Abstract

We study the zeros of L-functions over the rational function field. We consider the number $N(\beta,\chi)$ of zeros of an L-function $L(u,\chi)$ in an angular interval $[-\beta,\beta]$, where χ is an even primitive character modulo some monic polynomial Q, as a random variable by picking χ randomly. Our main interest is the limiting distribution of $N(\beta,\chi)$ as $\deg Q \to \infty$. We consider both the macroscopic scale (fixed β) and the mesoscopic scale ($\beta \to 0$ but $\beta \deg Q \to \infty$). We show that the fluctuating term has a gaussian limiting distribution as $\deg Q \to \infty$, when scaled by its standard deviation, which is calculated to be asymptotic to $\sqrt{\log(\beta \deg Q)}$, and obtain a bound on the fluctuations of $N(\beta,\chi)$ for fixed Q. Then we turn to the family of quadratic characters, and show that it, too, has gaussian limiting distribution of the fluctuating term in the zeros counting function.

Contents

1 Introduction

Let $N(t)$ be the number of zeros of the Riemann zeta function up to height t, and set $S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right)$. The Riemann - von Mangoldt formula asserts:

$$N(t) = \frac{1}{2\pi} t \log \frac{t}{2\pi e} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right)$$

so that $S(t)$ measures the fluctuations of $N(t)$ around its mean value. A well-known result of Selberg [Se1][Se2] states, that choosing t uniformly from $[0, T]$ induces standard gaussian distribution of

$$\frac{S(t)}{\sqrt{\frac{1}{2\pi^2} \log \log T}}$$

in the limit as $T \to \infty$.

A similar result was proved by Selberg [Se3] for the fluctuations in the number of zeros of Dirichlet L-functions up to a fixed height, for different primitive characters modulo k, as $k \to \infty$.

We will be concerned with L-functions associated with the polynomial ring over a finite field. Our goal is to prove a result fully analogous to that of Selberg, and then a similar result for the family of quadratic characters.
We work with the finite field $\mathbf{F}_q$, where q is a prime power, and we let $Q \in \mathbf{F}_q[x]$ be a monic polynomial.

A primitive Dirichlet character χ modulo Q is a group homomorphism $\chi : (\mathbf{F}_q[x]/Q\mathbf{F}_q[x])^* \to \mathbf{C}^*$, which is not "induced" by a character modulo some proper divisor of Q. Assume χ is primitive and even (i.e. equals 1 on constant polynomials). The associated L-function $L(u, \chi) = \sum_n u^{\deg n} = \prod_{p \text{ prime}} \left(1 - \chi(p) u^{\deg p}\right)^{-1}$ is a polynomial of degree $\deg Q - 1$, which has $\deg Q - 2$ roots of absolute value $\frac{1}{\sqrt{q}}$ and one trivial root which equals 1.

We write the number of non-trivial zeros of $L(u, \chi)$ in the angular interval $[-\beta, \beta]$ as:

$$\pi N(\beta, \chi) = \beta(\deg Q - 2) + 2\arg\left(1 - \frac{1}{\sqrt{q}} e^{i\beta}\right) + S(\beta, \chi)$$

$S(\beta, \chi)$ is treated as the fluctuating term of the counting function. The reason for this is explained in section 2 by a simple application of the argument principle.

Our first main result is a bound on the size of $S(\beta, \chi)$:

Theorem 1.

$$S(\beta, \chi) = O\left(\frac{\deg Q}{\log \deg Q}\right)$$

as q is fixed and $\deg Q \to \infty$.

The theorem is in complete agreement with the bound $S(T,\chi) = O\left(\frac{\log k}{\log\log k}\right)$ as T is fixed, and the conductor $k \to \infty$, obtained by Selberg [Se3] under GRH. It implies

Corollary 1. *The multiplicity of a non-trivial zero of $L(u,\chi)$ is $O\left(\frac{\deg Q}{\log \deg Q}\right)$.*

Then we investigate the fluctuations of $S(\beta,\chi)$ as χ varies.
We study at the same time both the macroscopic scale (fixed β) and the mesoscopic scale ($\beta \to 0$ but $\beta \deg Q \to \infty$).

Our goal is to prove the two theorems:

Theorem 2. *Let Q be a prime polynomial, $\deg Q = D \to \infty$. Let χ be a random non-principal character modulo Q.*
(1) Fix $0 < \beta < \pi$. The sequence of random variables

$$\frac{S(\beta,\chi)}{\sqrt{\log D}}$$

has a standard gaussian limiting distribution.
(2) Let $\beta = \beta(D)$ s.t. $0 \neq \beta \to 0$ and $\beta D \to \infty$. Then the sequence of r.v.

$$\frac{S(\beta,\chi)}{\sqrt{\log(\beta D)}}$$

has a standard gaussian limiting distribution.

Theorem 3. *Let $D \to \infty$ assume even values. Let χ_Q be the quadratic character modulo a random square-free monic polynomial Q of degree D.*
(1) Fix $0 < \beta < \pi$. The sequence of random variables

$$\frac{S(\beta,\chi_Q)}{\sqrt{2\log D}}$$

has a standard gaussian limiting distribution.
(2) Let $\beta = \beta(D)$ s.t. $0 \neq \beta \to 0$ and $\beta D \to \infty$. Then the sequence of r.v.

$$\frac{S(\beta,\chi_Q)}{\sqrt{2\log(\beta D)}}$$

has a standard gaussian limiting distribution.

The paper is organized as follows: In section 2 we review the basic definitions and facts concerning L-functions over rational function fields, and introduce the notation which will later be used.

In section 3 we sketch and motive the derivation of the approximation formula for $S(\beta,\chi)$, which has the form

$$S(\beta,\chi) = -\sum_{\deg n \le 2m} \frac{\Lambda(n) u_0^{\deg n} \sin(\beta \deg n) W_m(\deg n)}{\deg n} \left(\chi(n) + \overline{\chi(n)}\right) + Error$$

where $u_0 = q^{-1/2}e^{-1/m}$, and W_m is some particular function (precise definition is given later) which decreases from 1 to 0 between m and $2m$. Then in subsection 3.2 we give a full and detailed proof of this approximation formula. We obtain, as an intermediate result, theorem 1 and corollary 1.

Afterwards in section 4 we first calculate the moments of $T_m(\beta, \chi)$, which is defined as a sum over prime polynomials p :

$$T_m(\beta, \chi) = \sum_{\deg p \leq 2m} \frac{\sin(\beta \deg p)}{\sqrt{q}^{\deg p}}(\chi(p) + \overline{\chi}(p))$$

and then show that $T_m(\beta, \chi)$ is a good approximation of $S(\beta, \chi)$ for a suitable choice of m, and therefore they possess the same moments. This will imply that, in the limit $\deg Q \to \infty$, $S(\beta, \chi)$ has gaussian moments, and hence a gaussian value distribution, with variance asymptotic to $\log(\beta \deg Q)$.

Finally in section 5 we investigate a similar problem in a quadratic characters setting: instead of fixing Q and picking a random non-principal character modulo Q, we fix a degree D and take the quadratic character corresponding to a randomly chosen square-free polynomial Q of degree D. Then we study the fluctuations in the number of zeros of the corresponding L-function in the angular interval $[-\beta, \beta]$ as Q varies. In this case, too, a limiting gaussian distribution occurs.

2 Background

A complete survey of number theory over function fields can be found in [Ro1].

Let q be a power of a prime number. A monic irreducible polynomial will be called a prime polynomial. Choose a monic polynomial $Q \in \mathbf{F}_q[x]$ of degree D, and examine the ring $\mathbf{F}_q[x]/(Q\mathbf{F}_q[x])$.

A Dirichlet character modulo Q is a homomorphism

$$\chi : (\mathbf{F}_q[x]/(Q\mathbf{F}_q[x]))^* \to \mathbf{C}^*$$

We often will think of χ as acting on $\mathbf{F}_q[x]$, by extending it to equal 0 on polynomials not coprime to Q.

The number of different Dirichlet characters equals $\Phi(Q) = \# \left(\mathbf{F}_q[x]/(Q\mathbf{F}_q[x])\right)^*$.

The orthogonality relations state that

$$\sum_{\chi} \chi(n) = \begin{cases} \Phi(Q) & , n \equiv 1 \text{ mod} Q \\ 0 & , \text{otherwise} \end{cases}$$

$$\sum_{n} \chi(n) = \begin{cases} \Phi(Q) & , \chi \equiv 1 \\ 0 & , \text{otherwise} \end{cases}$$

An "even" Dirichlet character χ is a character s.t. $\chi(c) = 1$ for all constant polynomials $0 \neq c \in \mathbf{F}_q[x]$. The total number of even characters equals

$$\Phi_e(Q) = \left|(\mathbf{F}_q[x]/Q\mathbf{F}_q[x])^*/\mathbf{F}_q^*\right| = \frac{\Phi(Q)}{\#\mathbf{F}_q^*} = \frac{\Phi(Q)}{q-1}$$

The orthogonality relation for even characters reads

$$\sum_{\chi \text{ even}} \chi(n) = \begin{cases} \Phi_e(Q) & , n \equiv const \bmod Q \\ 0 & , \text{otherwise} \end{cases}$$

A character χ modulo Q is called non-primitive (and primitive otherwise) if it is induced by some character $\widehat{\chi}$ modulo a divisor $\widehat{Q}$ of Q, i.e.

$$\chi(n) = \begin{cases} \widehat{\chi}(n), & \gcd(n, Q) = 1 \\ 0, & \text{otherwise} \end{cases}$$

The L-function $L(u, \chi)$ associated with χ is defined as $L(u, \chi) = \sum_n \chi(n) u^{\deg n}$, the sum taken over monic polynomials. For a primitive character χ, $L(u, \chi)$ is actually a polynomial of degree $D - 1$. We denote the roots of $L(u, \chi)$ by $\eta_j(\chi) = \alpha_j(\chi)^{-1}$, $j = 1, ..., D - 1$. For even χ, $\eta_{D-1}(\chi) = 1$ is always a root of $L(u, \chi)$.
The Euler product formula states that for $|u| < \frac{1}{q}$

$$L(u, \chi) = \prod_{p \text{ prime}} \left(1 - \chi(p) u^{\deg p}\right)^{-1}$$

and the product converges absolutely and locally uniformly.
Taking logarithmic derivative of this yields

$$\frac{L'}{L}(u, \chi) = \sum_n \Lambda(n) \chi(n) u^{\deg n - 1}$$

where

$$\Lambda(n) = \begin{cases} \deg p & , n = p^k, p \text{ prime} \\ 0 & , \text{otherwise} \end{cases}$$

By a theorem of Weil [We1], for a primitive character χ, all zeros of $L(u, \chi)$ have absolute value $q^{-1/2}$, except for one zero which equals 1 when χ is even.

