(x, y) = \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m y}{b}\right), \quad n, m \in \mathbb{N}$$

and the eigenvalues are

$$\left\{ \pi^2 \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right) : n, m \in \mathbb{N} \right\}. \quad (1.2)$$

And so it can be seen that  $n(X)$ , the number of eigenvalues up to  $X$ , is related to the number of integer lattice points inside the ellipse  $\Omega : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq \frac{1}{\pi^2}$  dilated by a factor of  $\sqrt{X}$ . There are



still minor differences due to fact that the Dirichlet eigenvalues come from positive  $n, m$ . Nonetheless, it can still be shown that

$$N_{\Omega}(X) = 4n_{\Gamma}(X^2) + \frac{2}{\pi}(a+b)X + \mathcal{O}(1)$$

and as a result

$$E_{\Omega}(X) = 4e_{\Gamma}(X^2) + \mathcal{O}(1). \quad (1.3)$$

A similar thing happens in the case of  $\Gamma$  an equilateral triangle of side length  $\ell$ . While not a trivial matter as in the case of the rectangle, it was shown by Lamé [12] that the eigenvalues of  $\Gamma$  are of the form  $\frac{16\pi^2}{9\ell^2}(n^2 + nm + m^2)$  for  $n, m \in \mathbb{N}$ . A nice account of this was given by McCartin [14]. Denoting by  $\Omega$  the ellipse  $x^2 + xy + y^2 \leq \frac{9\ell^2}{16\pi^2}$  we once more get a connection

$$N_{\Omega}(X) = 6n_{\Gamma}(X^2) + \frac{9}{2\pi}\ell\sqrt{X} + \mathcal{O}(1).$$

which implies

$$E_{\Omega}(X) = 6e_{\Gamma}(X^2) + \mathcal{O}(1). \quad (1.4)$$

We see that for these cases, the study of the error term from the Dirichlet eigenvalue problem is equivalent to the study of the error term from the lattice point problem for ellipses.

### 1.3.2. The Window Count of Dirichlet Eigenvalues and the Lattice Point Problem in Thin Annuli

In the case of rectangles and special triangles, there is also an analogue for the window count function in terms of the lattice point problems. The number of lattice points in a thin annuli of shape  $\Omega$  and width  $h$ , is given by  $N_{\Omega}(R, h) = N_{\Omega}(R + h) - N_{\Omega}(R)$ . The expected number of points in this thin annuli is  $\text{Area}(\Omega)(2hR + h^2)$ . The error term is given by  $E(R, h) = E(R + h) - E(R)$ , and we define the normalized error term as  $F(R, h) = E(R, h)/R^{1/2}$ . Note that in the case of  $e(X, S)$ ,  $S$  is the area of the annulus, while in  $E(R, h)$ ,  $h$  is its width.

## 1.4. Main Results

In this paper we investigate further statistical properties of the error functions  $E(X)$  and  $E(X, h)$  coming from the lattice point problem for convex smooth domains. Specifically, we consider the covariance of such error function coming from two different domains. We give a more precise formula for the case of ellipses. These cases are of particular importance as they are also equivalent to error functions coming from the Dirichlet eigenvalue problem.

### 1.4.1. Global Covariance

Let  $\Omega_1, \Omega_2$  be two convex planar domains with a smooth boundary and positive curvature, and let  $F_1, F_2$  be their normalized error functions for the lattice counting problem as defined in (1.1). It was shown by Bleher that these functions have an asymptotic density, and he also gave a formula for their variance. We explore the covariance of these function, where the covariance of  $F_1$  and  $F_2$  is

**Definition 1.1** (Covariance).

$$\text{Cov}(F_1, F_2) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_1(t) F_2(t) dt$$

In [Theorem 1.2](#) we give a formula for  $\text{Cov}(F_1, F_2)$ .

**Theorem 1.2.** Denote  $\gamma_i = \partial\Omega_i$ ,  $i = 1, 2$ . For  $v \in \mathbb{R}^2$  we denote  $x_i(v)$  the point on  $\gamma_i$  with outer normal  $\frac{v}{\|v\|}$ . We define  $Y_i(v) = v \cdot x_i(v)$  and  $\rho_i(v)$  as the curvature radius of  $\gamma_i$  at  $x_i(v)$ . With these notations-

$$\text{Cov}(F_1, F_2) = \frac{1}{2\pi^2} \sum'_{n \in \mathbb{Z}^2} \sum'_{\substack{m \in \mathbb{Z}^2 \\ Y_2(m) = Y_1(n)}} \frac{\sqrt{\rho_1(n)} \sqrt{\rho_2(m)}}{|n|^{3/2} |m|^{3/2}}$$

where  $\sum'$  denotes a sum where 0 is omitted.

*Remark.* A typical pair of domains  $\Omega_1, \Omega_2$  will have no non-zero  $n, m \in \mathbb{Z}^2$  such that  $Y_1(n) = Y_2(m)$  (which means that the covariance will be 0). If we restrict to the case where the covariance is positive, then for the typical case the set of solutions  $n, m \in \mathbb{Z}^2$  to  $Y_1(n) = Y_2(m)$  will be of the form  $\{(kn_0, km_0) | k \in \mathbb{Z}\}$  for some initial solution  $n_0, m_0 \in \mathbb{Z}^2$ .

However, regardless of the number of summands, the series in [Theorem 1.2](#) always converges. This is a result of the function  $F_i$  being in the space  $\mathbb{B}^2$  of almost periodic functions, as will be discussed later.

Applying [Theorem 1.2](#) for the case of ellipses, we get the following corollary-

**Corollary 1.3.** Let  $\Omega_1, \Omega_2$  be general ellipses of the form

$$\Omega_1 : ax^2 + bxy + cy^2 \leq 1, \quad \Omega_2 : dx^2 + exy + fy^2 \leq 1$$

for some parameters  $a, b, c, d, e, f$ . Then the covariance between the normalized error functions  $F_{\Omega_1}(t), F_{\Omega_2}(t)$  is

$$\frac{16\pi}{\sqrt{4ac - b^2} \sqrt{4df - e^2}} \sum_v \frac{r_1(v) r_2(v)}{v^3}$$

where  $r_1(v)$  is the number of integer solutions  $n_1, n_2$  to

$$4\pi \frac{\sqrt{cn_1^2 - bn_1n_2 + an_2^2}}{\sqrt{4ac - b^2}} = v$$

and similarly  $r_2(v)$  is the number of integer solutions  $m_1, m_2$  to

$$4\pi \frac{\sqrt{f m_1^2 - e m_1 m_2 + d m_2^2}}{\sqrt{4df - e^2}} = v.$$

#### 1.4.2. Covariance in Short Intervals

We also consider the covariance of the error term for short intervals. Let  $F_1(R, h), F_2(R, h)$  be the normalized error functions in short intervals of the lattice point problem related to  $\Omega_1, \Omega_2$ . We assume that  $h(R)$  tends to 0 as  $R$  tends to infinity. We define the covariance of these functions as

**Definition 1.4** (Covariance in Short Intervals). for a given function  $h(R)$ , we say that the covariance between  $F_1(R, h)$  and  $F_2(R, h)$  is  $f(h)$  for some function  $f$  if

$$\frac{1}{T} \int_{t=0}^T F_1(t, h(T)) F_2(t, h(T)) dt \sim f(h(T)).$$

In [Theorem 3.8](#) we give a formula for the covariance in short intervals. However, our proof techniques require us to have  $h(R)$  decay sufficiently slowly towards 0. Furthermore, we also get a Diophantine-like condition that the two domains need to satisfy. For general domains this condition is somewhat vague.

We can however make these conditions more concrete if we restrict our attention to a small space of domains. In [section 4](#) we restrict our attention to ellipses. In this case, Bleher and Lebowitz showed [\[4\]](#) that the variance of the normalized error function  $F(R, h)$  of a "generic" ellipse of the form  $\{(x, y) : x^2 + \mu y^2 \leq 1\}$  has order  $h$ , provided that  $h(T) \gg T^{-1+\epsilon}$  for some  $\epsilon > 0$ . We show that for a generic pair of ellipses that have non-zero covariance in the global case, the covariance in a short interval  $h$  will be of order  $h^2 \log(h^{-1})$  provided that  $h(T) \gg T^{-1/110+\epsilon}$  for some  $\epsilon > 0$ . The main result for this is

**Theorem 1.5.** *Almost all pairs of ellipses*

$$\Omega_1 : ax^2 + bxy + cy^2 \leq 1, \quad \Omega_2 : dx^2 + exy + fy^2 \leq 1$$

*that have non-zero global covariance satisfy the following:*

*For any  $h(T)$  with  $h(T) \gg T^{-1/110+\epsilon}$  for some  $\epsilon > 0$ :*

$$\text{Cov}(F_1(T, h), F_2(T, h)) = Ch^2 \log(h^{-1})$$

*where  $F_1, F_2$  are the normalized error functions of  $\Omega_1, \Omega_2$ .*

*The constant  $C$  is given by*

$$C = \frac{16\pi}{v_0 \sqrt{4ac - b^2} \sqrt{4df - e^2}}$$

where  $v_0$  is the smallest positive number for which

$$4\pi \frac{\sqrt{cn_1^2 - bn_1n_2 + an_2^2}}{\sqrt{4ac - b^2}} = 4\pi \frac{\sqrt{fm_1^2 - em_1m_2 + dm_2^2}}{\sqrt{4df - e^2}} = v_0$$

has an integer solution  $n_1, n_2, m_1, m_2$ .

We also examine the covariance between two specific ellipses coming from the Dirichlet eigenvalue problem of an equilateral triangle and a rectangle. We get the following theorem:

**Theorem 1.6.** *Let  $F_1, F_2$  be the normalized error functions of the following two ellipses:*

$$\Omega_1 : x^2 + xy + y^2 \leq \frac{3}{4}, \quad \Omega_2 : x^2 + \alpha y^2 \leq 1$$

for some parameter  $\alpha \in \mathbb{R}^+$ .

1. *For almost all  $\alpha \in \mathbb{R}^+$ , if  $h(T) \gg T^{-1/86+\epsilon}$  for some  $\epsilon > 0$ , then the covariance of  $F_1$  and  $F_2$  is  $\frac{9}{\pi\sqrt{\alpha}} h^2 \log^2(h^{-1})$*
2. *If  $\alpha = \frac{p}{q}$  is a rational such that  $3pq$  is not a perfect square, then assuming  $h(T) \gg T^{-1/62}$ , the covariance of  $F_1$  and  $F_2$  is  $\frac{C\sqrt{3}}{\sqrt{\alpha}} h$  where  $C$  is a constant given in [Theorem A.3](#).*
3. *If  $\alpha = \frac{p}{q}$  is a rational and  $3pq$  is a perfect square then assuming  $h(T) \gg T^{-1/62}$ , the covariance of  $F_1$  and  $F_2$  is  $\frac{18}{\sqrt{pp'}} h \log(h^{-1})$  where  $p'$  is the square free part of  $p$ .*

Using (1.3), (1.4), we can restate [Theorem 1.6](#) in terms of the Dirichlet eigenvalue problem:

**Corollary 1.7.** *Let  $\Gamma_1$  be an equilateral triangle of side length  $\frac{2\pi}{\sqrt{3}}$ , and let  $\Gamma_2$  be a rectangle with side lengths  $\pi$  and  $\sqrt{\alpha}\pi$  for some  $\alpha > 0$ . Denote by  $n_i(t)$  the number of Dirichlet eigenvalues of  $\Gamma_i$  up to  $t^2$ , and denote*

$$e_i(t) = n_i(t) - \frac{1}{4\pi} \text{Area}(\Gamma_i) t^2 + \frac{1}{4\pi} \text{Length}(\partial\Gamma_i) t.$$

1. *For almost all  $\alpha$ , if  $h(X) \gg X^{-1/86+\epsilon}$  for some  $\epsilon > 0$ , then*

$$\frac{1}{X} \int_0^X \frac{(e_1(t+h) - e_1(t))(e_2(t+h) - e_2(t))}{t} dt \sim \frac{3}{8\pi\sqrt{\alpha}} h^2 \log^2(h^{-1}).$$

2. *If  $\alpha = \frac{p}{q}$  is a rational such that  $3pq$  is not a perfect square, and if  $h(T) \gg X^{-1/62}$ , then*

$$\frac{1}{X} \int_0^X \frac{(e_1(t+h) - e_1(t))(e_2(t+h) - e_2(t))}{t} dt \sim \frac{C}{8\sqrt{3\alpha}} h$$

where the constant  $C$  is given in [Theorem A.3](#).

3. If  $\alpha = \frac{p}{q}$  is a rational and  $3pq$  is a perfect square, and if  $h(T) \gg X^{-1/62}$ , then

$$\frac{1}{X} \int_0^X \frac{(e_1(t+h) - e_1(t))(e_2(t+h) - e_2(t))}{t} dt \sim \frac{3}{4\sqrt{pp'}} h \log(h^{-1})$$

where  $p'$  is the square free part of  $q$ .

Using (1.2), Theorem 1.5 can also be restated in terms of the error terms of the Dirichlet eigenvalue problem. This is not a direct corollary, but similar arguments to those given in subsection 4.1 result in the following:

**Corollary 1.8.** Let  $\Gamma_1 = [0, a] \times [0, b], \Gamma_2 = [0, c] \times [0, d]$  be two rectangles. Denote by  $n_i(t)$  the number of Dirichlet eigenvalues of  $\Gamma_i$  up to  $t^2$ , and denote

$$e_i(t) = n_i(t) - \frac{1}{4\pi} \text{Area}(\Gamma_i) t^2 + \frac{1}{4\pi} \text{Length}(\partial\Gamma_i) t.$$

For almost all pairs of rectangles  $\Gamma_1, \Gamma_2$  that have non-zero global covariance, we have that if  $h(T) \gg T^{-1/110+\epsilon}$  for some  $\epsilon > 0$ :

$$\frac{1}{X} \int_0^X \frac{(e_1(t+h) - e_1(t))(e_2(t+h) - e_2(t))}{t} dt \sim Ch^2 \log(h^{-1}).$$

The constant  $C$  is given by

$$C = \frac{abcd}{(2\pi)^3 v_0}$$

where  $v_0$  is the smallest positive number for which

$$\sqrt{b^2 n_1^2 + a^2 n_2^2} = \sqrt{d^2 m_1^2 + c^2 m_2^2} = v_0$$

has an integer solution  $n_1, n_2, m_1, m_2$ .

Similar results can also be obtained for a rectangle and equilateral triangle pair.

The reason we restrict our attention to rectangles is that their Dirichlet eigenvalues can be explicitly calculated. For general parallelograms the eigenvalues with Dirichlet or Neumann boundary conditions are not explicitly known (only with periodic boundary conditions). In fact, for parallelograms which are not rectangles, the Dirichlet eigenfunctions are not trigonometric, see [14, Theorems 4.1.2, 4.2.1].

## 2. Global Covariance

Let  $\Omega_1, \Omega_2$  be two convex planar domains with smooth boundary and positive curvature. In this section we look at the global normalized error functions  $F_1(R), F_2(R)$  of these domains

as defined in (1.1). In order to find the covariance of these functions we use the fact that these functions are Besicovitch almost periodic functions. In subsection 2.1 we introduce the theory of almost periodic functions. In subsection 2.2 we apply the theory to the normalized error functions to obtain a formula for the global covariance. In subsection 2.3 we consider the more specific case where  $\Omega_1, \Omega_2$  are ellipses.

## 2.1. Almost Periodic Functions

### 2.1.1. The space $\mathbb{B}^2$

**Definition 2.1.** The space  $\mathbb{B}^2$  of Besicovitch almost periodic functions (see [2]) is defined as the closure of trigonometric polynomials under the semi-norm

$$\|f\| = \left( \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T f^2(t) dt \right)^{1/2}.$$

In other words, a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is in  $\mathbb{B}^2$  if there exists a sequence of trigonometric polynomials  $f_N(t) = \sum_{n \leq N} c_n e^{i\lambda_n t}$ , ( $c_n \in \mathbb{C}, \lambda_n \in \mathbb{R}$ ) such that

$$\lim_{N \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T (f - f_N)^2(t) dt = 0.$$

For a function  $f \in \mathbb{B}^2$ , the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f^2(t) dt$$

exists.

### 2.1.2. Fourier Transform of Almost Periodic Functions

**Definition 2.2.** The Fourier transform of a function  $f \in \mathbb{B}^2$  is defined as

$$\hat{f}(\xi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) e^{-i\xi t} dt.$$

An important result in this regard is the following lemma-

**Lemma 2.3.**  $\hat{f}(\xi)$  is 0 for all  $\xi$  except at most a countable set.

If  $\hat{f}(\xi_0) \neq 0$ , we refer to  $\xi_0$  as a frequency of  $f$ .

**Definition 2.4** (Frequency Expansion). If  $\xi_1, \xi_2, \dots$  are all the frequencies of  $f \in \mathbb{B}^2$ , we denote

$$\sum_{n=1}^{\infty} \hat{f}(\xi_n) e^{i\xi_n t}$$

as the frequency expansion of  $f$ .

Note that this frequency expansion can be a divergent series. As such, it should be regarded as a formal way of encoding the Fourier transform of  $f$ , and not as a formula.

### 2.1.3. Parseval's Identity for Almost Periodic Functions

Our main gain from introducing Fourier transform of functions in  $\mathbb{B}^2$  is the following theorem

**Theorem 2.5** (Parseval's Identity). *Let  $f$  be a function in  $\mathbb{B}^2$ , and let  $\sum_{n=1}^{\infty} \hat{f}(\xi_n) e^{i\xi_n t}$  be its frequency expansion. Then-*

$$\|f\|^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f^2(t) dt = \sum_{n=1}^{\infty} |\hat{f}(\xi_n)|^2.$$

**Corollary 2.6.** *Let  $f, g \in \mathbb{B}^2$  with frequency expansions*

$$\sum_{n=1}^{\infty} \hat{f}(\alpha_n) e^{i\alpha_n t}, \quad \sum_{n=1}^{\infty} \hat{g}(\beta_n) e^{i\beta_n t}.$$

*Denote  $\{\xi_n\} = \{\alpha_n\} \cap \{\beta_n\}$  the set of common frequencies. Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) g(t) dt = \operatorname{Re} \left( \sum_{n=1}^{\infty} \hat{f}(\xi_n) \overline{\hat{g}(\xi_n)} \right).$$

*Proof.* Apply [Parseval's Identity](#) to  $f + g$ , which is also in  $\mathbb{B}^2$ . Note that the frequency expansion of  $f + g$  is the sum of the frequency expansions of  $f$  and  $g$ .  $\square$

## 2.2. Covariance of General Planar Domains

### 2.2.1. Definitions

We reintroduce some of the definitions given in [Theorem 1.2](#). Let  $\Omega$  be a convex planar domain with a smooth boundary  $\gamma = \partial\Omega$  and positive curvature. For  $v \in \mathbb{R}^2$  we denote  $x(v)$  the point on  $\gamma$  with outer normal  $\frac{v}{\|v\|}$ . We define  $Y(v) = v \cdot x(v)$  and  $\rho(v)$  as the curvature radius of  $\gamma$  at  $x(v)$ . Denote also  $F(R)$  the normalized error function associated with  $\Omega$  as defined in [\(1.1\)](#).

### 2.2.2. Almost Periodicity of the Error Function

In [\[3\]](#) Bleher proves that the normalized error function  $F(R)$  is in  $\mathbb{B}^2$ . More specifically, we have

**Theorem 2.7.** *Denote*

$$P_N(t) = \frac{1}{2\pi} \sum_{\substack{n \in \mathbb{Z}^2 \\ 0 < |n| < N}} \frac{\sqrt{\rho(n)}}{|n|^{3/2}} \left( \exp \left( i2\pi Y(n)t - i\frac{3\pi}{4} \right) + \exp \left( -i2\pi Y(n)t + i\frac{3\pi}{4} \right) \right)$$

which can be more compactly written as

$$P_N(t) = \frac{1}{\pi} \sum_{\substack{n \in \mathbb{Z}^2 \\ 0 < |n| < N}} \frac{\sqrt{\rho(n)}}{|n|^{3/2}} \cos\left(2\pi Y(n)t - \frac{3\pi}{4}\right).$$

For  $N \in \mathbb{N}$ ,  $T \in \mathbb{R}$  such that  $N^{5/2} \leq T$  we have

$$\frac{1}{T} \int_0^T (F(t) - P_N(t))^2 dt = \mathcal{O}(N^{-1/3}).$$

**Corollary 2.8.**

$$\lim_{N \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T (F(t) - P_N(t))^2 dt = 0.$$

which implies  $F(R) \in \mathbb{B}^2$ .

*Proof.* see [3], Theorem 3.1 and Lemmas 3.2, 3.3, 3.4. □

Furthermore, it can be shown that the frequency expansion of  $F(R)$  is

$$\frac{1}{2\pi} \sum'_{n \in \mathbb{Z}^2} \frac{\sqrt{\rho(n)}}{|n|^{3/2}} \left( \exp\left(i2\pi Y(n)t - i\frac{3\pi}{4}\right) + \exp\left(-i2\pi Y(n)t + i\frac{3\pi}{4}\right) \right). \quad (2.1)$$

### 2.2.3. A Formula For the Covariance

Consider once more the problem of computing the covariance of  $F_1(R)$ ,  $F_2(R)$ , the normalized error functions associated with  $\Omega_1$  and  $\Omega_2$ .

*Proof of Theorem 1.2.* From (2.1) we have an expression for the frequency expansions of  $F_1$  and  $F_2$ . Furthermore, from Corollary 2.6 we get a formula for the covariance of two functions in  $\mathbb{B}^2$ . putting these together we get that

$$\text{Cov}(F_1, F_2) = \frac{1}{2\pi^2} \sum'_{n \in \mathbb{Z}^2} \sum'_{\substack{m \in \mathbb{Z}^2 \\ Y_2(m) = Y_1(n)}} \frac{\sqrt{\rho_1(n)} \sqrt{\rho_2(m)}}{|n|^{3/2} |m|^{3/2}}$$

which proves Theorem 1.2. □

*Remark.* Note that the covariance is always non-negative, and it is positive if and only if  $F_1$  and  $F_2$  share common frequencies.

**Corollary 2.9.** *Let  $\Omega_1, \Omega_2$  be convex planar domains with a smooth boundary and positive curvature. The covariance between the normalized error functions of  $\Omega_1$  and  $\alpha\Omega_2$  is 0 except for a countable set of  $\alpha$ 's for which the covariance is positive.*



*Proof.* The covariance is positive if and only if the normalized error functions share common frequencies. This happens only when  $\alpha$  is of the form

$$\alpha = Y_1(n)/Y_2(m), \quad 0 \neq n, m \in \mathbb{Z}^2.$$

□

## 2.3. Covariance of Ellipses

### 2.3.1. Calculations Related to Ellipses

We can get a more specific formula than that in [Theorem 1.2](#) if we restrict to the case of ellipses. Let  $\Omega$  be a general ellipse given by

$$\Omega: ax^2 + bxy + cy^2 \leq 1.$$

Denote  $H(x, y) = ax^2 + bxy + cy^2 - 1$ . The normal at a point  $(x_0, y_0)$  on  $\gamma = \partial\Omega$  is given by

$$(H_x(x_0, y_0), H_y(x_0, y_0)) = (2ax_0 + by_0, 2cy_0 + bx_0).$$

It follows that

$$x(n_1, n_2) = \left( \frac{2cn_1 - bn_2}{\sqrt{4ac - b^2} \sqrt{an_2^2 - bn_1n_2 + cn_1^2}}, \frac{2an_2 - bn_1}{\sqrt{4ac - b^2} \sqrt{an_2^2 - bn_1n_2 + cn_1^2}} \right)$$

and

$$Y(n_1, n_2) = (n_1, n_2) \cdot x(n_1, n_2) = \frac{2\sqrt{an_2^2 - bn_1n_2 + cn_1^2}}{\sqrt{4ac - b^2}}. \quad (2.2)$$

The formula for the radius of curvature at a point  $(x_0, y_0)$  on  $\gamma$  is

$$\frac{(H_x^2 + H_y^2)^{3/2}}{H_{xx}H_y^2 - 2H_{xy}H_xH_y + H_{yy}H_x^2}(x_0, y_0).$$

It follows that

$$\rho(n_1, n_2) = \frac{\sqrt{4ac - b^2}}{2} \left( \frac{n_1^2 + n_2^2}{an_2^2 - bn_1n_2 + cn_1^2} \right)^{3/2}. \quad (2.3)$$

It will be useful to note that

$$\begin{aligned} \frac{1}{2\pi} \frac{\sqrt{\rho(n_1, n_2)}}{(n_1^2 + n_2^2)^{3/4}} &= \frac{\sqrt[4]{4ac - b^2}}{2\sqrt{2}\pi (an_2^2 - bn_1n_2 + cn_1^2)^{3/4}} = \\ &= \frac{2\sqrt{2}\pi}{\sqrt{4ac - b^2}} \left( \frac{4ac - b^2}{16\pi^2 (an_2^2 - bn_1n_2 + cn_1^2)} \right)^{3/4} = \frac{\sqrt{8\pi}}{\sqrt{4ac - b^2}} (2\pi Y(n_1, n_2))^{-3/2}. \end{aligned} \quad (2.4)$$

### 2.3.2. A Formula For The Covariance of Ellipses

We can now prove the formula for the covariance of normalized error functions of ellipses.

*Proof of [Corollary 1.3](#).* Let  $\Omega_1, \Omega_2$  be general ellipses of the form

$$\Omega_1 : ax^2 + bxy + cy^2 \leq 1, \quad \Omega_2 : dx^2 + exy + fy^2 \leq 1$$

for some parameters  $a, b, c, d, e, f$ . We plug (2.2) and (2.3) into the formula of the covariance from [Theorem 1.2](#). We get that

$$\begin{aligned} \text{Cov}(F_1, F_2) = & \frac{1}{2\pi^2} \sum'_{n \in \mathbb{Z}^2} \sum'_{\substack{m \in \mathbb{Z}^2 \\ Y_2(m) = Y_1(n)}} \frac{\sqrt{\rho_1(n)} \sqrt{\rho_2(m)}}{|n|^{3/2} |m|^{3/2}} = 2 \sum'_{n \in \mathbb{Z}^2} \sum'_{\substack{m \in \mathbb{Z}^2 \\ Y_2(m) = Y_1(n)}} \left( \frac{\sqrt{\rho_1(n)}}{2\pi |n|^{3/2}} \right) \left( \frac{\sqrt{\rho_2(m)}}{2\pi |m|^{3/2}} \right). \end{aligned}$$

From (2.4), we get that this is equal to

$$\text{Cov}(F_1, F_2) = \sum'_{n \in \mathbb{Z}^2} \sum'_{\substack{m \in \mathbb{Z}^2 \\ Y_2(m) = Y_1(n)}} \frac{16\pi}{\sqrt{4ac - b^2} \sqrt{4df - e^2}} \frac{1}{(2\pi Y_1(n))^3}.$$

Denoting by  $r_1(v)$  the number of integer solutions  $n_1, n_2$  to

$$4\pi \frac{\sqrt{an_2^2 - bn_1n_2 + cn_1^2}}{\sqrt{4ac - b^2}} = v$$

and similarly  $r_2(v)$  the number of integer solutions  $m_1, m_2$  to

$$4\pi \frac{\sqrt{dm_2^2 - em_1m_2 + fm_1^2}}{\sqrt{4df - e^2}} = v$$

we get from (2.2) that

$$\text{Cov}(F_1, F_2) = \frac{16\pi}{\sqrt{4ac - b^2} \sqrt{4df - e^2}} \sum_v \frac{r_1(v) r_2(v)}{v^3}$$

where  $v$  goes over all positive common frequencies. □

## 3. Covariance in Short Intervals

In this section we compute the covariance in a short interval  $h(T)$  of general domains  $\Omega_1, \Omega_2$  as defined in [Definition 1.4](#). We will always assume that the global covariance of the normalized error functions  $F_1, F_2$  of  $\Omega_1, \Omega_2$  is non-zero. We begin in [subsection 3.1](#) by approximating our functions  $F_1(t, h), F_2(t, h)$  with trigonometric polynomials and calculate their covariance. Then, in [subsection 3.2](#) we use the fact that we found the covariance of the approximating trigonometric polynomials to deduce the covariance of  $F_1(t, h), F_2(t, h)$ .