From now on, we will be concerned only with even characters. Denote the roots of $L(u, \chi)$ by $\eta_j(\chi) = \frac{1}{\sqrt{q}} e^{i\phi_j}$, $j = 1, ..., D - 2$ and $\eta_{D-1}(\chi) = 1$.
Let $N(\beta, \chi)$ be the number of $\eta_j \neq 1$ with argument lying between $-\beta$ and β, where zeros of argument equal to $\pm\beta$ count one-half only.

Define $\xi(u, \chi) = \frac{u^{1 - \frac{\deg Q}{2}}}{1 - u} L(u, \chi)$. Then [We2] ξ satisfies the functional equation

$$\xi(u, \chi) = c_\chi \xi(\frac{1}{qu}, \overline{\chi}), \ |c_\chi| = 1$$

By the argument principle,

$$2\pi N(\beta, \chi) = \Delta_\Gamma \arg \xi = \Im \int_\Gamma \frac{\xi'}{\xi}(u) du$$

where Γ is the curve comprised of the two circular arcs of radii R, r obeying $r < q^{-1/2} < R$ and $rR = \frac{1}{q}$, and the straight radial intervals of arguments β and $-\beta$, traversed counter-clockwise. Now define

$$S(\beta, \chi) = \Im \int_{\gamma} \frac{L'}{L}(u, \chi) du$$

where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ is defined in figure 1 below.

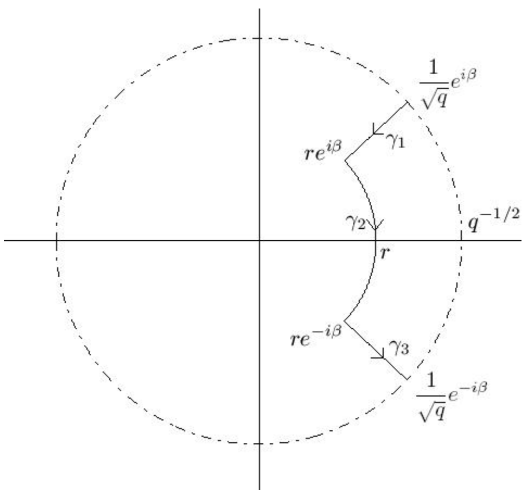

Figure 1: Integration curve γ

If some zeros have argument $\pm\beta$, define

$$S(\beta, \chi) = \lim_{\delta \to 0} \frac{S(\beta + \delta, \chi) + S(\beta - \delta, \chi)}{2} \tag{2.1}$$

By the functional equation, we can write

$$\pi N(\beta, \chi) = \Im \int_{\gamma} \frac{\xi'}{\xi}(u) du = \beta(D - 2) + 2\text{arg}\left(1 - \frac{1}{\sqrt{q}}e^{i\beta}\right) + S(\beta, \chi)$$

In the entire paper, sums over polynomials are understood as sums over monic polynomials.

3 The Approximation Formula

3.1 A Sketch of The Derivation

Here we present the ideas which lead to the approximation formula for $S(\beta, \chi)$, which will then be rigorously proved in subsection 3.1.

Write $S(\beta,\chi) = -A(\beta,\chi) + A(-\beta,\chi)$, where

$$A(\beta,\chi) = \Im \int_0^{\frac{1}{\sqrt{q}}e^{i\beta}} \frac{L'}{L}(u)du$$

The approximation formula can then be written in the following way:

$$A(\beta,\chi) = \sum_{\deg n \le 2m} \frac{W_m(\deg n)\Lambda(n)u_0^{\deg n}e^{i\beta\deg n}\chi(n)}{\deg n} +$$

$$+O\left(\frac{D}{m} + \frac{1}{m}\left|\sum_n \Lambda(n)\chi(n)u_0^{\deg n-1}e^{i\beta(\deg n-1)}W_m(\deg n)\right|\right)$$

We take a point $u_0e^{i\beta}$ near $\frac{1}{\sqrt{q}}e^{i\beta}$ where $u_0 = \frac{1}{\sqrt{q}}e^{-1/m}$ ($m = m(D)$ approaches infinity), and try to approximate $A(\beta,\chi)$ by truncating the integral at that point:

$$A(\beta,\chi) = J_1 + J_2 + J_3 \tag{3.1}$$

$$J_1 = \Im \int_0^{u_0e^{i\beta}} \frac{L'}{L}(u)du$$

$$J_2 = \Im\left((1/\sqrt{q} - u_0)e^{i\beta}\frac{L'}{L}(u_0e^{i\beta})\right)$$

$$J_3 = \Im \int_{u_0e^{i\beta}}^{\frac{1}{\sqrt{q}}e^{i\beta}} \frac{L'}{L}(u)du$$

Both J_2 and J_3 should be error terms. From

$$\frac{L'}{L}(u) = \sum \frac{1}{u-\eta_j}$$

we get by an elementary calculation (which is done in part 3.2 - see (3.13)),

$$J_3 \ll \frac{1}{m^2}\sum_j \frac{1}{|u_0e^{i\beta} - \eta_j|^2}$$

Next we need to approximate the logarithmic derivative of $L(u,\chi)$ on $[0, u_0e^{i\beta}]$. Start by writing for $|u| < 1/q$ the absolutely convergent series

$$\frac{L'}{L}(u,\chi) = \sum_n \Lambda(n)\chi(n)u^{\deg n-1}$$

We integrate $\frac{L'}{L}\left(\frac{u}{z}\right)$ against some function $f_m(z)$ over a circle of radius $R > q|u|$:

$$\frac{1}{2\pi i}\int_{|z|=R} \frac{L'}{L}(\frac{u}{z})f_m(z)dz = \sum_n \Lambda(n)\chi(n)u^{\deg n-1}\frac{1}{2\pi i}\int_{|z|=R}\frac{f_m(z)}{z^{\deg n-1}}dz$$

and require f_m to have a simple pole at $z = 1$. We will denote

$$M_m(k) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f_m(z)}{z^{k-1}} dz$$

and

$$A_m = -Res(f_m, 1)$$

The value of A_m can be arbitrarily chosen; we prefer to leave it intact for now. Since

$$Res\left(\frac{L'}{L}\left(\frac{u}{z}\right), z = u\alpha_j\right) = -u\alpha_j^2$$

by Cauchy's residues theorem

$$-A_m \frac{L'}{L}(u) - \sum_j u\alpha_j^2 f_m(u\alpha_j) = \sum_n \Lambda(n)\chi(n)u^{\deg n - 1} M_m(\deg n)$$

For the approximation formula to be finite, $M_m(k)$ has to be zero for large k. If we expand f_m into Taylor series around 0: $f_m(z) = \sum a_j z^j$, we get

$$M_m(k) = a_{k-2} - A_m$$

In particular, all a_k are equal for $k \geq d(m)$, which implies

$$f_m(z) = P_{d-1}(z) + \frac{a_d}{1-z} = \frac{B_d(z)}{1-z}$$

where $B_d = b_0 + ... + b_d z^d$ is a polynomial of degree $d = d(m)$, and $A_m = B_d(1)$. It holds that $a_k = b_0 + ... + b_k$, and $b_k = a_k - a_{k-1}$.

The approximation now reads

$$\frac{L'}{L}(u) = \sum_n \Lambda(n)\chi(n)u^{\deg n - 1} W_m(\deg n) - \frac{1}{A_m} \sum_j u\alpha_j^2 \frac{B_d(u\alpha_j)}{1 - u\alpha_j} \tag{3.2}$$

where

$$W_m(k) = -\frac{M_m(k)}{A_m} = 1 - a_{k-2}/A_m$$

We will treat the truncated series as the main term, and the sum over zeros as the remainder term.

Returning to (3.1),

$$J_2 \ll \frac{1}{m} \left| \frac{L'}{L}(u_0 e^{i\beta}) \right| \ll$$

$$\ll \frac{1}{m} \left| \sum_n \Lambda(n)\chi(n)u_0^{\deg n - 1} e^{i\beta(\deg n - 1)} W_m(\deg n) \right| + \frac{1}{m|A_m|} \left| \sum_j u_0 e^{i\beta} \alpha_j^2 \frac{B_d(u_0 e^{i\beta}\alpha_j)}{1 - u_0 e^{i\beta}\alpha_j} \right|$$

The first term is what is supposed to estimate $\frac{L'}{L}(u_0 e^{i\beta})$, so only a good choice of $m = m(D)$ will make it small. The second term we require to be same order of magnitude as the bound for J_3:

$$\frac{1}{m|A_m|}\left|\sum_j u_0 e^{i\beta}\alpha_j^2 \frac{B_d(u_0 e^{i\beta}\alpha_j)}{1-u_0 e^{i\beta}\alpha_j}\right| \ll \frac{1}{m^2}\sum_j \frac{1}{|u_0 e^{i\beta}-\eta_j|^2}$$

This would certainly be satisfied if we had

$$|b_0| + \sum_{k=1}^{d+1} |b_k - b_{k-1}| e^{-k/m} \ll \frac{|A_m|}{m} \tag{3.3}$$

We want to use (3.2) to approximate J_1. To achieve this we first need to bound the error term of it, again we require it to be the same bound of J_3:

$$\left|\frac{1}{A_m}\sum_j \int_0^{u_0 e^{i\beta}} u\alpha_j^2 \frac{B_d(u\alpha_j)}{1-u\alpha_j} du\right| \ll \frac{1}{m^2}\sum_j \frac{1}{|u_0 e^{i\beta}-\eta_j|^2}$$

We will simplify this requirement for the polynomial B_d into a stronger but simpler one: It is enough to require

$$\frac{1}{|A_m|}\int_0^{u_0} t|B_d(te^{i\beta}\alpha_j)(1-te^{i\beta}\alpha_j)|dt \ll \frac{1}{m^2}$$

which would hold if

$$\frac{|b_0|}{2}e^{-2/m} + \sum_{k=1}^{d+1} \frac{|b_k - b_{k-1}| e^{-\frac{k+2}{m}}}{k+2} \ll \frac{|A_m|}{m^2} \tag{3.4}$$

We have obtained the approximation formula for $A(\beta, \chi)$:

$$A(\beta,\chi) = \sum_{\deg n < d} \frac{W_m(\deg n)\Lambda(n) u_0^{\deg n} e^{i\beta \deg n}\chi(n)}{\deg n} + O\left(\frac{1}{m^2}\sum_j \frac{1}{|u_0 e^{i\beta}-\eta_j|^2}\right)$$

Now we want to choose f_m so that the error term would be small compared to the main term, for a suitable choice of $m = m(D)$.