### 3.1. Covariance of Approximating Trigonometric Polynomials

#### 3.1.1. The Trigonometric Polynomials $P_{i,N}$

The functions  $F_i(t)$ , ( $i = 1, 2$ ) are approximated by

$$P_{i,N}(t) = \frac{1}{\pi} \sum_{\substack{n \in \mathbb{Z}^2 \\ 0 < |n| < N}} \frac{\sqrt{\rho_i(n)}}{|n|^{3/2}} \cos\left(2\pi Y_i(n)t - \frac{3\pi}{4}\right)$$

in the sense of [Theorem 2.7](#).

It will be useful to introduce the following notation:

**Definition 3.1** (Averaging Operator). For a function  $F$ , we define

$$\langle F \rangle_T = \frac{1}{T} \int_0^T F(t) dt.$$

[Theorem 2.7](#) then states that

$$\left\langle (F_i(t) - P_{i,N}(t))^2 \right\rangle_T = \mathcal{O}(N^{-1/3})$$

provided that  $N \leq T^{2/5}$ .

Consider then the trigonometric polynomials

$$P_N(t, h) = P_N(t + h) - P_N(t).$$

Using the trigonometric identity

$$\cos(\alpha) - \cos(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\beta - \alpha}{2}\right)$$

we get that

$$P_{i,N}(t, h) = \frac{2}{\pi} \sum_{\substack{n \in \mathbb{Z}^2 \\ 0 < |n| < N}} \frac{\sqrt{\rho_i(n)}}{|n|^{3/2}} \sin\left(2\pi Y_i(n) \frac{h}{2}\right) \sin\left(2\pi Y_i(n) \left(t + \frac{h}{2}\right) + \frac{\pi}{4}\right).$$

From the triangle inequality we get that, for sufficiently small  $h$ ,  $P_{i,N}(t, h)$  is an approximation of  $F_i(t, h)$  in much the same way as  $P_{i,N}(t)$  is an approximation of  $F_i(t)$ , since

$$\begin{aligned} \sqrt{\left\langle (F_i(t, h) - P_{i,N}(t, h))^2 \right\rangle_T} &\leq \\ &\sqrt{\left\langle (F_i(t + h) - P_{i,N}(t + h))^2 \right\rangle_T} + \sqrt{\left\langle (F_i(t) - P_{i,N}(t))^2 \right\rangle_T} \asymp \\ &\sqrt{\left\langle (F_i(t) - P_{i,N}(t))^2 \right\rangle_T}. \end{aligned}$$

We continue by computing the covariance of  $P_{1,N}(t, h)$  and  $P_{2,N}(t, h)$ .

### 3.1.2. Covariance of $P_{1,N}$ and $P_{2,N}$

In this section we calculate  $\langle P_{1,N}(t, h) P_{2,N}(t, h) \rangle_T$  where  $N = N(T)$  is sufficiently small. We introduce the following notation:

$$D(M) = \min_{\substack{0 \leq |n|, |m| \leq M \\ Y_1(n) \neq Y_2(m)}} |Y_1(n) - Y_2(m)|.$$

**Definition 3.2.** Let  $\Omega_1, \Omega_2$  be two convex planar domains with smooth boundary. We say that  $\Omega_1, \Omega_2$  are a  $\kappa$ -Diophantine pair for some  $\kappa > 0$  if  $D(M) \gg M^{-\kappa}$ .

**Lemma 3.3.** Assume that  $\Omega_1, \Omega_2$  are a  $\kappa$ -Diophantine pair. Let  $N(T) = \mathcal{O}\left(T^{\left(\frac{25}{6} + \kappa\right)^{-1}}\right)$ . Under these assumptions

$$\begin{aligned} \langle P_{1,N}(t, h(T)) P_{2,N}(t, h(T)) \rangle_T = \\ \frac{2}{\pi^2} \sum'_{|n| < N} \sum'_{\substack{|m| < N \\ Y_1(n) = Y_2(m)}} \frac{\sqrt{\rho_1(n)}}{|n|^{3/2}} \frac{\sqrt{\rho_2(m)}}{|m|^{3/2}} \sin\left(2\pi Y_1(n) \frac{h}{2}\right)^2 + \mathcal{O}(N^{-1/6}) \end{aligned} \quad (3.1)$$

*Proof.* Let  $T > 0$ . From the linearity of  $\langle F \rangle_T$  as a function of  $F$ :

$$\begin{aligned} \langle P_{1,N}(t, h(T)) P_{2,N}(t, h(T)) \rangle_T = \\ \frac{4}{\pi^2} \sum'_{\substack{n \in \mathbb{Z}^2 \\ |n| < N}} \sum'_{\substack{m \in \mathbb{Z}^2 \\ |m| < N}} \frac{\sqrt{\rho_1(n)}}{|n|^{3/2}} \frac{\sqrt{\rho_2(m)}}{|m|^{3/2}} \sin\left(2\pi Y_1(n) \frac{h}{2}\right) \sin\left(2\pi Y_2(m) \frac{h}{2}\right) \times \\ \left\langle \sin\left(2\pi Y_1(n) \left(t + \frac{h}{2}\right) + \frac{\pi}{4}\right) \sin\left(2\pi Y_2(m) \left(t + \frac{h}{2}\right) + \frac{\pi}{4}\right) \right\rangle_T \end{aligned} \quad (3.2)$$

We can rewrite  $\sin(\alpha)$  as  $\frac{1}{2i} (\exp(i\alpha) - \exp(-i\alpha))$ . We get that

$$\begin{aligned} \left\langle \sin\left(2\pi Y_1(n) \left(t + \frac{h}{2}\right) + \frac{\pi}{4}\right) \sin\left(2\pi Y_2(m) \left(t + \frac{h}{2}\right) + \frac{\pi}{4}\right) \right\rangle_T = \\ \frac{1}{4} \left[ \left\langle \exp(2\pi i(Y_1(n) - Y_2(m))) \right\rangle_T e^{i\pi h(Y_1(n) - Y_2(m))} \right. \\ + \left\langle \exp(2\pi i(Y_2(m) - Y_1(n))) \right\rangle_T e^{i\pi h(Y_2(m) - Y_1(n))} \\ - \left\langle \exp(2\pi i(Y_1(n) + Y_2(m))) \right\rangle_T e^{-i\pi(1/2+h)(Y_1(n) + Y_2(m))} \\ \left. - \left\langle \exp(-2\pi i(Y_1(n) + Y_2(m))) \right\rangle_T e^{i\pi(1/2+h)(Y_1(n) + Y_2(m))} \right] \end{aligned} \quad (3.3)$$

If  $Y_1(n) = Y_2(m)$  we get that

$$\langle \exp(\pm 2\pi i(Y_1(n) - Y_2(m))) \rangle_T = 1.$$

Since both  $n, m$  satisfy  $|n|, |m| \leq N$ , in all other cases we get-

$$|\pm Y_1(n) \pm Y_2(m)| \geq D(N).$$

Since  $\Omega_1, \Omega_2$  are a  $\kappa$ -Diophantine pair, and from our choice of  $N$ , we get that

$$D(N) \gg N^{-\kappa} \gg T^{-\kappa/(\frac{25}{6}+\kappa)}.$$

Thus, since  $|\langle \exp(i\alpha t) \rangle_T| \leq \frac{2}{T\alpha}$ , we get that in these cases

$$|\langle \exp(2\pi i(\pm Y_1(n) \pm Y_2(m))) \rangle_T| \ll T^{\kappa/(\frac{25}{6}+\kappa)-1}$$

And so, since there are  $N^4$  pairs of  $n, m$  we get that-

$$\begin{aligned} \sum'_{\substack{\pm Y_1(n) \neq \pm Y_2(m) \\ |n|, |m| \leq N}} \langle \exp(2\pi i(\pm Y_1(n) \pm Y_2(m))) \rangle_T &\ll N^4 T^{\kappa/(\frac{25}{6}+\kappa)-1} \\ &\ll T^{(4+\kappa)/(\frac{25}{6}+\kappa)-1} = T^{-1/6(\frac{25}{6}+\kappa)} \ll N^{-1/6}. \end{aligned}$$

Thus, from (3.2), (3.3) we get that

$$\begin{aligned} \langle P_{1,N}(t, h(T)) P_{2,N}(t, h(T)) \rangle_T &= \\ \frac{2}{\pi^2} \sum'_{|n| < N} \sum'_{\substack{|m| < N \\ Y_1(n) = Y_2(m)}} \frac{\sqrt{\rho_1(n)}}{|n|^{3/2}} \frac{\sqrt{\rho_2(m)}}{|m|^{3/2}} \sin\left(2\pi Y_1(n) \frac{h}{2}\right)^2 &+ \mathcal{O}(N^{-1/6}) \end{aligned}$$

□

Our main result in this section is the following proposition, giving an expression for

$$\langle P_{1,N}(t, h(T)) P_{2,N}(t, h(T)) \rangle_T$$

which does not depend on  $N$ . There is an implicit dependence, as the formula requires  $T$  to be sufficiently larger than  $N$ , and  $N$  to be sufficiently larger than  $h$ .

**Proposition 3.4.** *Define the function*

$$f(h) = 4 \sum_v \hat{F}_1(v) \overline{\hat{F}_2(v)} \sin^2\left(\frac{h}{2}v\right)$$

where  $v$  goes over common frequencies of  $F_1$  and  $F_2$ .

Under the assumptions of [Lemma 3.3](#) and assuming  $h^{-12}(T) \ll N(T)$ , we have that

$$\langle P_{1,N}(t, h(T)), P_{2,N}(t, h(T)) \rangle_T = f(h(T)) (1 + o(1))$$

In order to prove [Proposition 3.4](#), we will first need the following two Lemmas regarding the function  $f$ -

**Lemma 3.5.** *Let  $F_1(t), F_2(t)$  be the normalized error functions corresponding to convex planar domains with smooth boundary. Assume also that  $F_1, F_2$  have a non-zero global covariance. Define*

$$r(M) = \sum_{|v| > M} \hat{F}_1(v) \overline{\hat{F}_2(v)}.$$

where  $v$  goes over all common frequencies. Then

$$r(M) \ll M^{-1/3}.$$

*Proof.* The function  $r(M)$  is weakly decreasing, since  $\hat{F}_1(v) \overline{\hat{F}_2(v)}$  is always positive. Furthermore, from [Parseval's Identity](#) we know that  $r(M)$  tends to 0 as  $M$  tends to  $\infty$ . Using [Parseval's Identity](#), namely [Corollary 2.6](#), we get that

$$r(M) = \sum_{v > M} \hat{F}_1(v) \overline{\hat{F}_2(v)} = \lim_{T \rightarrow \infty} \left\langle (F_1(t) - P_{1,M}(t)) (F_2(t) - P_{2,M}(t)) \right\rangle_T$$

where we used the fact  $P_{1,M}$  and  $P_{2,M}$  are the truncated frequency expansions of  $F_1$  and  $F_2$  respectively, as seen in [\(2.1\)](#). Using the Cauchy-Schwarz inequality we get that

$$r(M) \leq \lim_{T \rightarrow \infty} \sqrt{\left\langle (F_1(t) - P_{1,M}(t))^2 \right\rangle_T} \sqrt{\left\langle (F_2(t) - P_{2,M}(t))^2 \right\rangle_T}.$$

From [Theorem 2.7](#), we have that

$$\lim_{T \rightarrow \infty} \left\langle (F_i(t) - P_{i,M}(t))^2 \right\rangle_T = \mathcal{O}(M^{-1/3}).$$

Thus we have that  $r(M) \ll M^{-1/3}$ . □

**Lemma 3.6.** *Let  $F_1(t), F_2(t)$  be the normalized error functions corresponding to convex planar domains with smooth boundary. Assume also that  $F_1, F_2$  have a non-zero global covariance. Then*

$$\sum_{|v| \leq h^{-1}} \hat{F}_1(v) \overline{\hat{F}_2(v)} \sin^2\left(\frac{h}{2}v\right) \gg h^2 \log(h^{-1}).$$

*Proof.* We assume that the global covariance of  $F_1$  and  $F_2$  is positive. This means that there exist a common frequency  $v_0$  such that  $\hat{F}_1(v_0) \overline{\hat{F}_2(v_0)} > 0$ . Let  $k \in \mathbb{N}$ , and consider the frequency  $kv_0$ . From the equalities

$$Y_i(kn) = kY(n), \quad \rho_i(kn) = \rho_i(n), \quad |kn|^{3/2} = k^{3/2}|n|^{3/2}$$

we get that  $\hat{F}_1(k\nu_0)\overline{\hat{F}_2(k\nu_0)} \geq \frac{1}{k^3}\hat{F}_1(\nu_0)\overline{\hat{F}_2(\nu_0)}$ . This shows that-

$$\begin{aligned} f(h) &= \sum_{\nu} \hat{F}_1(\nu)\overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right) \gg \sum_{k \leq \nu_0^{-1}h^{-1}} \hat{F}_1(k\nu_0)\overline{\hat{F}_2(k\nu_0)} (hk\nu_0)^2 \\ &\gg h^2 \left( \nu_0^2 \hat{F}_1(\nu_0)\overline{\hat{F}_2(\nu_0)} \right) \left( \sum_{k \leq h^{-1}} \frac{1}{k} \right) \gg h^2 \log(h^{-1}) \end{aligned}$$

□

We can now prove [Proposition 3.4](#):

*Proof of Proposition 3.4.* From [Lemma 3.3](#) we have that

$$\begin{aligned} \langle P_{1,N}(t, h(T)), P_{2,N}(t, h(T)) \rangle_T &= \\ &= \frac{2}{\pi^2} \sum'_{|n| < N} \sum'_{\substack{|m| < N \\ Y_1(n) = Y_2(m)}} \frac{\sqrt{\rho_1(n)}}{|n|^{3/2}} \frac{\sqrt{\rho_2(m)}}{|m|^{3/2}} \sin\left(2\pi Y_1(n) \frac{h}{2}\right)^2 + \mathcal{O}(N^{-1/6}). \end{aligned}$$

We are going to show that  $f(h) \gg h^2 \log(h^{-1})$ , and since  $N > h^{-12}$  the error term  $\mathcal{O}(N^{-1/6})$  is going to be negligible.

We use the fact that  $Y_i(n) \asymp |n|$  and rewrite the RHS as

$$\frac{2}{\pi^2} \sum_{\nu \leq N} \sin^2(\pi \nu h) \left( \sum_{n: Y_1(n) = \nu} \frac{\sqrt{\rho_1(n)}}{|n|^{3/2}} \right) \left( \sum_{m: Y_2(m) = \nu} \frac{\sqrt{\rho_2(m)}}{|m|^{3/2}} \right)$$

where  $\nu$  goes over all common values of  $Y_1(n)$  and  $Y_2(m)$ . Using [\(2.1\)](#), this is equal to

$$4 \sum_{|\nu| \leq 2\pi N} \hat{F}_1(\nu)\overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right).$$

where  $\nu$  goes over all common frequencies.

We now wish to prove that this expression is asymptotically equal to  $f(h)$ . We do this by showing that the terms in  $f$  corresponding to frequencies larger than  $N$  are negligible. And so, it is enough to show that

$$\sum_{|\nu| > N} \hat{F}_1(\nu)\overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right) \ll \sum_{|\nu| \leq h^{-1}} \hat{F}_1(\nu)\overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right).$$

From [Lemma 3.6](#) we have that

$$\sum_{|\nu| \leq h^{-1}} \hat{F}_1(\nu)\overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right) \gg h^2 \log(h^{-1}).$$

Using the bound  $\sin(x) \leq 1$  and from [Lemma 3.5](#) we get

$$\sum_{|\nu| > N} \hat{F}_1(\nu)\overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right) \ll r(N) \ll N^{-1/3}.$$

Since  $N > h^{-12}$  we have  $N^{-1/3} \ll h^2 \log(h^{-1})$  which completes the proof. □

**Corollary 3.7.** *The function  $f(h)$  defined in [Proposition 3.4](#) satisfies  $f(h) \gg h^2 \log(h^{-1})$ . Furthermore, if we assume that all common frequencies of  $F_1, F_2$  are of the form  $kv_0$  for some initial  $v_0$  and  $k \in \mathbb{Z}$ , and we assume that  $2\pi Y_1(n) = 2\pi Y_2(m) = kv_0$  only when  $n = kn_0, m = km_0$  for some solution  $n_0, m_0$  of  $2\pi Y_1(n_0) = 2\pi Y_2(m_0) = v_0$  (i.e. the number of solutions is constant), then:*

$$f(h) = Ch^2 \log(h^{-1}) + \mathcal{O}(h^2)$$

where

$$C = 2v_0^2 \hat{F}_1(v_0) \overline{\hat{F}_2(v_0)}.$$

*Proof.* the bound  $f(h) \gg h^2 \log(h^{-1})$  follows from [Lemma 3.6](#).

As for the second part of the statement, The fact that  $2\pi Y_1(n) = 2\pi Y_2(m) = kv_0$  only when  $n = kn_0, m = km_0$  for some solution  $n_0, m_0$  of  $2\pi Y_1(n_0) = 2\pi Y_2(m_0) = v_0$  implies that

$$\hat{F}_1(kv_0) = \frac{1}{k^3} \hat{F}_1(v_0), \quad \hat{F}_2(kv_0) = \frac{1}{k^3} \hat{F}_2(v_0).$$

We then have that-

$$\begin{aligned} f(h) &= 4 \sum'_{k \in \mathbb{Z}} \hat{F}_1(kv_0) \overline{\hat{F}_2(kv_0)} \sin^2\left(\frac{h}{2} kv_0\right) = \\ &= 8 \hat{F}_1(v_0) \overline{\hat{F}_2(v_0)} \sum_{k \in \mathbb{N}} \frac{1}{k^3} \sin^2\left(\frac{h}{2} kv_0\right) = \\ &= Ch^2 \sum_{k \ll h^{-1}} \frac{1}{k} + \mathcal{O}\left(\sum_{k \gg h^{-1}} \frac{1}{k^3}\right) + \mathcal{O}(h^2) = Ch^2 \log(h^{-1}) + \mathcal{O}(h^2). \end{aligned}$$

□

## 3.2. Covariance of the Error Term in Short Intervals of General Domains

We now compute the covariance of  $F_1(t, h(T)), F_2(t, h(T))$ .

**Theorem 3.8.** *Let  $\Omega_1, \Omega_2$  be convex planar domains with a smooth boundary and positive curvature and let  $F_1(t), F_2(t)$  be their normalized error functions. Assume that the global covariance of  $F_1(t), F_2(t)$  is non-zero. Assume also that  $\Omega_1, \Omega_2$  are a  $\kappa$ -Diophantine pair. Let  $h(T)$  be a function of  $T$  tending to 0 such that  $h(T) \gg T^{-1/(50+12\kappa)}$ . Then-*

$$\text{Cov}(F_1(t, h(T)), F_2(t, h(T))) = f(h)$$

where

$$f(h) = 4 \sum_v \hat{F}_1(v) \overline{\hat{F}_2(v)} \sin^2\left(\frac{h}{2} v\right).$$



*Proof.* Take  $N(T) = h^{-12}(T) \ll T^{1/(\frac{25}{6} + \kappa)}$ . Note that the conditions on  $h$  imply that  $N(T) \ll T^{2/5}$ . From [Proposition 3.4](#) we have

$$\langle P_{1,N}(t, h(T)) P_{2,N}(t, h(T)) \rangle_T = f(h)(1 + o(1)).$$

Thus, in order to prove the theorem it is enough to show that-

$$\lim_{T \rightarrow \infty} \frac{1}{f(h(T))} (\langle F_1(t, h) F_2(t, h) \rangle_T - \langle P_{1,N}(t, h) P_{2,N}(t, h) \rangle_T) = 0.$$

From the equality

$$P_{1,N} P_{2,N} - F_1 F_2 = (F_1 - P_{1,N})(F_2 - P_{2,N}) - (F_1 - P_{1,N}) F_2 - (F_2 - P_{2,N}) F_1.$$

We get-

$$\begin{aligned} \frac{1}{f(h(T))} |\langle F_1(t, h) F_2(t, h) \rangle_T - \langle P_{1,N}(t, h) F_2(t, h) \rangle_T| \leq \\ \frac{1}{f(h(T))} |\langle (F_1 - P_{1,N})(F_2 - P_{2,N}) \rangle_T + \langle (F_1 - P_{1,N}) F_2 \rangle_T + \langle (F_2 - P_{2,N}) F_1 \rangle_T|. \end{aligned}$$

Using the Cauchy–Schwarz inequality we get that

$$\begin{aligned} \frac{1}{f(h(T))} |\langle F_1(t, h) F_2(t, h) \rangle_T - \langle P_{1,N}(t, h) F_2(t, h) \rangle_T| \leq \\ \frac{1}{f(h)} \left( \sqrt{\langle (F_1 - P_{1,N})^2 \rangle_T} \sqrt{\langle (F_2 - P_{2,N})^2 \rangle_T} \right. \\ \left. + \sqrt{\langle (F_1 - P_{1,N})^2 \rangle_T} \sqrt{\langle F_2^2(t, h) \rangle_T} \right. \\ \left. + \sqrt{\langle (F_2 - P_{2,N})^2 \rangle_T} \sqrt{\langle F_1^2(t, h) \rangle_T} \right). \end{aligned} \quad (3.4)$$

From [Theorem 2.7](#) we get that  $\langle (F_i - P_{i,N})^2 \rangle_T = \mathcal{O}(N^{-1/3})$ . As for the terms  $\sqrt{\langle F_i^2(t, h) \rangle_T}$ , using the triangle inequality we get

$$\sqrt{\langle F_i^2(t, h) \rangle_T} = \sqrt{\langle (F_i(t+h) - F_i(t))^2 \rangle_T} \leq \sqrt{\langle F_i^2(t+h) \rangle_T} + \sqrt{\langle F_i^2(t) \rangle_T} \asymp \sqrt{\langle F_i^2(t) \rangle_T}$$

which is bounded in  $T$  since  $F_i$  have finite variance (see [\[3\]](#)).

And so, from [\(3.4\)](#) we get that

$$\begin{aligned} \frac{1}{f(h(T))} (\langle F_1(t, h) F_2(t, h) \rangle_T - \langle P_{1,N}(t, h) F_2(t, h) \rangle_T) = \\ \frac{1}{f(h)} (\mathcal{O}(N^{-1/3}) + \mathcal{O}(N^{-1/6}) + \mathcal{O}(N^{-1/6})). \end{aligned}$$

From [Lemma 3.6](#) we get that  $f(h) \gg h^2 \log(h^{-1})$ . And so,

$$\frac{1}{f(h(T))} (\langle F_1(t, h) F_2(t, h) \rangle_T - \langle P_{1,N}(t, h) F_2(t, h) \rangle_T) = \mathcal{O}\left(\frac{N^{-1/6}}{h^2 \log(h^{-1})}\right) = o(1)$$

where the last equality used the fact that  $N = h^{-12}$ . □

## 4. Covariance of Ellipses in Short Intervals

In this section we consider the covariance in short intervals of normalized error functions of ellipses. In [subsection 4.1](#) we prove a result concerning the covariance of 'generic' ellipses. In [subsection 4.2](#) we look at the covariance of two specific ellipses. Namely,  $x^2 + \alpha y^2 \leq 1$  and  $x^2 + xy + y^2 \leq \frac{3}{4}$  for various  $\alpha$ 's. This example is of particular interest since it related to error terms of the Dirichlet eigenvalue problem for a rectangle and for an equilateral triangle.

### 4.1. Covariance of Generic Ellipses

Our main goal in this section is to prove [Theorem 1.5](#) regarding the covariance of a generic ellipse pair. By this, we mean that the set of ellipses that don't satisfy the desired properties has measure 0. In order to make this statement precise, we will first need to define the measure space that we are referring too. We then proceed by proving that for  $\kappa > 5$  almost all ellipse pairs are  $\kappa$ -Diophantine, which allows us to invoke [Theorem 3.8](#) and get a formula for the covariance in short intervals (with some restriction on the decay rate of the short interval). We also prove that almost all pairs of ellipses satisfy the conditions of [Corollary 3.7](#), proving that for almost all pairs the covariance has order  $h^2 \log(h^{-1})$ .

#### 4.1.1. Definition of the Measure Space

Let

$$\Omega_1 : ax^2 + bxy + c \leq 1, \quad \Omega_2 : dx^2 + exy + fy^2 \leq 1$$

be two ellipses for some parameters  $a, b, c, d, e, f$ .

**Definition 4.1.** We define  $\Gamma$  as the set of ellipse pairs. We identify  $\Gamma$  with a subset of  $\mathbb{R}^6$  (with its standard Lebesgue measure) defined by

$$\Gamma = \{(a, b, c, d, e, f) \in \mathbb{R}^6 \mid 4ac > b^2, 4de > f^2\}$$

where we identify  $(a, b, c, d, e, f)$  with  $(\Omega_1, \Omega_2)$ .

We will however be interested in ellipse pairs that have non-zero global covariance. To this end, for any non-zero  $n = (n_1, n_2), m = (m_1, m_2) \in \mathbb{Z}^2$  we define  $\Gamma_{(n,m)} \subset \Gamma$  as the zero set of

$$(Y_1(n) - Y_2(m))(a, b, c, d, e, f) = \frac{2\sqrt{an_2^2 - bn_1n_2 + cn_1^2}}{\sqrt{4ac - b^2}} - \frac{2\sqrt{dm_2^2 - em_1m_2 + fm_1^2}}{\sqrt{4df - e^2}}$$

where we used [\(2.2\)](#) for the formulas of  $Y_1$  and  $Y_2$ .

**Definition 4.2.** We define the space of ellipse pairs with non-zero global covariance as

$$\Gamma' = \bigcup_{\substack{n, m \in \mathbb{Z}^2 \\ n, m \neq 0}} \Gamma_{(n, m)} \quad (4.1)$$

with the inherited metric from  $\Gamma$ .

#### 4.1.2. Diophantine Property for Ellipse Pairs

**Proposition 4.3.** *For all  $\kappa > 5$ , almost all ellipse pairs that have non-zero global covariance are a  $\kappa$ -Diophantine pair.*

We will first need the following lemma.

Define a function  $\psi : \mathbb{Z}^4 \rightarrow \mathbb{Z}^6$  by

$$\psi(n_1, n_2, m_1, m_2) = (n_2^2, n_1 n_2, n_1^2, m_2^2, m_1 m_2, m_2^2). \quad (4.2)$$

**Definition 4.4.** We say that  $\mathbf{p} \in \mathbb{R}^6$  is V.W.A (very well approximated) if there is some  $\epsilon > 0$  such that there are infinitely many  $(n_1, n_2, m_1, m_2) \in \mathbb{Z}^4$  for which

$$0 < |\mathbf{p} \cdot \psi(n_1, n_2, m_1, m_2)| \leq (\max(n_1, n_2, m_1, m_2))^{-4-\epsilon}.$$

**Lemma 4.5.** *Let  $\mathbf{q} \neq 0 \in \mathbb{Z}^6$  and let  $H = \mathbf{q}^\perp$  be the hyper-plane perpendicular to  $\mathbf{q}$ . Almost all  $\mathbf{p} \in H$  are not V.W.A.*

*Proof.* It is enough to prove the statement for some local compact subset of  $H$ , which we denote  $\mathcal{C}$ . Let  $n = (n_1, n_2), m = (m_1, m_2) \in \mathbb{Z}^2$ . Denote by  $\mathcal{C}_{n, m}$  the set of points  $\mathbf{p} \in \mathcal{C}$  which satisfy

$$0 < |\mathbf{p} \cdot \psi(n, m)| \leq (\max(n_1, n_2, m_1, m_2))^{-4-\epsilon}.$$

If  $\psi(n, m)$  is proportional to  $\mathbf{q}$ , then  $\mathcal{C}_{n, m} = \emptyset$  has measure 0. Otherwise, the points in  $\mathcal{C}_{n, m}$  satisfy

$$0 < \left| \mathbf{p} \cdot \frac{\psi(n, m)}{\|\psi(n, m)\|} \right| \ll \delta$$

where  $\delta = (\max(n_1, n_2, m_1, m_2))^{-6-\epsilon}$ . This means that  $\mathbf{p}$  is in a  $\delta$  neighbourhood of the hyper-plane  $\psi(n, m)^\perp$  in  $\mathbb{R}^6$ . We now show that this also means that  $\mathbf{p}$  is in a small neighbourhood of  $H \cap \psi(n, m)^\perp$  in  $H$ .