Since

$$\sum_{j=1}^{D-1} \frac{1}{1-\alpha_j u} = -u\frac{L'}{L}(u) + D - 1$$

we can rewrite formula (3.2) as following

$$\sum_{j=1}^{D-1} \frac{1}{1-\alpha_j u} = D - 1 - u\sum_n \Lambda(n)\chi(n)u^{\deg n - 1} W_m(\deg n) - \frac{u^2}{A_m}\sum_j \alpha_j^2 \frac{B_d(u\alpha_j)}{1-\alpha_j u} \tag{3.5}$$

It is straightforward to get the inequality

$$\frac{1}{|1-\alpha_j u_0 e^{i\beta}|^2} \leq \frac{1}{1-e^{-1/m}} \Re \frac{1}{1-u_0 e^{i\beta}\alpha_j}$$

Substituting $u = u_0 e^{i\beta}$ and taking real parts of both sides in (3.5) yields

$$\Re \sum_{j=1}^{D-1} \frac{1}{1-\alpha_j u_0 e^{i\beta}} \leq D-1+O\left(\left|\sum_n \Lambda(n)\chi(n)u_0^{\deg n-1} e^{i\beta(\deg n-1)} W_m(\deg n)\right|\right)+$$

$$+\left|\frac{u_0^2 e^{2i\beta}}{A_m} \sum_j \alpha_j^2 \frac{B_d(u_0 e^{i\beta}\alpha_j)}{1-\alpha_j u_0 e^{i\beta}}\right|$$

Now

$$\left|\frac{u_0^2 e^{2i\beta}}{A_m}\alpha_j^2 \frac{B_d(u_0 e^{i\beta}\alpha_j)}{1-\alpha_j u_0 e^{i\beta}}\right| = \frac{e^{-2/m}}{|A_m|} \frac{|B_d(u_0 e^{i\beta}\alpha_j)|1-u_0 e^{i\beta}\alpha_j|}{|1-u_0 e^{i\beta}\alpha_j|^2} \leq$$

$$\leq \frac{e^{-2/m}}{1-e^{-1/m}} \frac{1}{|A_m|} |B_d(u_0 e^{i\beta}\alpha_j)||1-u_0 e^{i\beta}\alpha_j| \Re \frac{1}{1-\alpha_j u_0 e^{i\beta}} \leq$$

$$\leq \frac{m}{|A_m|} |B_d(u_0 e^{i\beta}\alpha_j)||1-u_0 e^{i\beta}\alpha_j| \Re \frac{1}{1-\alpha_j u_0 e^{i\beta}}$$

So if we refine condition (3.3) to be

$$\frac{m}{|A_m|} |B_d(u_0 e^{i\beta}\alpha_j)||1-u_0 e^{i\beta}\alpha_j| < c \leq 1$$

for some absolute constant c, which is certainly true when

$$|b_0| + \sum_{k=1}^{d+1} |b_k - b_{k-1}| e^{-k/m} \leq c \frac{|A_m|}{m} \tag{3.6}$$

then this would guarantee

$$\frac{1}{m} \sum_j \frac{1}{|u_0 e^{i\beta} - \eta_j|^2} \ll \Re \sum_j \frac{1}{1-u_0 e^{i\beta}\alpha_j} \ll$$

$$\ll D + \left|\sum_n \Lambda(n)\chi(n) u_0^{\deg n-1} e^{i\beta(\deg n-1)} W_m(\deg n)\right|$$

Conditions (3.4) and (3.6) suggest that most b_k should equal. Also, we want the main term of $A(\beta,\chi)$ (which has degree $d(m)-1$ in u_0) to be as short a sum as possible. Choosing $d = 2m-1$, $B_d(z) = z^m + ... + z^{2m-1}$ would satisfy all requirements.

3.2 A Rigorous Derivation

In the following m will be denoting some natural number, and $u_0 = q^{-1/2}e^{-1/m}$. We also will use the notation $\alpha_j(\chi) = \eta_j(\chi)^{-1}$.
Throughout this section, we assume $\phi_j \neq \pm\beta$, $\forall 1 \leq j \leq D-2$. This restriction is removed in the end of the section.

The approximation formula will be based on a formula derived by Soundararajan as an analogue to Selberg's formula. This is precisely formula (3.2) with $B_d(z) = z^m + ... + z^{2m-1}$:

Lemma 3.1. *Let m be a natural number, and define*

$$W_m(k) = \begin{cases} 1 & \text{if } k \leq m+1 \\ 2-(k-1)/m & \text{if } m+1 < k \leq 2m \\ 0 & \text{if } k > 2m \end{cases}$$

Then

$$\frac{L'}{L}(u,\chi) = \sum_n \Lambda(n)u^{\deg n - 1}W_m(\deg n)\chi(n) + R(u,\chi)$$

where $R(u,\chi)$ is given by

$$R(u,\chi) = \sum_{j=1}^{D-1} \frac{\alpha_j(\chi)^2 u}{(\alpha_j(\chi)u-1)^2} \frac{[\alpha_j(\chi)u]^{2m} - [\alpha_j(\chi)u]^m}{m}$$

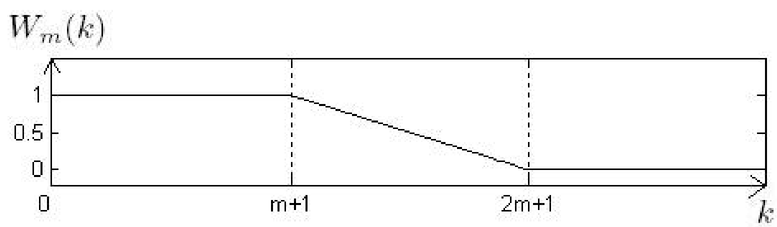

Figure 2: A plot of W_m

Now we bound the error term $R(u,\chi)$:

Lemma 3.2. *Let $|u| \leq u_0$, $\arg u = \beta$. Then*

$$\left|\frac{L'}{L}(u,\chi) - \sum_n \Lambda(n)\chi(n)u^{\deg n - 1}W_m(\deg n)\right| \leq 2q^{m/2}|u|^{m-1}\Re\sum_{j=1}^{D-1}\frac{1}{1-\alpha_j(\chi)u_0e^{i\beta}}$$

Proof. From lemma 3.1 we have

$$\left|\frac{L'}{L}(u,\chi) - \sum_n \Lambda(n)\chi(n)u^{\deg n - 1}W_m(\deg n)\right| \leq \sum_{j=1}^{D-1}\frac{2q^{1+m/2}|u|^{m-1}}{m}\left|\frac{1}{u} - \alpha_j(\chi)\right|^{-2} \leq$$

$$\leq \frac{2q^{1+m/2}|u|^{m-1}}{m} \sum_{j=1}^{D-1} \left| \frac{1}{u_0 e^{i\beta}} - \alpha_j(\chi) \right|^{-2}$$

The last inequality is true since all $\alpha_j(\chi)$ have absolute value $\sqrt{q}$ or 1, and $\sqrt{q} < u_0^{-1} \leq |u|^{-1}$.

Finally,

$$\Re \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}} = \frac{\Re(1 - \overline{\alpha_j(\chi)} u_0 e^{-i\beta})}{|1 - \alpha_j(\chi) u_0 e^{i\beta}|^2} \geq \frac{1 - e^{-1/m}}{|1 - \alpha_j(\chi) u_0 e^{i\beta}|^2} =$$

$$= \frac{q(1 - e^{-1/m})}{e^{-2/m}} \left| \frac{1}{u_0 e^{i\beta}} - \alpha_j(\chi) \right|^{-2} \geq \frac{q}{m} \left| \frac{1}{u_0 e^{i\beta}} - \alpha_j(\chi) \right|^{-2}$$

i.e.

$$\left| \frac{1}{u_0 e^{i\beta}} - \alpha_j(\chi) \right|^{-2} \leq \frac{m}{q} \Re \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}} \tag{3.7}$$

□

Next we derive an estimate

Lemma 3.3.

$$\Re \sum_{j=1}^{D-1} \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}} = O\left(D + \left| \sum_n \Lambda(n) \chi(n) u_0^{\deg n} e^{i\beta \deg n} W_m(\deg n) \right| \right) \tag{3.8}$$

Proof. We observe that

$$\sum_{j=1}^{D-1} \frac{1}{1 - \alpha_j(\chi) u} = -u \frac{L'}{L}(u, \chi) + D - 1$$

and

$$\Re \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}} > 0$$

By taking $u = u_0 e^{i\beta}$ and using lemma 3.2 we can obtain

$$\Re \sum_{j=1}^{D-1} \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}} \leq D - 1 + \left| \sum_n \Lambda(n) \chi(n) u_0^{\deg n} e^{i\beta \deg n} W_m(\deg n) \right| + \frac{2}{e} \Re \sum_{j=1}^{D-1} \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}}$$

which implies

$$\Re \sum_{j=1}^{D-1} \frac{1}{1 - \alpha_j(\chi) u_0 e^{i\beta}} \leq 4 \left(D - 1 + \left| \sum_n \Lambda(n) \chi(n) u_0^{\deg n} e^{i\beta \deg n} W_m(\deg n) \right| \right) \tag{3.9}$$

and the lemma is proved. □

A simple substitution of (3.9) into lemma 3.2 gives

Corollary 3.4. *Let* $|u| \le u_0, \arg u = \beta$. *Then*

$$\frac{L'}{L}(u,\chi) = \sum_n \Lambda(n)\chi(n)u^{\deg n - 1}W_m(\deg n) + O\left(q^{m/2}|u|^{m-1}D\right) +$$

$$+O\left(q^{m/2}|u|^{m-1}\left|\sum_n \Lambda(n)\chi(n)u_0^{\deg n}e^{i\beta\deg n}W_m(\deg n)\right|\right) \tag{3.10}$$

and in particular

$$\frac{L'}{L}(u_0e^{i\beta},\chi) = O(D) + O\left(\left|\sum_n \Lambda(n)\chi(n)u_0^{\deg n}e^{i\beta\deg n}W_m(\deg n)\right|\right) \tag{3.11}$$

Next we prove a lemma which will be used in the proof of the approximation formula for $S(\beta,\chi)$ (specifically, it is used to bound J_3 from subsection 3.1):

Lemma 3.5. *Let* $\eta_j = \mu_j + i\nu_j$, $1 \le j \le D-1$. *Then*

$$I = \int_{u_0}^{q^{-1/2}} \frac{|\nu_j\cos\beta - \mu_j\sin\beta||t-u_0|\left|t+u_0-2\Re(\overline{\eta_j}e^{i\beta})\right|}{|te^{i\beta}-\eta_j|^2}dt = O\left(\frac{1}{m^2}\right) \tag{3.12}$$

Proof. If $\eta_j = 1$, the denominator is $\Omega(1)$, and the numerator is $O(\frac{1}{m})$. Since $t - u_0 = O(1/m)$, integration over the interval $[u_0, q^{-1/2}]$ provides the required bound.