Denote by  $\theta$  the angle between  $\mathbf{q}$  and  $\psi(n, m)$ . Since  $\mathbf{q} \cdot \psi(n, m)$  is an integer, and  $\mathbf{q}$  is not proportional to  $\psi(n, m)$  we get that

$$\cos^2(\theta) = \frac{(\mathbf{q} \cdot \psi(n, m))^2}{\|\mathbf{q}\|^2 \|\psi(n, m)\|^2} \leq \frac{\|\mathbf{q}\|^2 \|\psi(n, m)\|^2 - 1}{\|\mathbf{q}\|^2 \|\psi(n, m)\|^2} \leq 1 - \frac{1}{\|\mathbf{q}\|^2 \|\psi(n, m)\|^2}. \quad (4.3)$$

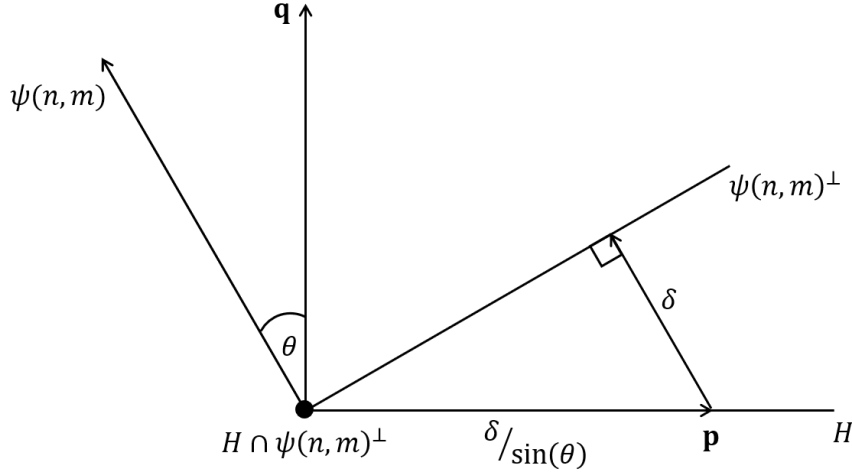


Figure 4.1: A projection on to  $\text{span}(\mathbf{q}, \psi(n, m))$

It can be seen that any point on  $H$  that is in a  $\delta$  neighbourhood of  $\psi(n, m)^\perp$ , is in a  $\delta / \sin(\theta)$  neighbourhood of  $H \cap \psi(n, m)^\perp$  (see Figure 4.1).

Thus, from (4.3) we get that  $\mathbf{p}$  is in a

$$\delta / \sin(\theta) \asymp_{\mathbf{q}} \delta \|\psi(n, m)\| \asymp (\max(n_1, n_2, m_1, m_2))^{-4-\epsilon}$$

neighbourhood of  $H \cap \psi(n, m)^\perp$ .

Since we restricted our attention to a compact subset of  $H$ , it follows that

$$\mu(C_{n,m}) \ll_{\mathcal{C}} (\max(n_1, n_2, m_1, m_2))^{-4-\epsilon}$$

where  $\mu$  is the standard Lebesgue measure on  $H$ . We now have that

$$\sum_{\substack{n,m \in \mathbb{Z}^2 \\ n,m \neq 0}} \mu(C_{n,m}) \ll \sum_{\substack{n,m \in \mathbb{Z}^2 \\ n,m \neq 0}} (\max(n_1, n_2, m_1, m_2))^{-4-\epsilon} < \infty.$$

The statement then follows from the Borel–Cantelli lemma.  $\square$

*Proof of Proposition 4.3.* Denote by  $\mathcal{A}$  the set of points in  $\Gamma'$  which are not a  $\kappa$ -Diophantine pair. We will show that the set  $\mathcal{A}$  has measure zero. Since (4.1) expresses  $\Gamma'$  as a countable union, it is enough to prove the statement for each set in the union. Thus, we set some non-zero  $n^s = (n_1^s, n_2^s), m^s = (m_1^s, m_2^s) \in \mathbb{Z}^2$  and we wish to show that  $\mathcal{A} \cap \Gamma_{(n^s, m^s)}$  has measure 0.

In order for a point to lie in  $\mathcal{A}$ , we must have that for infinitely many  $n = (n_1, n_2), m = (m_1, m_2) \in \mathbb{Z}^2$

$$Y_1(n) \neq Y_2(m), \quad |Y_1(n) - Y_2(m)| \ll_{a,b,c,d,e,f} (\max(n_1, n_2, m_1, m_2))^{-\kappa}.$$

Since  $Y_1(n) \asymp_{a,b,c,d,e,f} |n|$ , we get that this is equivalent to having infinitely many  $n, m \in \mathbb{Z}^2$  with

$$Y_1^2(n) - Y_2^2(m) = (Y_1(n) - Y_2(m))(Y_1(n) + Y_2(m)) \ll_{a,b,c,d,e,f} (\max\{n_1, n_2, m_1, m_2\})^{-\kappa+1}. \quad (4.4)$$

We define the functions  $A, B, C, D, E, F$  of  $a, b, c, d, e, f$  by

$$A = \frac{a}{4ac - b^2}, \quad B = \frac{-b}{4ac - b^2}, \quad C = \frac{c}{4ac - b^2}, \\ D = \frac{-d}{4df - e^2}, \quad E = \frac{e}{4df - e^2}, \quad F = \frac{-f}{4df - e^2}.$$

With these notations we have that

$$Y_1^2(n) - Y_2^2(m) = (A, B, C, D, E, F) \cdot \psi(n, m).$$

Consider the function  $\Phi : \Gamma \rightarrow \mathbb{R}^6$  defined by  $\Phi : (a, b, c, d, e, f) \mapsto (A, B, C, D, E, F)$ . It is nowhere degenerate since

$$\left| \frac{\partial(A, B, C, D, E, F)}{\partial(a, b, c, d, e, f)} \right| = \frac{-1}{(4ac - b^2)^3 (4df - e^2)^3} < 0.$$

The restriction of  $\Phi$  to  $\Gamma_{(n^s, m^s)}$  takes elements from the zero set of  $Y_1^2(n^s) - Y_2^2(m^s)$  to the hyper-plane

$$H : \{(A, B, C, D, E, F) \mid (A, B, C, D, E, F) \cdot \psi(n^s, m^s) = 0\}.$$

From [Lemma 4.5](#) we get that almost all points  $(A, B, C, D, E, F) \in H$  are not V.W.A. Thus, since  $\Phi$  is nowhere degenerate, it follows that for almost all  $(a, b, c, d, e, f) \in \Gamma_{(n^s, m^s)}$ , the corresponding  $(A, B, C, D, E, F) \in H$  is not V.W.A. Since  $\kappa > 5$ , this means that the set of  $(a, b, c, d, e, f) \in \Gamma_{(n^s, m^s)}$  that satisfy (4.4) for infinitely many  $(n, m)$  has measure 0. It follows that almost all ellipse pairs  $(a, b, c, d, e, f) \in \Gamma_{(n^s, m^s)}$  are a  $\kappa$ -Diophantine pair.  $\square$

#### 4.1.3. Common Frequencies of Generic Ellipses

In this section we prove a result regarding common frequencies of generic ellipses that have non-zero global covariance. We will use the same notations we used in the proof of [Proposition 4.3](#).

**Proposition 4.6.** *Almost all ellipse pairs in  $\Gamma'$  satisfy the following: The only  $n, m \in \mathbb{Z}^2$  for which  $Y_1(n) = Y_2(m)$  are  $n = kn_0, m = \pm km_0$  for  $k \in \mathbb{Z}$  and some initial  $n_0, m_0 \in \mathbb{Z}^2$ .*

We will first note the following-

**Lemma 4.7.** *Let  $\psi$  be the function defined in (4.2). Then  $\psi(n_1, n_2, m_1, m_2)$  is proportional to  $\psi(\ell_1, \ell_2, k_1, k_2)$  if and only if  $(n_1, n_2, m_1, m_2)$  is proportional to either  $(\ell_1, \ell_2, k_1, k_2)$  or to  $(\ell_1, \ell_2, -k_1, -k_2)$ .*

*Proof.* This follows from the fact that  $(x, y) \mapsto (y^2, xy, x^2)$  is invertible up to  $\pm$  sign.  $\square$

*Proof of Proposition 4.6.* As in the proof of Proposition 4.3 it is enough to prove this for the case of  $\Gamma_{(n^s, m^s)}$  for some  $n^s, m^s \in \mathbb{Z}^2$ . We wish to show that for almost all  $(a, b, c, d, e, f) \in \Gamma_{(n^s, m^s)}$ , the only  $n, m \in \mathbb{Z}^2$  for which  $Y_1(n) = Y_2(m)$  are integer multiples of  $n^s, m^s$  or of  $n^s, -m^s$ . In other words, we wish to show that

$$\mathcal{A} = \bigcup_{\substack{n, m \in \mathbb{Z}^2 \\ n, m \neq kn_0, \pm km_0}} \{F_{(n, m)}(a, b, c, d, e, f) = 0\} \quad (4.5)$$

has measure 0 in  $\Gamma_{(n^s, m^s)}$  where

$$\begin{aligned} F_{(n, m)}(a, b, c, d, e, f) = Y_1^2(n) - Y_2^2(m) = \\ \frac{a}{4ac - b^2} n_2^2 - \frac{b}{4ac - b^2} n_1 n_2 + \frac{c}{4ac - b^2} n_1^2 \\ - \frac{d}{4df - e^2} m_2^2 + \frac{e}{4df - e^2} m_1 m_2 - \frac{f}{4df - e^2} m_1^2. \end{aligned}$$

The expression (4.5) expresses  $\mathcal{A}$  as a countable union, so once more it is enough to show that

$$\{F_{(n, m)}(a, b, c, d, e, f) = 0\} \cap \Gamma_{(n^s, m^s)}$$

has measure 0 for some specific  $n, m$  which is not a multiple of  $(n^s, \pm m^s)$ . We once again use the map  $\Phi$  defined in Proposition 4.3. The map  $\Phi$  takes  $\{F_{(n, m)}(a, b, c, d, e, f) = 0\}$  to the hyper-plane  $H_1$  perpendicular to  $\psi(n, m)$ . However, since we are restricting our attention to  $\Gamma_{(n^s, m^s)}$ , the image is also contained in the hyper-plane  $H_2$  perpendicular to  $\psi(n^s, m^s)$ . Since  $n, m$  is not a multiple of  $n^s, \pm m^s$ , we get from Lemma 4.7 that  $H_1$  and  $H_2$  are not equal. This means that  $H_1 \cap H_2$  has co-dimension 1 in  $H_2$ . From the non-degeneracy of  $\Phi$ , this means that  $\{F_{(n, m)}(a, b, c, d, e, f) = 0\} \cap \Gamma_{(n^s, m^s)}$  is a sub-manifold of co-dimension 1 in  $\Gamma_{(n^s, m^s)}$ . Thus, it has measure 0 as required.  $\square$

#### 4.1.4. Covariance of a Generic Ellipse Pair

We now prove the main result of this section

*Proof of Theorem 1.5.* Let  $\epsilon > 0$ . From Proposition 4.3 we have that almost all ellipse pairs  $(\Omega_1, \Omega_2) \in \Gamma'$  are a  $(5 + \epsilon)$ -Diophantine pair. Thus, from Theorem 3.8 we get if  $h(T) \gg T^{-1/110+\epsilon'}$ , then

$$\text{Cov}(F_1(t, h), F_2(t, h)) = 4 \sum_{\nu} \hat{F}_1(\nu) \overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right).$$

Denote by  $\nu_0$  the smallest positive common frequency of  $F_1$  and  $F_2$ . From Proposition 4.6 we get that almost all ellipse pairs satisfy the conditions of Corollary 3.7. Namely, we get that if  $2\pi Y_1(\pm n_0) = 2\pi Y_2(\pm m_0) = \nu_0$ , then the only solutions  $n, m$  to  $2\pi Y_1(n) = 2\pi Y_2(m) = k\nu_0$  are  $n = \pm kn_0, m = \pm km_0$ . Thus, from Corollary 3.7 we have that

$$4 \sum_{\nu} \hat{F}_1(\nu) \overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right) = Ch^2 \log(h^{-1}) + \mathcal{O}(h^2)$$

where  $C$  is given by  $2\nu_0^2 \hat{F}_1(\nu_0) \overline{\hat{F}_2(\nu_0)}$ . From (2.1), (2.2), (2.4) we get that

$$C = \frac{16\pi}{\nu_0 \sqrt{4ac - b^2} \sqrt{4df - e^2}}.$$

□

## 4.2. Covariance Between a Rectangle and an Equilateral Triangle

We now consider the covariance between the normalized error functions  $F_1, F_2$  of the following two ellipses:

$$\Omega_1 : x^2 + xy + y^2 \leq \frac{3}{4}, \quad \Omega_2 : x^2 + \alpha y^2 \leq 1.$$

These normalized error functions are related to the Dirichlet eigenvalue error functions corresponding to an equilateral triangle of side length  $\frac{2\pi}{\sqrt{3}}$ , and a rectangle with side lengths  $\pi$  and  $\sqrt{\alpha}\pi$ . The covariance can behave differently depending on  $\alpha$ . Our main result from this section is Theorem 1.6. We begin by examining the 'generic' case, and then proceed to explore the case of rational  $\alpha$ .

### 4.2.1. The Generic Case

**Lemma 4.8.** *For any  $\kappa > 3$ , for almost all  $\alpha \in \mathbb{R}$ ,  $\Omega_1, \Omega_2$  is a  $\kappa$ -Diophantine pair.*

*Proof.* Assume that  $\alpha \notin \mathbb{Q}$ . Denote  $K = (\kappa - 1)/2$ . Since  $K > 1$ , it is known that almost all  $\alpha$  satisfy

$$|k - \ell\alpha| \gg \frac{1}{\ell^K} \tag{4.6}$$

for almost all  $k, \ell \in \mathbb{Z}$ . We show that if  $\alpha$  is such a number, then  $\Omega_1, \Omega_2$  is a  $\kappa$ -Diophantine pair.

From (2.2) we get that for  $n = (n_1, n_2), m = (m_1, m_2) \in \mathbb{Z}^2$ :

$$Y_1(n) = \sqrt{n_1^2 - n_1 n_2 + n_2^2}, \quad Y_2(m) = \frac{1}{\sqrt{\alpha}} \sqrt{\alpha m_1^2 + m_2^2}.$$

If  $n_1, n_2, m_1, m_2 \leq M$  then we have that-

$$Y_1(n) - Y_2(m) = \frac{1}{Y_1(n) + Y_2(m)} (n_1^2 - n_1 n_2 + n_2^2 - m_2^2 - \alpha m_1^2) \gg \frac{1}{M} (n_1^2 - n_1 n_2 + n_2^2 - m_2^2 - \alpha m_1^2). \quad (4.7)$$

We wish to prove that if  $Y_1(n) \neq Y_2(m)$  then  $|Y_1(n) - Y_2(m)| \gg M^{-\kappa}$ . Since  $n_1^2 - n_1 n_2 + n_2^2 - m_2^2$  and  $m_1^2$  are integers of size at most  $M^2$ , and since we assume that  $\alpha$  satisfies (4.6), it follows that

$$n_1^2 - n_1 n_2 + n_2^2 - m_2^2 - \alpha m_1^2 \gg (M^2)^{-K} = M^{-2K}.$$

From (4.7) we get that

$$|Y_1(n) - Y_2(m)| \gg M^{-(2K+1)} = M^{-\kappa}$$

which implies that  $(\Omega_1, \Omega_2)$  is a  $\kappa$ -Diophantine pair.  $\square$

*Proof of Theorem 1.6, Part 1.* From Lemma 4.8 we know that almost all  $\alpha \notin \mathbb{Q}$  will result in  $(\Omega_1, \Omega_2)$  being a  $\kappa$ -Diophantine pair for any  $\kappa > 3$ . Let  $\alpha$  be such a number, and let  $h(T) \gg T^{-1/86+\epsilon}$  for some  $\epsilon > 0$ . We can find  $\kappa$  sufficiently close to 3 such that

$$h(T) \gg T^{-1/86+\epsilon} \gg T^{-(50+12\kappa)^{-1}}.$$

Thus, from Theorem 3.8 we get that

$$\text{Cov}(F_1(t, h), F_2(t, h)) = 4 \sum_{\nu} \hat{F}_1(\nu) \overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right).$$

Since  $\alpha \notin \mathbb{Q}$ , we get that the only common frequencies of  $F_1$  and  $F_2$  are of the form  $2\pi k, k \in \mathbb{Z}$ .

From (2.1), (2.4) we get that for  $k \in \mathbb{N}$ :

$$\hat{F}_1(\pm 2\pi k) = e^{\mp 3\pi i/4} \frac{\sqrt{8\pi}}{\frac{4}{\sqrt{3}}} \frac{r_1(2\pi k)}{(2\pi k)^{3/2}} = e^{\mp 3\pi i/4} \frac{\sqrt{3}}{4\pi} \frac{r_1(2\pi k)}{k^{3/2}}$$

where

$$r_1(\nu) = \#\left\{n \in \mathbb{Z}^2 \mid 2\pi \sqrt{n_1^2 + n_1 n_2 + n_2^2} = \nu\right\}.$$

Similarly, noting that there are exactly two solutions  $m \in \mathbb{Z}^2$  to  $2\pi Y_2(m) = 2\pi k$ , we get that



$$\hat{F}_2(\pm 2\pi k) = e^{\mp 3\pi i/4} \frac{\sqrt{8\pi}}{2\sqrt{\alpha}} \frac{2}{(2\pi k)^{3/2}} = e^{\mp 3\pi i/4} \frac{1}{\pi\sqrt{\alpha}k^{3/2}}$$

Thus, the covariance is given by

$$\text{Cov}(F_1(t, h), F_2(t, h)) = 4 \sum_{\nu} \hat{F}_1(\nu) \overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2}\nu\right) = \frac{2\sqrt{3}}{\pi^2\sqrt{\alpha}} \sum_{k \in \mathbb{N}} \frac{r_1(2\pi k)}{k^3} \sin^2(\pi h k). \quad (4.8)$$

We can further estimate this sum by looking at the first  $h^{-1}$  terms and the rest of the terms separately. We will use the fact that

$$\sum_{k \leq N} r_1(2\pi k) = \frac{3\sqrt{3}}{\pi} N \log N + \mathcal{O}(N)$$

which we prove in [Theorem A.1](#). For the terms with  $k \gg h^{-1}$  we use the inequality  $\sin(x) \leq 1$  and summation by parts to get

$$\sum_{k \gg h^{-1}} \frac{r_1(2\pi k)}{k^3} \sin^2(\pi h k) = \mathcal{O}(h^2 \log(h^{-1})). \quad (4.9)$$

As for the first  $h^{-1}$  terms, using the approximation  $\sin(x) = x + \mathcal{O}(x^2)$ , and once more using summation by parts, we get

$$\begin{aligned} \sum_{k \ll h^{-1}} \frac{r_1(2\pi k)}{k^3} \sin^2(\pi h k) &= 3\sqrt{3}\pi h^2 \sum_{k \ll h^{-1}} \frac{\log(k)}{k} + \mathcal{O}(h^2 \log(h^{-1})) = \\ &= \frac{3\sqrt{3}\pi}{2} h^2 \log^2(h^{-1}) + \mathcal{O}(h^2 \log(h^{-1})). \end{aligned} \quad (4.10)$$

From (4.8), (4.9), (4.10) we get that

$$\text{Cov}(F_1(t, h), F_2(t, h)) = \frac{9}{\pi\sqrt{\alpha}} h^2 \log^2(h^{-1}).$$

□

#### 4.2.2. The Case of Rational $\alpha$

Assume now that  $\alpha \in \mathbb{Q}$  and denote  $\alpha = p/q$  where  $p, q$  are co-prime.

**Lemma 4.9.** *The pair  $(\Omega_1, \Omega_2)$  is a 1-Diophantine pair.*

*Proof.* Let  $n, m \in \mathbb{Z}^2$ ,  $0 < |n|, |m| < M$ . From (4.7) we get that if  $Y_1(n) \neq Y_2(m)$  then

$$|Y_1(n) - Y_2(m)| \gg \frac{1}{M} |n_1^2 - n_1 n_2 + n_2^2 - m_2^2 - \alpha m_1^2|.$$

Writing  $\alpha = p/q$ , we get that  $|Y_1(n) - Y_2(m)| \gg_q \frac{1}{M}$ .

□

*Proof of Theorem 1.6, Parts 2,3.* Let  $\alpha = \frac{p}{q} \in \mathbb{Q}$ . From Lemma 4.9 we have that  $\Omega_1, \Omega_2$  are a 1-Diophantine pair. Thus, from Theorem 3.8, if  $h(T) \gg T^{-1/62}$  then

$$\text{Cov}(F_1(t, h), F_2(t, h)) = 4 \sum_{\nu} \hat{F}_1(\nu) \overline{\hat{F}_2(\nu)} \sin^2\left(\frac{h}{2} \nu\right).$$

We denote

$$r_2(\nu) = \# \left\{ (n_1, n_2) \in \mathbb{Z}^2 \mid 2\pi \sqrt{m_1^2 + \frac{q}{p} m_2^2} = \nu \right\}.$$

Note that the common frequencies of  $F_1$  and  $F_2$  are supported on the set  $\{\pm 2\pi\sqrt{\ell} : \ell \in \mathbb{N}\}$ , and so we get that

$$\text{Cov}(F_1(t, h), F_2(t, h)) = \frac{\sqrt{3}}{\pi^2 \sqrt{\alpha}} \sum_{\ell \in \mathbb{N}} \frac{r_1(2\pi\sqrt{\ell}) r_2(2\pi\sqrt{\ell})}{\ell^{3/2}} \sin^2(\pi h \sqrt{\ell}). \quad (4.11)$$

In the appendix (Theorems A.2, A.3) we show that

$$\sum_{\ell \leq N} r_1(2\pi\sqrt{\ell}) r_2(2\pi\sqrt{\ell}) = \begin{cases} \frac{3\sqrt{3}}{\sqrt{p'q}} N \log N + \mathcal{O}(N) & \text{if } 3pq \text{ is a square} \\ CN + \mathcal{O}(N^{3/4+\epsilon}) & \text{otherwise} \end{cases} \quad (4.12)$$

where  $p'$  is the square free part of  $p$ , and  $C$  is given in Theorem A.3.

We now use (4.11), (4.12) and summation by parts. We denote  $u(x) = \frac{\sin^2(\pi\sqrt{x})}{x^{3/2}}$ . We consider the case of  $3pq$  not a square. In this case we get

$$\begin{aligned} \text{Cov}(F_1(t, h), F_2(t, h)) &= \frac{\sqrt{3}}{\pi^2 \sqrt{\alpha}} \sum_{\ell \in \mathbb{N}} \frac{r_1(2\pi\sqrt{\ell}) r_2(2\pi\sqrt{\ell})}{\ell^{3/2}} \sin^2(\pi h \sqrt{\ell}) = \\ &= \frac{\sqrt{3}}{\pi^2 \sqrt{\alpha}} \sum_{\ell \in \mathbb{N}} h^5 (C\ell + \mathcal{O}(\ell^{3/4+\epsilon})) \frac{u((\ell+1)h^2) - u(\ell h^2)}{h^2}. \end{aligned}$$

As  $h$  tends to 0, using the definitions of the derivative and of the Riemann sum, we get that

$$\text{Cov}(F_1(t, h), F_2(t, h)) = Ch \frac{\sqrt{3}}{\pi^2 \sqrt{\alpha}} \int_0^\infty x u'(x) dx + \mathcal{O}(h^{5/4-\epsilon}).$$

Using integration by parts and the variable change  $y = \sqrt{x}$ , it follows that

$$\text{Cov}(F_1(t, h), F_2(t, h)) = 2Ch \frac{\sqrt{3}}{\pi^2 \sqrt{\alpha}} \int_0^\infty \frac{\sin^2(\pi y)}{y^2} dy = 2Ch \frac{\sqrt{3}}{\pi^2 \sqrt{\alpha}} \frac{\pi^2}{2} = \frac{C\sqrt{3}}{\sqrt{\alpha}} h.$$

Similar arguments show that in the case where  $3pq$  is a square:

$$\text{Cov}(F_1(t, h), F_2(t, h)) = \frac{18}{\sqrt{pp'}} h \log(h^{-1})$$

where  $p'$  is the square free part of  $p$  (which is either 3 or 1).

□

## A. Appendix

We denote

$$r_{\omega}(k) = \#\{n_1, n_2 \in \mathbb{Z} \mid n_1^2 - n_1 n_2 + n_2^2 = k\}$$

$$r_{a,b}(k) = \#\left\{n_1, n_2 \in \mathbb{Z} \mid n_1^2 + \frac{a}{b} n_2^2 = k\right\}.$$

In this section we prove several results regarding these functions that were used in [Theorem 1.6](#). We start by stating the main theorems.

**Theorem A.1.**

$$\sum_{k \leq N} r_{\omega}(k^2) = \frac{3\sqrt{3}}{\pi} N \log N + \mathcal{O}(N).$$

**Theorem A.2.** *Assume that  $3ab$  is an integer square. Then*

$$\sum_{k \leq N} r_{\omega}(k) r_{a,b}(k) = \frac{3\sqrt{3}}{\sqrt{ab'}} N \log N + \mathcal{O}(N)$$

where  $b'$  is the square free part of  $b$ .

**Theorem A.3.** *Assume that  $3ab$  is not an integer square. Then*

$$\sum_{k \leq N} r_{\omega}(k) r_{a,b}(k) = CN + \mathcal{O}(N^{3/4+\epsilon})$$

for any  $\epsilon > 0$ .

The constant  $C$  is given by

$$C = \left( \frac{2\pi^2}{\sqrt{3ab'}} \right) \left( \frac{L(1, \chi)}{L(2, \chi)} \right) \sigma_2 \prod_{\substack{p^r \parallel 3ab' \\ p \geq 3, r \geq 2}} \sigma_p$$

where-

- $b'$  is the square free part of  $b$ .
- $\chi$  is the Kronecker symbol  $\chi(*) = \left( \frac{12ab'}{*} \right)$ .
- The value of  $\sigma_2$  is given in [Lemma A.9](#).
- The value of  $\sigma_p$  for  $p \geq 3$  which divides  $3ab'$  with multiplicity is given in lemmas [A.6](#), [A.7](#), [A.11](#).

We begin with a proof of [Theorem A.1](#).