Otherwise, $\eta_j = q^{-1/2}e^{i\phi_j}$. It holds for $t \in [u_0, q^{-1/2}]$ that

$$|te^{i\beta} - \eta_j| \ge \frac{1}{\sqrt{q}}\sin|\beta - \phi_j|$$

Fix some $\epsilon_0 = 0.01$. Then if $|\beta - \phi_j| \ge \epsilon_0$, we'll have $|te^{i\beta} - \eta_j|^2 \ge \frac{1}{q}\sin^2\epsilon_0$, and the integrand in (3.12) is $O(q^{-1/2} - u_0)$, and therefore the entire integral is $O\left(\frac{1}{m^2}\right)$.

Assume now that $|\beta - \phi_j| < \epsilon_0$.

$$|t + u_0 - 2\Re(\overline{\eta_j}e^{i\beta})| \le |t - \frac{1}{\sqrt{q}}\cos(\beta-\phi)| + |u_0 - \frac{1}{\sqrt{q}}\cos(\beta-\phi)|$$

Now

$$\left|u_0 - \frac{1}{\sqrt{q}}\cos(\beta-\phi)\right| = \frac{1}{\sqrt{q}}|e^{-\frac{1}{m}} - \cos(\beta-\phi_j)| \le \frac{1}{\sqrt{q}}(|1-e^{-\frac{1}{m}}| + |1-\cos(\beta-\phi_j)|) \le$$

$$\le \frac{1}{\sqrt{q}}\left(\frac{1}{m} + \frac{(\beta-\phi)^2}{2}\right)$$

Similarly

$$\left|t - \frac{1}{\sqrt{q}}\cos(\beta-\phi)\right| \leq \frac{1}{\sqrt{q}}\left(\frac{1}{m} + \frac{(\beta-\phi)^2}{2}\right)$$

and thus

$$|t + u_0 - 2\Re(\overline{\eta_j}e^{i\beta})| \leq \frac{1}{\sqrt{q}}\left(\frac{2}{m} + (\beta-\phi)^2\right)$$

Putting everywhere $t - u_0 \ll \frac{1}{m}$, we obtain

$$I = O\left(\frac{1}{m^2}\right)\int_{u_0}^{q^{-1/2}} \frac{|\nu_j\cos\beta - \mu_j\sin\beta|}{|te^{i\beta}-\eta_j|^2}dt + O\left(\frac{(\beta-\phi_j)^2}{m}\right)\int_{u_0}^{q^{-1/2}} \frac{|\nu_j\cos\beta - \mu_j\sin\beta|}{|te^{i\beta}-\eta_j|^2}dt$$

To estimate the second term, we simply note that $|\nu_j\cos\beta - \mu_j\sin\beta| = O(1)$ and that

$$|te^{i\beta}-\eta_j|^2 \geq \frac{1}{q}\sin^2|\beta-\phi_j| \geq \frac{1}{2q}|\beta-\phi_j|^2$$

so that the second term is $O\left(\frac{1}{m}\right)O(q^{-1/2}-u_0) = O\left(\frac{1}{m^2}\right)$.

We are left to show that

$$I_1 = \int_{u_0}^{q^{-1/2}} \frac{|\nu_j\cos\beta - \mu_j\sin\beta|}{|te^{i\beta}-\eta_j|^2}dt = O(1)$$

Rewrite I_1 as

$$I_1 = \int_{u_0}^{q^{-1/2}} \frac{|\nu_j\cos\beta - \mu_j\sin\beta|}{(t-(\mu_j\cos\beta+\nu_j\sin\beta))^2 + \frac{1}{q} - (\mu_j\cos\beta+\nu_j\sin\beta)^2}dt =$$

$$= \int_{u_0}^{q^{-1/2}} \frac{\frac{1}{\sqrt{q}}\sin|\phi_j-\beta|\,dt}{(t-\frac{1}{\sqrt{q}}\cos(\phi_j-\beta))^2 + \frac{1}{q}\sin^2(\phi_j-\beta)} \leq \int_{-\infty}^{\infty}\frac{dt}{t^2+1} = \pi \qquad \square$$

We now state and prove the approximation formula for $S(\beta,\chi)$. Denote

$$E_m(\beta) = \sum_n \Lambda(n)\chi(n)u_0^{\deg n}e^{i\beta\deg n}W_m(\deg n)$$

Theorem 3.6.

$$S(\beta,\chi) = -\sum_{\deg n\leq 2m}\frac{\Lambda(n)u_0^{\deg n}\sin(\beta\deg n)W_m(\deg n)}{\deg n}\left(\chi(n)+\overline{\chi(n)}\right) +$$

$$+O\left(\frac{1}{m}(|E_m(\beta)|+|E_m(-\beta)|)\right) + O\left(\frac{D}{m}\right)$$

Proof. First we will shrink γ_2 to the origin, therefore obtaining

$$S(\beta,\chi) = \Im \int_\gamma \frac{L'}{L}(u,\chi)du$$

where

$$\gamma = \left[q^{-1/2}e^{i\beta}, 0\right] \cup \left[0, q^{-1/2}e^{-i\beta}\right]$$

Thus $S(\beta,\chi) = -A(\beta,\chi) + A(-\beta,\chi)$, where

$$A(\beta,\chi) = \Im \int_0^{\frac{1}{\sqrt{q}}e^{i\beta}} \frac{L'}{L}(u)du$$

Write $A(\beta,\chi) = J_1 + J_2 + J_3$

$$J_1 = \Im \int_0^{u_0 e^{i\beta}} \frac{L'}{L}(u)du$$

$$J_2 = \Im \left(\left(\frac{1}{\sqrt{q}} - u_0 \right) e^{i\beta} \frac{L'}{L}(u_0 e^{i\beta}) \right)$$

$$J_3 = \Im \int_{u_0}^{\frac{1}{\sqrt{q}}} \left(\frac{L'}{L}(te^{i\beta}) - \frac{L'}{L}(u_0 e^{i\beta}) \right) e^{i\beta} dt$$

To calculate J_1, we apply (3.4)

$$J_1 = \Im \sum_n \frac{\Lambda(n)\chi(n)u_0^{\deg n} e^{i\beta \deg n} W_m(\deg n)}{\deg n} + O\left(\frac{D}{m}\right) +$$

$$+O\left(\frac{1}{m} \left| \sum_n \Lambda(n)\chi(n)u_0^{\deg n} e^{i\beta \deg n} W_m(\deg n) \right| \right)$$

Next, $q^{-1/2} - u_0 = O(1/m)$, and so by (3.11)

$$J_2 = O\left(\frac{1}{m} \left| \sum_n \Lambda(n)\chi(n)u_0^{\deg n} e^{i\beta \deg n} W_m(\deg n) \right| \right) + O\left(\frac{D}{m}\right)$$

We now bound J_3. Since $du = \cos\beta dt + i \sin\beta dt$, we have

$$J_3 = J_{3x} \cos\beta + J_{3y} \sin\beta$$

where

$$J_{3x} = \int_{u_0}^{\frac{1}{\sqrt{q}}} \Im \left(\frac{L'}{L}(te^{i\beta}) - \frac{L'}{L}(u_0 e^{i\beta}) \right) dt$$

and

$$J_{3y} = \int_{u_0}^{\frac{1}{\sqrt{q}}} \Re \left(\frac{L'}{L}(te^{i\beta}) - \frac{L'}{L}(u_0 e^{i\beta}) \right) dt$$

Note that

$$\frac{L'}{L}(u,\chi) = \sum_{j=1}^{D-1} \frac{1}{u-\eta_j(\chi)}$$

Therefore

$$J_{3x} = \sum_j \int_{u_0}^{q^{-1/2}} \frac{-t\sin\beta+\nu_j}{|te^{i\beta}-\eta_j|^2} - \frac{-u_0\sin\beta+\nu_j}{|u_0e^{i\beta}-\eta_j|^2} =$$

$$= -\sum_j \int_{u_0}^{q^{-1/2}} \frac{\sin\beta(t-u_0)}{|u_0e^{i\beta}-\eta_j|^2} + \sum_j \int_{u_0}^{q^{-1/2}} \frac{(\nu_j - t\sin\beta)(|u_0e^{i\beta}-\eta_j|^2 - |te^{i\beta}-\eta_j|^2)}{|u_0e^{i\beta}-\eta_j|^2|te^{i\beta}-\eta_j|^2}$$

the first summand is simply

$$-\sum_j \int_{u_0}^{q^{-1/2}} \frac{\sin\beta(t-u_0)}{|u_0e^{i\beta}-\eta_j|^2} = -\sum_j \frac{\sin\beta}{|u_0e^{i\beta}-\eta_j|^2} \int_{u_0}^{q^{-1/2}} (t-u_0)dt =$$

$$= -\sum_j \frac{(q^{-1/2}-u_0)^2\sin\beta}{2|u_0e^{i\beta}-\eta_j|^2} = O\left(\frac{1}{m^2}\sum_j \frac{1}{|u_0e^{i\beta}-\eta_j|^2}\right)$$