*Proof.* Let  $\omega = e^{2\pi i/3}$  and let  $\mathbb{Z}[\omega]$  be the Eisenstein integers. Denote also  $U = \{\pm 1, \pm\omega, \pm\omega^2\}$  the units of  $\mathbb{Z}[\omega]$ . Then  $r_\omega(k^2)$  is the number of elements in  $\mathbb{Z}[\omega]$  with norm  $k^2$ . Denote  $a(k) = \frac{1}{6}r_\omega(k^2)$ . Then  $a(k)$  counts the number of Eisenstein integers of norm  $k^2$  up to multiplication by a unit in  $U$ . From unique factorization in  $\mathbb{Z}[\omega]$  we have that  $a(k)$  is a multiplicative function. From properties of primes in  $\mathbb{Z}[\omega]$  we get that  $a(p^e)$  for a prime power  $p^e$  is:

$$a(p^e) = \begin{cases} 2e+1 & p \equiv 1 \pmod{3} \\ 1 & p \equiv 2 \pmod{3} \\ 1 & p = 3. \end{cases} \quad (\text{A.1})$$

By the Euler product expansion, it can be seen that the Dirichlet series of  $a(k)$  is given by

$$D_a(s) = \sum_{n \in \mathbb{N}} \frac{a(n)}{n^s} = \zeta^2(s) \frac{L(s, \chi_3)}{\zeta(2s)(1-3^{-s})}.$$

From [17, Chapter II.5, Theorem 3] we get that

$$\sum_{n \leq N} a(n) = CN \log N + \mathcal{O}(N)$$

with

$$C = \frac{L(1, \chi_3)}{\zeta(2)(1-\frac{1}{3})} = \frac{\sqrt{3}}{2\pi}.$$

The result follows after noting that  $r_\omega(k) = 6a(k)$ . □

We now present the proofs of [Theorem A.2](#) and [Theorem A.3](#). These use results regarding the singular integral and singular series, which we prove in lemmas [A.4](#) through [A.11](#).

*Proof of Theorem A.2.* The sum

$$S_N = \sum_{k \leq N} r_\omega(k) r_{a,b}(k)$$

is equal to the number of solutions  $x, y, z, t \in \mathbb{Z}^4$ ,  $G(x, y, z, t) = 0$  to the quadratic form

$$G(x, y, z, t) = x^2 - xy + y^2 - z^2 - \frac{a}{b}t^2.$$

which satisfy  $x^2 - xy + y^2 \leq N$ . Denote by  $b'$  the square free part of  $b$ . Then any solution to  $G(x, y, z, t) = 0$  must have  $(b' \sqrt{\frac{b}{b'}}) \mid t$ . Writing  $t = (b' \sqrt{\frac{b}{b'}}) w$  we get that counting solutions to  $G = 0$  is equivalent to counting solutions  $x, y, z, w \in \mathbb{Z}^4$  to  $F(x, y, z, w) = 0$  where  $F$  is the quadratic form

$$F(x, y, z, w) = x^2 - xy + y^2 - z^2 - ab'w^2.$$

The determinant of the form  $F$  is  $12ab'$ . If we assume that  $3ab'$  is a square, then from [9, Theorem 7] we get that

$$S_N = CN \log N + \mathcal{O}(N).$$

Furthermore, the constant  $C$  is given by

$$C = \frac{1}{2} \sigma_\infty \prod_p \left( \left( 1 - \frac{1}{p} \right) \sigma_p \right) \quad (\text{A.2})$$

where  $\sigma_\infty$  is the singular integral, and  $\sigma_p$  are the  $p$ -adic densities. Since  $3a$  is a square, from lemmas A.5, A.6, A.9, A.11, we get that  $\sigma_p = 1 + \frac{1}{p}$  for all  $p \neq 3$ , and  $\sigma_3 = 2$ . Also, from Lemma A.4 we get that  $\sigma_\infty = \frac{2\pi^2}{\sqrt{3ab'}}$ . Plugging these into (A.2), we get that

$$C = \frac{1}{2} \left( \frac{2\pi^2}{\sqrt{3ab'}} \right) \left( \frac{6}{\pi^2} \right) \left( \frac{2}{4/3} \right) = \frac{3\sqrt{3}}{\sqrt{ab'}}.$$

□

*Proof of Theorem A.3.* Once more, the sum

$$S_N = \sum_{k \leq N} r_w(k) r_{a,b}(k)$$

is equivalent to the number of solutions  $x, y, z, w \in \mathbb{Z}^4$ ,  $F(x, y, z, w) = 0$  to the quadratic form

$$F(x, y, z, w) = x^2 - xy + y^2 - z^2 - ab'w^2$$

where  $b'$  is the square free part of  $b$ . Assuming  $3ab'$  is not a square, we get from [9, Theorem 6] that  $S_N = CN + \mathcal{O}(N^{3/4+\epsilon})$  for any  $\epsilon > 0$ . Furthermore, we get a formula for the constant  $C$ . Denote by  $\chi$  the Kronecker symbol  $\chi(*) = \left( \frac{\det(F)}{*} \right)$  where  $\det(F) = 12ab'$ . Then

$$C = \sigma_\infty L(1, \chi) \prod_p (1 - \chi(p) p^{-1}) \sigma_p.$$

From lemmas A.5, A.7 A.11 we get that

$$\sigma_p = \begin{cases} 1 + \frac{1}{p} & \chi(p) = 1 \\ \left( 1 + \frac{1}{p^2} \right) \left( 1 + \frac{1}{p} \right) & \chi(p) = -1 \\ 1 & p \neq 2 \quad p \mid 3a'b \text{ and } p^2 \nmid 3a'b. \end{cases}$$

The cases of  $p = 2$  and  $p^r \mid 3ab'$  for some  $r > 1$  are handled in lemmas A.6, A.7, A.9, A.11. It follows that

$$\prod_p (1 - \chi(p) p^{-1}) \sigma_p = L(2, \chi)^{-1} \prod_{\substack{p^r \parallel 12ab' \\ r \geq 2}} \sigma_p.$$

where the values of  $\sigma_p$  for  $p = 2$  and  $p \geq 3$  with  $p^r \parallel 3ab'$  for some  $r \geq 2$  are explicitly given in lemmas A.6, A.7, A.9, A.11.

As for the singular integral, from Lemma A.4 we get  $\sigma_\infty = \frac{2\pi^2}{\sqrt{3ab'}}$ . Thus, overall we get that

$$C = \left( \frac{2\pi^2}{\sqrt{3ab'}} \right) \left( \frac{L(1, \chi)}{L(2, \chi)} \right) \sigma_2 \prod_{\substack{p^r \parallel 3ab' \\ p \geq 3, r \geq 2}} \sigma_p.$$

For example, if  $3ab'$  is square free and  $ab' \equiv 1, 5 \pmod{8}$ , then we get that

$$C = \left( \frac{2\pi^2}{\sqrt{3a'b}} \right) \left( \frac{L(1, \chi)}{L(2, \chi)} \right).$$

□

We now turn to the calculations of the singular integral and  $p$ -adic densities. Throughout, we denote

$$F(x, y, z, w) = x^2 - xy + y^2 - z^2 - ab'w^2$$

where  $\gcd(a, b') = 1$  and  $b'$  is square free. We will at times denote  $\alpha = ab'$ .

**Lemma A.4.** *Let*

$$R(x, y, z, w) = \begin{cases} 1 & x^2 - xy + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Then the singular integral  $\sigma_\infty$ , which is given by*

$$\sigma_\infty = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon} \int_{|F(x, y, z, w)| < \epsilon} R(x, y, z, w) dx dy dz dw,$$

*has value*

$$\sigma_\infty = \frac{2\pi^2}{\sqrt{3\alpha}}.$$

*Proof.* We first consider the following coordinate change:

$$(x, y, z, w) \mapsto (x', y', z', w') = \left( \frac{x-2y}{2}, \frac{\sqrt{3}}{2}x, z, \sqrt{\alpha}w \right)$$

with Jacobian

$$\frac{\partial(x', y', z', w')}{\partial(x, y, z, w)} = \frac{\sqrt{3\alpha}}{2}.$$

Note that

$$F(x', y', z', w') = x'^2 + y'^2 - z'^2 - w'^2.$$

We then have

$$\sigma_\infty = \frac{2}{\sqrt{3\alpha}} \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \text{Vol}(V_\epsilon)$$

where

$$V_\epsilon = \left\{ (x', y', z', w') \left| \left| x'^2 + y'^2 - z'^2 - w'^2 \right| \leq \epsilon, \quad x'^2 + y'^2 \leq 1 \right\}.$$

We now switch to polar coordinates in both  $x'y'$ -plane and  $z'w'$ -plane:

$$(x', y') \mapsto (r, \theta),$$

$$(z', w') \mapsto (\rho, \psi).$$

We then get

$$\begin{aligned} \frac{1}{2\epsilon} \text{Vol}(V_\epsilon) &= \frac{1}{\epsilon} \int_{\theta=0}^{2\pi} d\theta \int_{\psi=0}^{2\pi} d\psi \int_{r=0}^{\sqrt{1-\epsilon}} r \int_{\rho=r}^{\sqrt{r^2+\epsilon}} \rho d\rho dr = \\ &= \frac{4\pi^2}{\epsilon} \left( \frac{\rho^2}{2} \right) \Big|_r^{\sqrt{r^2+\epsilon}} \left( \frac{r^2}{2} \right) \Big|_0^{\sqrt{1-\epsilon}} \xrightarrow{\epsilon \rightarrow 0} \pi^2. \end{aligned}$$

It follows that

$$\sigma_\infty = \frac{2\pi^2}{\sqrt{3\alpha}}.$$

□

We now proceed to compute the  $p$ -adic densities:

$$\sigma_p = \lim_{k \rightarrow \infty} \frac{1}{p^{3k}} \# \left\{ (x, y, z, w) \pmod{p^k} \mid F(x, y, z, w) = 0 \pmod{p^k} \right\}.$$

We first introduce some notations. Let  $p$  be a prime. Assume that  $p^i \mid \alpha$  and denote  $\alpha' = \frac{ab'}{p^i}$ .

We denote

$$N_k^r = \# \left\{ (x, y, z, w) \pmod{p^k} \mid x^2 - xy + y^2 - z^2 - p^r \alpha' w^2 = 0 \pmod{p^k} \right\}.$$

We also denote  $N_k = N_k^0$ , and we define  $N_0 = 1$ .

**Lemma A.5.** *Let  $p$  be a prime,  $p \neq 2, 3$ ,  $p \nmid \alpha$ . Then*

$$\sigma_p = \begin{cases} 1 + \frac{1}{p} & 3\alpha \text{ is a square in } \mathbb{F}_p \\ \left(1 + \frac{1}{p^2}\right) \left(1 + \frac{1}{p}\right)^{-1} & \text{otherwise.} \end{cases}$$

*Proof.* Since  $p \neq 2, 3$ , and

$$x^2 - xy + y^2 = \frac{1}{4} ((x - 2y)^2 + 3x^2),$$

we can consider the equivalent form  $x^2 + 3y^2 - z^2 - \alpha w^2$ .

We first find the number of solutions mod  $p$ . We can consider an equivalent form

$$F'(x, y, z, h) : (x + h)^2 + 3y^2 - h^2 - \alpha w^2$$

which can be rewritten as

$$2xh + x^2 + 3y^2 - \alpha w^2.$$

For any value of  $x, y, w \pmod p$  for which  $x \neq 0$  we can pick  $h$  accordingly to solve the equation. This gives us  $(p-1)p^2$  solutions.

If however we assume  $x = 0 \pmod p$ , then (after picking arbitrary  $h$  which has  $p$  options) our equation becomes

$$3y^2 = \alpha w^2 \pmod p. \quad (\text{A.3})$$

If  $3\alpha$  is a square mod  $p$ , then this has  $2p-1$  solutions. Otherwise, there is only the single 0 solution. And so, we get an additional  $p(2p-1)$  solutions if  $3\alpha$  is a square mod  $p$ , and only an additional  $p$  solutions otherwise.

Thus, the number of non-zero solutions mod  $p$  satisfies

$$N_1 - 1 = \begin{cases} p^3 + p^2 - p - 1 & 3\alpha \text{ is a square} \pmod p \\ p^3 - p^2 + p - 1 & \text{otherwise.} \end{cases}$$

We now use Hensel's lemma to calculate the number of solutions mod  $p^k$ . We say that a solution mod  $p^k$  is non-degenerate if it is not the zero solution mod  $p$ . From Hensel's lemma, every non-degenerate solution can be lifted from  $p^i$  to  $p^{i+1}$  in  $p^3$  ways. This means that we have  $p^{3(k-1)}(N_1 - 1)$  non-degenerate solutions mod  $p^k$ .

For degenerate solution mod  $p^k$ , we can divide each variable by  $p$ . This gives us a solution mod  $p^{k-2}$ , with the most significant  $p$ -adic digit of each variable chosen arbitrarily. It follows that the number of degenerate solutions mod  $p^k$  is  $p^4 N_{k-2}$ .

Thus we have the recursion

$$\frac{N_k}{p^{3k}} = \frac{N_1 - 1}{p^3} + \frac{1}{p^2} \left( \frac{N_{k-2}}{p^{3(k-2)}} \right).$$

This recursion can be solved to give

$$\lim_{k \rightarrow \infty} \frac{N_k}{p^{3k}} = \left( 1 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3} \right) \left( 1 - \frac{1}{p^2} \right)^{-1} = 1 + \frac{1}{p}$$

if  $3\alpha$  is a square mod  $p$ , and

$$\lim_{k \rightarrow \infty} \frac{N_k}{p^{3k}} = \left( 1 - \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3} \right) \left( 1 - \frac{1}{p^2} \right)^{-1} = \left( 1 + \frac{1}{p^2} \right) \left( 1 + \frac{1}{p} \right)^{-1}$$

otherwise. □



**Lemma A.6.** Let  $p \neq 2, 3$ . Assume that  $p^r \mid \alpha$ , where  $r > 0$  is even and  $p^{r+1} \nmid \alpha$ . Then

$$\sigma_p = \begin{cases} 1 + \frac{1}{p} & \frac{3\alpha}{p^r} \text{ is a square mod } p \\ 1 + \frac{1}{p} - \frac{2}{p^{r/2}(p+1)} & \text{otherwise.} \end{cases}$$

*Proof.* We consider the equation

$$x^2 - xy + y^2 - z^2 - p^r \alpha w^2 = 0 \pmod{p^k}. \quad (\text{A.4})$$

Modulo  $p$ , this equation become:

$$x^2 - xy + y^2 - z^2 = 0 \pmod{p} \quad (\text{A.5})$$

(and  $w$  can be chosen arbitrarily). We will call a solution mod  $p$  good if either  $x, y$  or  $z$  is inevitable. From Hensel's lemma, these good solutions can be lifted to a solution of (A.4) mod  $p^k$  in  $p^{3(k-1)}$  ways. It can be shown that the number of good solutions to (A.5) is  $p^2 - 1$  (times  $p$  options for  $w$ ). So the number of solutions mod  $p^k$  which come from a good solution is  $(p^3 - p)p^{3(k-1)}$ .