Similarly,

$$J_{3y} = \sum_j \int_{u_0}^{q^{-1/2}} \frac{t\cos\beta-\mu_j}{|te^{i\beta}-\eta_j|^2} - \frac{u_0\cos\beta-\mu_j}{|u_0e^{i\beta}-\eta_j|^2} =$$

$$= \sum_j \int_{u_0}^{q^{-1/2}} \frac{\cos\beta(t-u_0)}{|u_0e^{i\beta}-\eta_j|^2} + \sum_j \int_{u_0}^{q^{-1/2}} \frac{(t\cos\beta-\mu_j)(|u_0e^{i\beta}-\eta_j|^2 - |te^{i\beta}-\eta_j|^2)}{|u_0e^{i\beta}-\eta_j|^2|te^{i\beta}-\eta_j|^2}$$

and again the first summand is

$$O\left(\frac{1}{m^2}\sum_j \frac{1}{|u_0e^{i\beta}-\eta_j|^2}\right)$$

Therefore, up to this error, J_3 is given by

$$\sum_j \frac{1}{|u_0e^{i\beta-\eta_j}|^2} \int_{u_0}^{q^{-1/2}} \frac{(|u_0e^{i\beta}-\eta_j|^2 - |te^{i\beta}-\eta_j|^2)\left((\nu_j - t\sin\beta)\cos\beta + (t\cos\beta-\mu_j)\sin\beta\right)}{|te^{i\beta}-\eta_j|^2} =$$

$$= \sum_j \frac{1}{|u_0e^{i\beta-\eta_j}|^2} \int_{u_0}^{q^{-1/2}} \frac{(t-u_0)(t+u_0-2\Re(\overline{\eta_j}e^{i\beta}))(\nu_j\cos\beta-\mu_j\sin\beta)}{|te^{i\beta}-\eta_j|^2}$$

since

$$|te^{i\beta}-\eta_j|^2 - |u_0e^{i\beta}-\eta_j|^2 = (t-u_0)\left(t+u_0-2\Re(\overline{\eta_j}e^{i\beta})\right)$$

By lemma 3.5, we have proved that

$$J_3 = O\left(\frac{1}{m^2}\sum_{j=1}^{D-1}\frac{1}{|u_0e^{i\beta}-\eta_j|^2}\right) \tag{3.13}$$

Now we use inequality (3.7):

$$\frac{1}{|u_0e^{i\beta}-\eta_j|^2} \ll \frac{1}{\left|\alpha_j - \frac{1}{u_0e^{i\beta}}\right|^2} \ll m\Re\frac{1}{1-\alpha_j u_0 e^{i\beta}}$$

and by (3.8) conclude that

$$\sum_{j=1}^{D-1}\frac{1}{|u_0e^{i\beta}-\eta_j|^2} = O\left(mD + m\left|\sum_n \Lambda(n)\chi(n)u_0^{\deg n}e^{i\beta\deg n}W_m(\deg n)\right|\right)$$

i.e,

$$J_3 = O\left(\frac{D}{m} + \frac{1}{m}\left|\sum_n \Lambda(n)\chi(n)u_0^{\deg n}e^{i\beta\deg n}W_m(\deg n)\right|\right)$$

Finally, since $S(\beta,\chi) = -A(\beta,\chi) + A(-\beta,\chi)$ and

$$\Im\left(\chi(n)\left(-e^{i\beta\deg n} + e^{-i\beta\deg n}\right)\right) = -\sin(\beta\deg n)\left(\chi(n) + \overline{\chi(n)}\right)$$

we can conclude the proof. □

We note that if there are zeros of argument $\pm\beta$, theorem 3.6 is still valid, from definition (2.1), since the estimate of $S(\beta,\chi)$ in theorem 3.6 is continuous in β.

Following Selberg, we will use theorem 3.6 to establish the bound on $S(\beta,\chi)$:

Theorem 1.

$$S(\beta,\chi) = O\left(\frac{D}{\log D}\right)$$

Proof. We start by observing that

$$\int_1^x \frac{e^t dt}{t} = \int_e^{e^x}\frac{ds}{\log s} = O\left(\frac{e^x}{x}\right)$$

Now we take $m = \log_q D$ and use theorem 3.6 The main term can be bounded by

$$2\sum_{k\le 2m}\frac{q^{-k/2}e^{-k/m}|\sin(\beta k)|}{k}\sum_{\deg n = k}\Lambda(n) \le 2\sum_{k\le 2m}\frac{q^{k/2}}{k} \ll \int_1^m \frac{q^t dt}{t} = O\left(\frac{D}{\log D}\right)$$

The error terms in theorem 3.6 are easily seen to satisfy this bound. □

Since $N(\beta,\chi) = \frac{1}{\pi}\left(\beta(\deg Q - 2) + 2\arg\left(1 - \frac{1}{\sqrt{q}}e^{i\beta}\right) + S(\beta,\chi)\right)$, we have

Corollary 1. *The multiplicity of a non-trivial zero of $L(u,\chi)$ does not exceed $O\left(\frac{D}{\log D}\right)$.*

4 The Moments of $S(\beta, \chi)$

In the following, $\sum_\chi$ denotes a sum over even primitive characters, and $\sum_p$ denotes a sum over prime polynomials. Q will be a prime polynomial of degree D. Next we prove:

Lemma 4.1. *Fix $k \geq 1$ and let $m = m(D) < \frac{D}{k}$.*
(1) Let $\{a_p\} \subset \mathbf{C}$ be a set of coefficients indexed by prime polynomials p which satisfy $|a_p| < A^{\frac{\deg p}{m}}$ for some constant A. Then

$$\sum_\chi \left| \sum_{\deg p \leq m} \frac{a_p}{\sqrt{\|p\|}} \chi(p) \right|^{2k} = O(q^D)$$

(2) Let $\{a'_p\} \subset \mathbf{C}$ be a set of coefficients indexed by prime polynomials p which satisfy $|a_p| < A$ for some constant A. Then

$$\sum_\chi \left| \sum_{\deg p \leq m} \frac{a'_p}{\|p\|} \chi(p^2) \right|^{2k} = O(q^D)$$

Proof. (1) Define b_n by

$$\left(\sum_{\deg p \leq m} \frac{a_p}{\sqrt{\|p\|}} \chi(p) \right)^k = \sum_{\deg n \leq km} \frac{b_n}{\sqrt{\|n\|}} \chi(n)$$

Then

$$\sum_\chi \left| \sum_{\deg p \leq m} \frac{a_p}{\sqrt{\|p\|}} \chi(p) \right|^{2k} = \sum_{\deg n_j \leq km} \frac{b_{n_1} \overline{b_{n_2}}}{\sqrt{\|n_1\|}\sqrt{\|n_2\|}} \sum_\chi \chi(n_1) \overline{\chi(n_2)}$$

Since $\deg n_j < D$, $n_1 \equiv n_2$ (mod$Q$) iff $n_1 = n_2$. Also, $|b_n| < A^k$. Thus this expression becomes

$$(\Phi_e(Q) - 1) \sum_{\deg n \leq km} \frac{|b_n|^2}{\|n\|} - \left| \sum_{\deg n \leq km} \frac{b_n}{\sqrt{\|n\|}} \chi_1(n) \right|^2 \leq$$

$$\leq A^k q^D \sum_{\deg n \leq km} \frac{|b_n|}{\|n\|} = A^k q^D \left(\sum_{\deg p \leq m} \frac{|a_p|}{\|p\|} \right)^k \ll q^D \left(\sum_{d=1}^{m} \frac{d/m}{q^d} \frac{q^d}{d} \right)^k \ll q^D$$

(2) Similarly, we get

$$\sum_\chi \left| \sum_{\deg p \leq m} \frac{a'_p}{\|p\|} \chi(p^2) \right|^{2k} \leq A^k q^D \left(\sum_{\deg p \leq m} \frac{|a'_p|}{\|p\|^2} \right)^k \ll q^D \left(\sum_{d=1}^{m} \frac{q^d}{d} \frac{1}{q^{2d}} \right)^k \ll q^D$$

$\square$

Now we calculate the moments of an expression which will later be shown to approximate $S(\beta,\chi)$ well. Define

$$T_m(\beta,\chi) = \sum_{\deg p \leq 2m} \frac{\sin(\beta \deg p)}{\sqrt{q}^{\deg p}}(\chi(p) + \overline{\chi}(p))$$

Lemma 4.2. *Let $m = m(Q) \to \infty$, fix $r \geq 0$, and assume $0 < \beta < \pi$ (in mesoscopic scale we assume that for all Q). Then for both scales*
(1) When $2m(2r+1) < D$

$$\sum_{\chi} T_m(\beta,\chi)^{2r+1} \ll \frac{q^{m(2r+1)}}{m^{2r+1}}$$

(2) Assume $4mr < D$, and $m\beta \to \infty$ (for mesoscopic scale). Then

$$\sum_{\chi} T_m(\beta,\chi)^{2r} = \Phi_e(Q)\frac{(2r)!}{2^r r!}\log^r(m\beta) + O(q^D \log^{r-1}(m\beta))$$

Proof. The inequalities on m ensure that

$$\sum_{\chi} \chi\left(\prod_{j\leq v} p_j\right)\overline{\chi}\left(\prod_{j\geq v+1} p_j\right) = \begin{cases} \Phi_e(Q)-1, & \prod_{j\leq v} p_j = \prod_{j \geq v+1} p_j \\ -1, & \text{otherwise} \end{cases}$$

(1)

$$\sum_{\chi} T_m(\beta,\chi)^{2r+1} \ll \sum_{\substack{\deg p_j \leq 2m \\ j=1..2r+1}} \prod_j \frac{1}{\sqrt{q}^{\deg p_j}} \ll \left(\sum_{\deg p \leq 2m} \frac{1}{\sqrt{q}^{\deg p}}\right)^{2r+1} \ll \frac{q^{m(2r+1)}}{m^{2r+1}}$$

(2) We will use induction on r. The case of $r=0$ is trivial. Write

$$\sum_{\chi} T_m(\beta,\chi)^{2r} = T_{\text{diag}} + T_{\text{non-diag}}$$

where

$$T_{\text{diag}} = (\Phi_e(Q)-1)\sum_{S\subset\{1,\dots,2r\}} \sum_{\substack{\prod_{j\in S} p_j = \prod_{j\in\overline{S}} p_j \\ \deg p_j \leq 2m}} \prod_{j=1}^{2r} \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}}$$

and

$$T_{\text{non-diag}} = -\sum_{S\subset\{1,\dots,2r\}} \sum_{\substack{\prod_{j\in S} p_j \neq \prod_{j\in\overline{S}} p_j \\ \deg p_j \leq 2m}} \prod_{j=1}^{2r} \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}}$$

The non-diagonal term is easy to bound:

$$T_{\text{non-diag}} \ll \left(\sum_{\deg p \le 2m} \frac{1}{\sqrt{q}^{\deg p}} \right)^{2r} \ll \frac{q^{2mr}}{m^{2r}} \ll q^D$$

To calculate the diagonal term, we will take all p_j to be different, and bound the sum of the remaining terms:

$$T_{\text{diag}} = (\Phi_e(Q) - 1) \begin{pmatrix} 2r \\ r \end{pmatrix} \sum_{\substack{\prod_{j=1}^{r} p_j = \prod_{j=r+1}^{2r} p_j \\ \text{all } p_j \text{ different}}} \prod_{j=1}^{2r} \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} + Error_1$$

where

$$Error_1 \ll q^D \sum_{p_1 = p_2 = p_{2r-1} = p_{2r}} \frac{\sin^4(\beta \deg p_1)}{q^{2 \deg p_1}} \sum_{\prod_{j=3}^{r} p_j = \prod_{j=r+1}^{2r-2} p_j} \prod_{j=3}^{2r-2} \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} \ll$$

$$\ll q^D \sum_{\prod_{j=3}^{r} p_j = \prod_{j=r+1}^{2r-2} p_j} \prod_{j=3}^{2r-2} \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} \ll \sum_{\chi} T_m(\beta, \chi)^{2r-4} \ll q^D \log^{r-2}(m\beta)$$

The last inequality was obtained by induction on r. Also notice that the products of the form $\prod \frac{\sin(\beta \deg p)}{\sqrt{q}^{\deg p}}$ are always positive, since the p-s come in equal pairs. Also,

$$\sum_{\substack{\prod_{j=1}^{r} p_j = \prod_{j=r+1}^{2r} p_j \\ \text{all } p_j \text{ different}}} \prod_{j=1}^{2r} \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} = r! \sum_{\substack{\deg p_j \le 2m \\ j = 1, \dots, r \\ \text{all } p_j \text{ different}}} \prod_{j=1}^{r} \frac{\sin^2(\beta \deg p_j)}{q^{\deg p_j}} =$$

$$= r! \left(\sum_{\deg p \le 2m} \frac{\sin^2(\beta \deg p)}{q^{\deg p}} \right)^r + Error_2$$

and

$$Error_2 \ll \sum_{\substack{\deg p_j \le 2m \\ j = 1, \dots, r \\ p_1 = p_2}} \prod_{j=1}^{r} \frac{\sin^2(\beta \deg p_j)}{q^{\deg p_j}} \ll \sum_{\substack{\deg p_j \le 2m \\ j = 3, \dots, r}} \prod_{j=3}^{r} \frac{\sin^2(\beta \deg p_j)}{q^{\deg p_j}} =$$

$$= \left(\sum_{\deg p \le 2m} \frac{\sin^2(\beta \deg p)}{q^{\deg p}} \right)^{r-2}$$

By PNT and lemma A.1

$$\sum_{\deg p \leq 2m} \frac{\sin^2(\beta \deg p)}{q^{\deg p}} = \sum_{d=1}^{2m} \frac{\sin^2(\beta d)}{d} + O\left(\sum_{d=1}^{2m} \frac{\sin^2(\beta d)}{q^{d/2}}\right) = \frac{1}{2}\log(m\beta) + O(1)$$

Therefore,

$$\left(\sum_{\deg p \leq 2m} \frac{\sin^2(\beta \deg p)}{q^{\deg p}}\right)^r = \frac{1}{2^r}\log^r(m\beta) + O(\log^{r-1}(m\beta))$$

and summarizing

$$\sum_{\chi} T_m(\beta,\chi)^{2r} = \Phi_e(Q)\frac{(2r)!}{r!2^r}\log^r(m\beta) + O(q^D \log^{r-1}(m\beta)) \qquad \square$$

In particular, for $4m < D$ it holds for both scales that

$$\sum_{\chi} T_m(\beta,\chi)^2 = (1+o(1))\Phi_e(Q)\log(m\beta)$$

Next we want to estimate how good an approximation of $S(\beta,\chi)$ is $-T_m(\beta,\chi)$. Apply theorem 3.6 to write:

$$|S(\beta,\chi) + T_m(\beta,\chi)| = O\left(1 + E_1 + E_2 + E_3 + E_4 + E_5^+ + E_5^- + E_6^+ + E_6^- + \frac{D}{m}\right)$$

where

$$E_1 = \left|\sum_p W_m(\deg p)\left(\frac{1}{(\sqrt{q})^{\deg p}} - u_0^{\deg p}\right)\sin(\beta \deg p)\left(\chi(p) + \overline{\chi}(p)\right)\right|$$

$$E_2 = \left|\sum_{\deg p \leq 2m}(1 - W_m(\deg p))\frac{1}{(\sqrt{q})^{\deg p}}\sin(\beta \deg p)(\chi(p) + \overline{\chi}(p))\right|$$

$$E_3 = \left|\sum_p W_m(2\deg p)\frac{1}{(\sqrt{q})^{2\deg p}}\sin(2\beta \deg p)(\chi(p^2) + \overline{\chi}(p^2))\right|$$

$$E_4 = \left|\sum_p W_m(2\deg p)\left(\frac{1}{(\sqrt{q})^{2\deg p}} - u_0^{2\deg p}\right)\sin(2\beta \deg p)(\chi(p^2) + \overline{\chi}(p^2))\right|$$

$$E_5^{\pm} = \left|\frac{1}{m}\sum_p W_m(\deg p)\deg(p)u_0^{\deg p}e^{\pm i\beta \deg p}\chi(p)\right|$$

$$E_6^{\pm} = \left|\frac{1}{m}\sum_p W_m(2\deg p)\deg(p)u_0^{2\deg p}e^{\pm 2i\beta \deg p}\chi(p^2)\right|$$

Observe that

$$|E_4| \ll \left| \sum_{\deg p \le m} \frac{1}{(\sqrt{q})^{2\deg p}} \left(1 - e^{-2\deg p/m}\right) \right| \ll \left| \sum_{d \le m} \frac{1}{q^d} \left(1 - e^{-2d/m}\right) \frac{q^d}{d} \right| \le$$

$$\le \sum_{d \le m} \frac{2}{m} \ll 1$$

and

$$|E_6^{\pm}| \ll \frac{1}{m} \sum_{\deg p \le m} \deg(p) u_0^{2\deg p} \ll \frac{1}{m} \sum_{d \le m} d \frac{1}{q^d} \frac{q^d}{d} \ll 1$$

Also

$$|E_1| = \left| \int_{u_0}^{q^{-1/2}} \sum_p \deg(p) t^{\deg p - 1} W_m(\deg p) \sin(\beta \deg p)(\chi(p) + \overline{\chi}(p)) \right|$$

$$\le \int_{u_0}^{q^{-1/2}} dt \left| \sum_p t^{\deg p - 1} \deg(p) W_m(\deg p) \sin(\beta \deg p)(\chi(p) + \overline{\chi}(p)) \right| =$$

$$= O\left(\frac{1}{m}\right) \left| \sum_p t^{\deg p} \deg(p) W_m(\deg p) \sin(\beta \deg p)(\chi(p) + \overline{\chi}(p)) \right|$$

For $u_0 \le t \le q^{-1/2}$, we want to estimate

$$S^* = \left| \sum_p t^{\deg p} \deg(p) W_m(\deg p) e^{i\beta \deg p} \chi(p) \right|$$

Write

$$t^{\deg p} = (t\sqrt{q})^{-m} (\deg p + m) \int_0^t q^{\frac{m}{2}} s^{\deg p + m - 1} ds$$

Then

$$S^* = (t\sqrt{q})^{-m} \left| \int_0^t (s\sqrt{q})^m \sum_p s^{\deg p - 1} W_m(\deg p) \deg(p)(m + \deg p) e^{i\beta \deg p} \chi(p) ds \right| \le$$

$$\le (u_0\sqrt{q})^{-m} \int_0^{q^{-1/2}} (s\sqrt{q})^m \left| \sum_p s^{\deg p - 1} W_m(\deg p) \deg(p)(m + \deg p) e^{i\beta \deg p} \chi(p) \right| ds$$

and $(u_0\sqrt{q})^{-m} = e$. Denote this last estimate by $eE_0(\chi)$, and the internal sum by $U_p(s, \chi)$:

$$E_0(\chi) = \int_0^{q^{-1/2}} (s\sqrt{q})^m \left| U_p(s, \chi) \right| ds$$

And so it holds that $E_1(\chi) = \frac{1}{m} O(E_0(\chi))$ and also $E_5^{\pm}(\chi) = \frac{1}{m} O(E_0(\chi))$.