If  $(x_0, y_0, z_0, w_0)$  is a solution that doesn't come from a good solution, then we can divide  $x_0, y_0, z_0$  by  $p$  and the coefficient of  $w_0$  by  $p^2$  and get a solution mod  $p^{k-2}$ . We can then pick the most significant p-adic digit of  $x_0, y_0, z_0$  arbitrarily, and the two most significant p-adic digit of  $w_0$ . Thus, the number of non-good solutions is  $p^5 N_{k-2}^{r-2}$ .

Overall, we get the following recursion:

$$\frac{N_k^r}{p^{3k}} = \left(1 - \frac{1}{p^2}\right) + \frac{1}{p} \left(\frac{N_{k-2}^{r-2}}{p^{3(k-2)}}\right). \quad (\text{A.6})$$

By repeating (A.6)  $r/2$  times we get

$$\begin{aligned} \frac{N_k^r}{p^{3k}} &= \left(1 - \frac{1}{p^2}\right) \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^{r/2-1}}\right) + \frac{1}{p^{r/2}} \left(\frac{N_{k-r}}{p^{3(k-r)}}\right) = \\ &= \left(1 - \frac{1}{p^2}\right) \frac{1 - \frac{1}{p^{r/2}}}{1 - \frac{1}{p}} + \frac{1}{p^{r/2}} \left(\frac{N_{k-r}}{p^{3(k-r)}}\right) = \left(1 + \frac{1}{p}\right) + \frac{1}{p^{r/2}} \left(\frac{N_{k-r}}{p^{3(k-r)}} - 1 - \frac{1}{p}\right). \end{aligned} \quad (\text{A.7})$$

From Lemma A.5, we have

$$\lim_{k \rightarrow \infty} \frac{N_{k-r}}{p^{3(k-r)}} = \begin{cases} 1 + \frac{1}{p} & 3\alpha \text{ is a square mod } p \\ \left(1 + \frac{1}{p^2}\right) \left(1 + \frac{1}{p}\right)^{-1} & \text{otherwise.} \end{cases}$$

Plugging this into (A.7) we get

$$\lim_{k \rightarrow \infty} \frac{N_k^r}{p^{3k}} = \begin{cases} 1 + \frac{1}{p} & 3\alpha \text{ is a square mod } p \\ 1 + \frac{1}{p} - \frac{2}{p^{r/2}(p+1)} & \text{otherwise.} \end{cases}$$

□

**Lemma A.7.** Let  $p \neq 2, 3$ . Assume that  $p^r \mid \alpha$ ,  $p^{r+1} \nmid \alpha$  where  $r$  is odd. Then

$$\sigma_p = 1 + \frac{1}{p} - \frac{1}{p^{(r+1)/2}}.$$

*Proof.* We start with the case  $r = 1$ . That is, we show that  $\frac{N_k^1}{p^{3k}} \rightarrow 1$ .

For this, we first find the non-degenerate solutions mod  $p$ . Our equation mod  $p$  is

$$x^2 + 3y^2 - z^2 = 0 \pmod{p}.$$

There are then  $p(p^2 - 1)$  solutions  $x_0, y_0, z_0, w_0$  with at least one of  $x_0, y_0$ , or  $z_0$  not zero. From Hensel's lemma, these give us  $p(p^2 - 1)p^{3(k-1)}$  solution mod  $p^k$ .

We now consider the rest of the solutions. Let  $x_0, y_0, z_0, w_0$  be a solution mod  $p^k$  such that  $x_0, y_0$  and  $z_0$  are 0 mod  $p$ . Assume also that  $k > 1$ . Then since

$$p\alpha w_0^2 = x_0^2 + 3y_0^2 - z_0^2 \pmod{p^k}$$

We must have that  $p \mid w_0$  as well (since the RHS is divisible by  $p^2$ ). Dividing  $x_0, y_0, z_0, w_0$  by  $p$  will then give us a solution mod  $p^{k-2}$ .

This gives us the relation

$$\frac{N_k^1}{p^{3k}} = \left(1 - \frac{1}{p^2}\right) + \frac{1}{p^2} \left(\frac{N_{k-2}^1}{p^{3(k-2)}}\right).$$

From this we can deduce that

$$\lim_{k \rightarrow \infty} \frac{N_k^1}{p^{3k}} = \left(1 - \frac{1}{p^2}\right) \left(\frac{1}{1 - \frac{1}{p^2}}\right) = 1.$$

We now consider the case  $r > 1$ . It can be shown in the same way we did in [Lemma A.6](#) that

$$\frac{N_k^r}{p^{3k}} = \left(1 - \frac{1}{p^2}\right) + \frac{1}{p} \left(\frac{N_{k-2}^{r-2}}{p^{3(k-2)}}\right).$$

Repeating this recursion  $(r-1)/2$  times gives us

$$\begin{aligned} \frac{N_k^r}{p^{3k}} &= \left(1 - \frac{1}{p^2}\right) \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^{(r-3)/2-1}}\right) + \frac{1}{p^{(r-1)/2}} \left(\frac{N_{k-r+1}^1}{p^{3(k-r+1)}}\right) = \\ &\left(1 - \frac{1}{p^2}\right) \frac{1 - \frac{1}{p^{(r-1)/2}}}{1 - \frac{1}{p}} + \frac{1}{p^{(r-1)/2}} \left(\frac{N_{k-r+1}^1}{p^{3(k-r+1)}}\right) = \left(1 + \frac{1}{p}\right) + \frac{1}{p^{(r-1)/2}} \left(\frac{N_{k-r+1}^1}{p^{3(k-r+1)}} - 1 - \frac{1}{p}\right). \end{aligned}$$

Using the fact that

$$\frac{N_{k-r+1}^1}{p^{3(k-r+1)}} \xrightarrow{k \rightarrow \infty} 1$$

gives us

$$\frac{N_k^r}{p^{3k}} \xrightarrow{k \rightarrow \infty} 1 + \frac{1}{p} - \frac{1}{p^{(r+1)/2}}.$$

□

**Lemma A.8.** Assume that  $4 \nmid \alpha$ , then

$$\sigma_2 = \begin{cases} 1.5 & \alpha \equiv 3 \pmod{8} \\ 7/6 & \alpha \equiv 7 \pmod{8} \\ 1 & \alpha \equiv 1, 2, 5, 6 \pmod{8}. \end{cases}$$

*Proof.* The difference from previous case is that we can only use Hensel's lemma starting at  $2^3$ . Thus, every non-degenerate solution mod  $2^3$  can be lifted to a non-degenerate solution mod  $2^k$  in  $2^{3(k-3)}$  ways. This is also true if  $2 \mid \alpha$ ,  $4 \nmid \alpha$  since then every non-degenerate solution mod  $2^3$  has to have one of  $x, y$  or  $z$  be inevitable.

Denote by  $n_3$  the number of non-degenerate solutions mod  $2^3$ . In a similar way to previous lemmas, we get the recursion-

$$\frac{N_k}{2^{3k}} = \frac{n_3}{8^3} + \frac{1}{4} \left( \frac{N_{k-2}}{2^{3(k-2)}} \right).$$

This implies

$$\frac{N_k}{2^{3k}} \xrightarrow{k \rightarrow \infty} \frac{4}{3} \left( \frac{n_3}{8^3} \right).$$

It remains to find the value of  $\frac{n_3}{8^3}$ . This can be done by direct computation:

$$\frac{n_3}{8^3} = \begin{cases} 1.125 & \alpha \equiv 3 \pmod{8} \\ 0.875 & \alpha \equiv 7 \pmod{8} \\ 0.75 & \alpha \equiv 1, 2, 5, 6 \pmod{8}. \end{cases}$$

The result then follows. □

**Lemma A.9.** If  $4^s \mid \alpha$ ,  $4^{s+1} \nmid \alpha$ , then

$$\sigma_2 = 1.5 + \frac{1}{2^s} (\sigma'_2 - 1.5)$$

where  $\sigma'_2$  is the value of  $\sigma_2$  from [Lemma A.8](#) for  $\frac{\alpha}{2^{2s}} \pmod{8}$ .

*Remark.* If  $3\alpha$  is a square in the integers, then  $\frac{\alpha}{2^{2s}} \equiv 3 \pmod{8}$ . This means that  $\sigma'_2 = 1.5$ . Thus, for this case we get  $\sigma_2 = 1.5$ .

*Proof.* In similar ways to previous lemmas, we get the recursion

$$\frac{N_k^r}{2^k} = \frac{n_3}{8^3} + \frac{1}{2} \left( \frac{N_{k-2}^{r-2}}{2^{3(k-2)}} \right). \quad (\text{A.8})$$

The result then follows in a similar way to [Lemma A.7](#), by repeating (A.8)  $s = \lfloor r/2 \rfloor$  times and then using [Lemma A.8](#).  $\square$

**Lemma A.10.** Assume that  $3^2 \nmid \alpha$ . Then-

$$\sigma_3 = \begin{cases} 1 & 3 \nmid \alpha \\ 2 & \frac{\alpha}{3} \equiv 1 \pmod{3} \\ 1.5 & \frac{\alpha}{3} \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* Mod  $3^k$ , the quadratic form  $F$  is equivalent to

$$x^2 + 3y^2 - z^2 - \alpha w^2.$$

Assuming that  $3^2 \nmid \alpha$ , we can use Hensel's starting at  $3^3$ . Denote by  $n$  the number of non-degenerate solutions mod  $3^3$ . We can get the recursion

$$\frac{N_k}{3^{3k}} = \frac{n}{3^3} + \frac{1}{p^2} \left( \frac{N_{k-2}}{3^{3(k-2)}} \right).$$

From this we get that

$$\lim_{k \rightarrow \infty} \frac{N_k}{3^{3k}} = \frac{9}{8} \left( \frac{n}{3^3} \right).$$

From direct computation we get that

$$\frac{9}{8} \left( \frac{n}{3^3} \right) = \begin{cases} 1 & 3 \nmid \alpha \\ 2 & 3 \mid \alpha, \quad \frac{\alpha}{3} \equiv 1 \pmod{3} \\ 1.5 & 3 \mid \alpha, \quad \frac{\alpha}{3} \equiv 2 \pmod{3} \end{cases}$$

$\square$

**Lemma A.11.** Assume  $9^s \mid \alpha$ ,  $9^{s+1} \nmid \alpha$  for some  $s \geq 0$ , then

$$\sigma_3 = 2 - \frac{1}{3^s} (2 - \sigma'_3)$$

where  $\sigma'_3$  is the value from [Lemma A.10](#) corresponding to  $\frac{\alpha}{9^s}$ .

*Proof.* We have the equation

$$x^2 + 3y^2 - z^2 - \alpha w^2 = 0 \pmod{3^k} \quad (\text{A.9})$$

for some  $k \geq 3$ . We will call a solution  $x_0, y_0, z_0, w_0$  of (A.9) good if at least one of  $x_0, y_0$  or  $z_0$  is invertible. From Hensel's lemma, every good solution mod  $3^3$  can be lifted in  $3^{3(k-3)}$  ways to a solution mod  $p^k$ . From direct calculation we get that the number  $n$  of good solutions mod  $3^3$  satisfies  $\frac{n}{3^3} = \frac{4}{3}$ .

Every non-good solution comes from a solution in  $N_{k-2}^{r-2}$  in  $p^5$  ways. This gives us the recursion

$$\frac{N_k^r}{3^k} = \frac{4}{3} + \frac{1}{3} \left( \frac{N_{k-2}^{r-2}}{3^{3(k-2)}} \right).$$

Repeating this recursion  $s = \lfloor r/2 \rfloor$  times and using Lemma A.10 we get

$$\sigma_3 = 2 - \frac{1}{3^s} (2 - \sigma'_3)$$

where  $\sigma'_3$  is the value from Lemma A.10 corresponding to  $\frac{\alpha}{9^s}$ .

□

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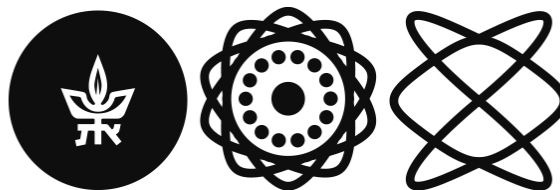
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# תקציר

בעבודה זו נבחן את השונות המשותפת של פונקציות שגיאה שמגיעות מההשערת Weyl לגבי ספירת כמות הערכים העצמיים של בעיית דיריכלה עד גודל  $X$ . נבחן את הבעיה הזו גם במקרה של קטעים קצרים, משמע נסתכל על השגיאה שמגיעה מספירת העי"ע בקטע  $[X, X + S]$  עבור איזושהי פונקציה  $S(X)$ . אנו נתבונן בפונקציות שגיאה אלו עבור תחומים מישוריים שבהם העי"ע של בעיית דיריכלה ניתנים ע"י נוסחה מפורשת. במקרים האלה, פונקציית השגיאה קשורה לפונקציית השגיאה מהבעיה הקלאסית של ספירת נקודות שריג בתוך תחומים מישוריים מתרחבים. אנחנו נמצא נוסחה עבור השונות המשותפת של פונקציות שגיאה אלו עבור תחומים מישוריים כלליים. בנוסף, נימצא נוסחה עבור שונות משותפת בקטעים קצרים, עבור קטעים מספיק גדולים. בהקשר של בעיית העי"ע של דיריכלה, אנחנו נשתמש בכלים הני"ל כדי לחשב את השונות המשותפת עבור השגיאות שמגיעות משני מלבנים גנרים. נבחן גם דוגמה ספציפית - אנחנו נחשב את השונות המשותפת בין השגיאות שמגיעות ממשולש שווה צלעות ומגוון מלבנים.

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