Conclusion. We have proved that

$$|S(\beta,\chi)+T_m(\beta,\chi)| = O(1)+O\left(\frac{D}{m}\right)+O(E_2(\chi))+O(E_3(\chi))+\frac{1}{m}O(E_0(\chi))$$

Theorem 4.3. *Fix $r \geq 0$. For $\epsilon D < m < \frac{D}{2r}$*

$$\sum_{\chi} |S(\beta,\chi)+T_m(\beta,\chi)|^{2r} = O(q^D)$$

Proof. By Jensen's inequality for $x \mapsto x^{2r}$, we have

$$E_0(\chi)^{2r} \leq \left(\int_0^{q^{-1/2}} (s\sqrt{q})^m ds\right)^{2r-1} \int_0^{q^{-1/2}} (s\sqrt{q})^m |U_p(s,\chi)|^{2r} ds =$$

$$= \left(\frac{q^{-1/2}}{m+1}\right)^{2r-1} \int_0^{q^{-1/2}} (s\sqrt{q})^m |U_p(s,\chi)|^{2r} ds$$

Denote

$$a_p = \frac{1}{m^2}\sqrt{q}(\sqrt{q}s)^{\deg p-1} W_m(\deg p)\deg(p)(m+\deg p)e^{i\beta\deg p}$$

Then

$$|a_p| \leq 6\sqrt{q}\frac{\deg p}{2m}$$

and when $2m < \frac{D}{r}$ from lemma 4.1 part (1) we get,

$$\sum_{\chi} |U_p(s,\chi)|^{2r} = m^{4r}\sum_{\chi}\left|\sum_{\deg p\leq 2m}\frac{a_p}{\sqrt{\|p\|}}\chi(p)\right|^{2r} \ll m^{4r}q^D$$

so

$$\sum_{\chi} E_0(\chi)^{2r} \ll \frac{1}{m^{2r-1}}m^{4r}q^D(\sqrt{q})^m\frac{q^{-(m+1)/2}}{m+1} \ll m^{2r}q^D$$

and finally

$$\sum_{\chi}\left(\frac{1}{m}E_0(\chi)\right)^{2r} \ll q^D$$

From lemma 4.1 part (1) follows also that

$$\sum_{\chi}\left|\sum_{\deg p\leq 2m}(1-W_m(\deg p))\frac{1}{\sqrt{q}^{\deg p}}e^{i\beta\deg p}\chi(p)\right|^{2r} \ll q^D$$

which implies

$$\sum_{\chi} E_2(\chi)^{2r} = O(q^D)$$

and similarly from lemma 4.1 part (2)

$$\sum_{\chi} E_3(\chi)^{2r} = O(q^D)$$

Finally

$$\sum_{\chi} \left(\frac{D}{m}\right)^{2r} \ll \sum_{\chi} \frac{1}{\epsilon^{2r}} \ll q^D \qquad \square$$

Theorem 4.4. *Fix $r \geq 0$, and assume $0 < \beta < \pi$. Then for both scales*
(1)

$$\sum_{\chi} S(\beta,\chi)^{2r+1} = O(q^D \log^r(\beta D))$$

(2)

$$\sum_{\chi} S(\beta,\chi)^{2r} = \Phi_e(Q)\frac{(2r)!}{2^r r!}\log^r(\beta D) + O(q^D \log^{r-1/2}(\beta D))$$

Proof. (1) Take $m = \left\lfloor \frac{D}{8r+4} \right\rfloor$. Then

$$\sum_{\chi} S(\beta,\chi)^{2r+1} = \sum_{\chi} (-T_m(\beta,\chi) + S(\beta,\chi) + T_m(\beta,\chi))^{2r+1} =$$

$$= -\sum_{\chi} T_m(\beta,\chi)^{2r+1} + O\left(\sum_{\chi} T_m(\beta,\chi)^{2r}|S(\beta,\chi) + T_m(\beta,\chi)|^{2r+1}\right)$$

By lemma 4.2

$$\sum_{\chi} T_m(\beta,\chi)^{2r+1} \ll q^D$$

and by Cauchy-Schwartz inequality and lemma 4.2+theorem 4.3 we have

$$\sum_{\chi} T_m(\beta,\chi)^{2r}|S(\beta,\chi)+T_m(\beta,\chi)|^{2r+1} \ll \sqrt{\sum_{\chi} T_m(\beta,\chi)^{4r}}\sqrt{\sum_{\chi} |S(\beta,\chi) + T_m(\beta,\chi)|^{4r+2}} \ll$$

$$\ll \sqrt{q^D \log^{2r}(m\beta)}\sqrt{q^D} = q^D \log^r(m\beta) \ll q^D \log^r(\beta D)$$

and we proved part (1).
(2) Take $m = \lfloor \frac{D}{8r} \rfloor$. As in part (1), and then using lemma 4.2 and theorem 4.3 ,

$$\sum_{\chi} S(\beta,\chi)^{2r} = \sum_{\chi} T_m(\beta,\chi)^{2r} + O\left(\sqrt{\sum_{\chi} T_m(\beta,\chi)^{4r-2}}\sqrt{\sum_{\chi} |S(\beta,\chi) + T_m(\beta,\chi)|^{4r}}\right) =$$

Now this implies

$$\sum_{\chi} S(\beta,\chi)^{2r} = \Phi_e(Q)\frac{(2r)!}{2^r r!}\log^r(m\beta)+O(q^D\log^{r-1}(m\beta))+O(q^D\log^{r-1/2}(m\beta)) =$$

$$= \Phi_e(Q)\frac{(2r)!}{2^r r!}\log^r(\beta D) + O(q^D\log^{r-1/2}(\beta D)) \qquad \square$$

In particular, it holds for both scales that

$$\sum_{\chi} S(\beta,\chi) = O(q^D)$$

and

$$\sum_{\chi} S(\beta,\chi)^2 = (1+o(1))\Phi_e(Q)\log(\beta D)$$

As a consequence, we deduce:

Theorem 2. *Let $D \to \infty$.*
(1) Fix $0 < \beta < \pi$. The sequence of random variables

$$\frac{S(\beta,\chi)}{\sqrt{\log D}}$$

has a standard gaussian limiting distribution.
(2) Let $\beta = \beta(D)$ s.t. $0 \neq \beta \to 0$ and $\beta D \to \infty$. Then the sequence of r.v.

$$\frac{S(\beta,\chi)}{\sqrt{\log(\beta D)}}$$

has a standard gaussian limiting distribution.

5 The Family of Real Characters

In this section we establish an analogous result for the family of real quadratic characters. A detailed introduction can be found in [Ro1].

A monic polynomial $Q \in \mathbf{F}_q[x]$ is called separable if it has no multiple roots in some algebraic closure of $\mathbf{F}_q$, or, equivalently, if it is square-free. Let $K_s(D)$ denote the number of separable polynomials of degree D.

Lemma 5.1. *For $D \geq 2$, $K_s(D) = q^D\left(1-\frac{1}{q}\right)$*

Proof. We will exploit the following fact:

$$\sum_{A^2|Q} \mu(A) = \begin{cases} 1, & Q \text{ separable} \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

where $\mu(A)$ is the Mobius function for polynomials. The sum over monic polynomials of fixed degree satisfies

$$\sum_{\deg n=m} \mu(n) = \begin{cases} 1, & m=0 \\ -q, & m=1 \\ 0, & m \geq 2 \end{cases}$$

$$K_s(D) = \sum_{\deg Q=D} \sum_{A^2|Q} \mu(A) = \sum_{\deg A \leq D/2} \mu(A) \sum_{\deg B=D-2\deg A} 1 =$$

$$= q^D \sum_{\deg A \leq D/2} \frac{\mu(A)}{q^{2\deg A}} = q^D \sum_{\deg A=0}^{1} \frac{\mu(A)}{q^{2\deg A}} = q^D(1-1/q) \qquad \square$$

The quadratic reciprocity law states that for prime P_1, P_2

$$\left(\frac{P_1}{P_2}\right) = (-1)^{\frac{q-1}{2}\deg P_1 \deg P_2} \left(\frac{P_2}{P_1}\right)$$

We can generalize the Legendre symbol (the so called Jacobi symbol) by extending it multiplicatively to all monic polynomials N in

$$\chi_N = \left(\frac{\cdot}{N}\right)$$

which makes χ_N a real character modulo N. In the case when N is separable, χ_N is a primitive character. We also note that χ_N is an even character precisely when $\deg N$ is even. The quadratic reciprocity law still holds, namely

$$\left(\frac{N_1}{N_2}\right) = (-1)^{\frac{q-1}{2}\deg N_1 \deg N_2} \left(\frac{N_2}{N_1}\right)$$

for monic polynomials N_1, N_2.

Next we state a lemma which is analogous to the Polya-Vinogradov inequality.

Lemma 5.2. *Let χ be a non-principal character modulo N of degree D, and $m \geq 0$. Then*

$$\left|\sum_{\deg n=m} \chi(n)\right| \ll \binom{D}{m} q^{\frac{m}{2}}$$

Proof. This is straightforward from RH. Let $L(u,\chi) = \prod_{j=1}^{k}(1-\alpha_j u)$, with $k \leq D-1$. Then $|\alpha_j| = \sqrt{q}$ or $|\alpha_j| = 1$ for all j, and the coefficient at u^m is

$$O\left(\binom{k}{m} q^{\frac{m}{2}}\right)$$

on the other hand, this coefficient is exactly $\sum_{\deg n=m} \chi(n)$. $\square$

Remark. For $m \geq D$, $\sum_{\deg n = m} \chi(n) = 0$.

Now we apply this result to quadratic characters.

Lemma 5.3. *Let N be a monic polynomial of positive degree, and $D > 0$ even. Then*

$$\left| \sum_{\substack{\deg Q = D \\ Q \ \textit{separable}}} \left(\frac{N}{Q}\right) \right| = \begin{cases} O(q^D), & N \ \textit{perfect square} \\ O\left(q^{D/2} 2^{\deg N}\right), & \textit{otherwise} \end{cases}$$

Proof. If N is a perfect square, it is clear.
Assume otherwise. Then $\left(\frac{\cdot}{N}\right)$ is a non-principal character modulo N.
We use quadratic reciprocity law and formula (5.1) to obtain

$$\sum_{\substack{\deg Q = D \\ Q \text{ separable}}} \left(\frac{N}{Q}\right) = \sum_{\deg Q = D} \left(\frac{Q}{N}\right) \sum_{A^2 | Q} \mu(A) =$$

$$= \sum_{\deg A \leq D/2} \mu(A) \left(\frac{A^2}{N}\right) \sum_{\deg B = D - 2\deg A} \left(\frac{B}{N}\right)$$

from lemma 5.2 we get

$$\left| \sum_{\substack{\deg Q = D \\ Q \text{ separable}}} \left(\frac{N}{Q}\right) \right| \ll \sum_{j=0}^{D/2} q^j \binom{\deg N}{D - 2j} q^{D/2 - j} \ll q^{D/2} 2^{\deg N} \qquad \square$$

We will be studying the distribution of $N(\beta, \chi_Q)$, where Q ranges over all separable polynomials of even degree D, in the limit $D \to \infty$. We again consider both macroscopic and mesoscopic scales for β.
Through the rest of the section, $\sum_Q$ will denote a sum over (monic) separable polynomials Q.

Next we formulate a lemma analogous to lemma 4.1:

Lemma 5.4. *Fix $k \geq 1$ and let $m = m(D) < \frac{D}{6k}$. Let $\{a_p\} \subset \mathbf{C}$ be a set of coefficients indexed by prime polynomials p which satisfy $|a_p| < A^{\frac{\deg p}{m}}$ for some constant A. Then*

$$\sum_{\deg Q = D} \left| \sum_{\deg p \leq m} \frac{a_p}{\sqrt{\|p\|}} \chi_Q(p) \right|^{2k} = O(q^D)$$

Proof. Write

$$\sum_{\deg Q = D} \left| \sum_{\deg p \leq m} \frac{a_p}{\sqrt{\|p\|}} \chi(p) \right|^{2k} = A_{sq} + A_{nsq}$$

where

$$A_{sq} = \sum_{\substack{p_1,\,\dots,\,p_{2k} \\ \prod p_j \text{ is a square}}} \frac{\prod_{j=1}^{k} a_{p_j}\overline{a_{p_{k+j}}}}{\sqrt{\prod_{j=1}^{2k}\|p_j\|}} \sum_Q \left(\frac{\prod_{j=1}^{2k} p_j}{Q}\right)$$

$$A_{nsq} = \sum_{\substack{p_1,\,\dots,\,p_{2k} \\ \prod p_j \text{ is not a square}}} \frac{\prod_{j=1}^{k} a_{p_j}\overline{a_{p_{k+j}}}}{\sqrt{\prod_{j=1}^{2k}\|p_j\|}} \sum_Q \left(\frac{\prod_{j=1}^{2k} p_j}{Q}\right)$$

By Lemma 5.2,

$$A_{nsq} \ll q^{D/2}\left(\sum_{\deg p\le m}\frac{|a_p|2^{\deg p}}{\sqrt{\|p\|}}\right)^{2k} \ll q^{D/2}\left(\sum_{\deg p\le m}\frac{|a_p|q^{\deg p}}{\sqrt{\|p\|}}\right)^{2k} \ll$$

$$\ll q^{D/2}\left(\sum_{j=0}^{m}\frac{q^j}{jq^{j/2}}\frac{q^j j}{m}\right)^{2k} \ll \frac{q^{D/2+3mk}}{m^{2k}} \ll q^D$$

And

$$A_{sq} \ll q^D\left(\sum_{\deg p\le m}\frac{|a_p|^2}{\|p\|}\right)^{k} \ll q^D\left(\sum_{j=0}^{m}\frac{q^j}{jq^j}\frac{j^2}{m^2}\right)^{k} \ll q^D \qquad \square$$

Following the scheme of section 4, we define for separable Q

$$T_m(\beta, Q) = 2\sum_{\deg p\le 2m}\frac{\sin(\beta\deg p)}{\sqrt{q^{\deg p}}}\left(\frac{p}{Q}\right)$$

We will require a lemma:

Lemma 5.5. *For any $k \ge 0$ define*

$$W(k,D) = \sum_{\substack{\deg Q = D \\ Q \text{ separable}}}\left(\sum_{p|Q}\frac{1}{q^{\deg p}}\right)^{k}$$

Then $W(k,D) \ll q^D$.

Proof.

$$W(k,D) = \sum_{p_1,\dots,p_k}\frac{1}{q^{\sum\deg p_j}}\#\{\deg Q = D : \forall j, p_j|Q\}$$

Separate this into two sums, a sum over all k-tuples with all p_j distinct S_{dist}, and the remainder S_{rem}. First bound S_{rem}:

$$S_{rem} \ll \sum_{p_1=p_2}\frac{1}{q^{2\deg p_1}}\#\{\deg Q = D : \forall j\ge 3, p_j|Q\} = \sum_p \frac{1}{q^{2\deg p}}W(k-2,D) \ll$$

$$\ll W(k-2, D)$$

Next,

$$S_{dist} \leq \sum_{\substack{\deg p_j \leq D \\ \text{all different}}} \frac{K_s\left(D - \sum \deg p_j\right)}{q^{\sum \deg p_j}} \ll q^D \sum_{\deg p_j \leq D} \frac{1}{q^{2 \sum \deg p_j}} \ll$$

$$\ll q^D \left(\sum_{\deg p \leq D} \frac{1}{q^{2 \deg p}} \right)^k \ll q^D \left(\sum_{j=1}^{D} \frac{1}{j q^j} \right)^k \ll q^D$$

Since for $k = 0$ the claim is obvious, and for $k = 1$ S_{rem} is an empty sum, induction completes the proof. □

Next we prove an analogue of lemma 4.2 for the family of quadratic characters:

Lemma 5.6. *Let $m = m(D) \to \infty$, fix $r \geq 0$, and assume $0 < \beta < \pi$ (in mesoscopic scale we assume that for all D). Then for both scales*
(1)

$$\sum_{\deg Q = D} T_m(\beta, Q)^{2r+1} \ll q^{D/2} \frac{q^{3m(2r+1)}}{m^{2r+1}}$$

(2) Assume $12mr < D$, and that $m\beta \to \infty$ (for mesoscopic scale). Then

$$\sum_Q T_m(\beta, Q)^{2r} = K_s(D) \frac{(2r)!}{r!} \log^r(m\beta) + O(q^D \log^{r-1}(m\beta))$$

Proof.

$$\sum_{\deg Q = D} \chi_Q \left(\prod p_j \right) = \begin{cases} O(q^D), & \prod p_j = \text{square} \\ O\left(q^{D/2} 2^{\sum \deg p_j}\right), & \text{otherwise} \end{cases}$$

(1) The product of an odd number of primes is not a perfect square, hence

$$\sum_{\deg Q = D} T_m(\beta, Q)^{2r+1} = 2^{2r+1} \sum_{\substack{p_1, \ldots, p_{2r+1} \\ \deg p_j \leq 2m}} \prod_j \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} \sum_{\deg Q = D} \left(\frac{\prod p_j}{Q} \right) \ll$$

$$\ll q^{D/2} \sum_{p_1, \ldots, p_{2r+1}} \left(\frac{2}{\sqrt{q}} \right)^{\sum \deg p_j} = q^{D/2} \left(\sum_{\deg p \leq 2m} \left(\frac{2}{\sqrt{q}} \right)^{\deg p} \right)^{2r+1} \ll$$

$$\ll q^{D/2} \left(\sum_{j=1}^{2m} \frac{q^j}{j} \frac{q^j}{\sqrt{q}^j} \right)^{2r+1} \ll q^{D/2} \frac{q^{3m(2r+1)}}{m^{2r+1}}$$

(2) Write

$$\sum_{Q} T_m(\beta, \chi_Q)^{2r} = 2^{2r}(T_{sq} + T_{nsq})$$

where T_{sq} and T_{nsq} are the terms for which $\prod p_j$ is a perfect square, or not a perfect square, respectively. Now T_{nsq} is estimated as in part (1):

$$T_{nsq} = \sum_{\substack{p_1,\ldots,p_{2r} \\ \deg p_j \le 2m \\ \prod p_j \ne \text{square}}} \prod_j \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} \sum_{\deg Q = D} \left(\frac{\prod p_j}{Q} \right) \ll q^{D/2} \frac{q^{6mr}}{m^{2r}}$$

Turning to T_{sq}, we have

$$T_{sq} = \sum_{\substack{p_1,\ldots,p_{2r} \\ \deg p_j \le 2m \\ \prod p_j = \text{square}}} \prod_j \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} \sum_{\substack{\deg Q = D \\ (Q, \prod p_j) = 1}} 1$$

Notice all summands are non-negative, as primes appear in equal pairs. We will use lemma 5.5 to show that the restriction $(Q, \prod p_j) = 1$ can be neglected:

$$\sum_{\substack{p_1,\ldots,p_{2r} \\ \deg p_j \le 2m \\ \prod p_j = \text{square}}} \prod_j \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}} \sum_{\substack{\deg Q = D \\ (Q, \prod p_j) \ne 1}} 1 \ll$$

$$\ll \sum_{\substack{p_1,\ldots,p_r \\ \deg p_j \le 2m}} \prod_j \frac{\sin^2(\beta \deg p_j)}{q^{\deg p_j}} \sum_{\substack{\deg Q = D \\ (Q, \prod p_j) \ne 1}} 1 \ll$$

$$\ll \sum_{\deg Q = D} \sum_{p_1 | Q} \frac{1}{q^{\deg p_1}} \sum_{p_2,\ldots,p_r} \prod_j \frac{\sin^2(\beta \deg p_j)}{q^{\deg p_j}} \ll q^D \log^{r-1}(m\beta)$$

We are left to calculate

$$T = \sum_{\substack{p_1,\ldots,p_{2r} \\ \deg p_j \le 2m \\ \prod p_j = \text{square}}} \prod_j \frac{\sin(\beta \deg p_j)}{\sqrt{q}^{\deg p_j}}$$

Induction shows that we can assume every prime pair appears exactly once, and the error introduced will be $O(q^D \log^{r-2}(m\beta))$. Thus up to this error

$$T = \frac{(2r)!}{r! 2^r} \sum_{p_1,\ldots,p_r} \prod_j \frac{\sin^2(\beta \deg p_j)}{q^{\deg p_j}} = \frac{(2r)!}{r! 2^r} \left(\sum_{\deg p \le 2m} \frac{\sin^2(\beta \deg p)}{q^{\deg p}} \right)^r$$

This was already calculated during the proof of lemma 4.2:

$$T = \frac{(2r)!}{r! 2^r} \left(\frac{1}{2} \log(m\beta) + O(1) \right)^r$$

Bringing all together,

$$\sum_{Q} T_m(\beta, Q)^{2r} = K_s(D)\frac{(2r)!}{r!}\log^r(m\beta) + O(q^D \log^{r-1}(m\beta)) \qquad \square$$

In particular, for $12m < D$ it holds for both scales that

$$\sum_{\deg Q = D} T_m(\beta, Q)^2 = 2(1 + o(1))K_s(D)\log(m\beta)$$

Now we proceed as in section 4 and prove the real version of theorem 4.3.

Theorem 5.7. *Fix $r \geq 0$. For $\epsilon D < m < \frac{D}{12r}$*

$$\sum_{\deg Q = D} |S(\beta, \chi_Q) + T_m(\beta, \chi_Q)|^{2r} = O(q^D)$$

Proof. The only difference would be in bounding $\sum_Q |E_3(\chi_Q)|^{2r}$ by q^D, where

$$E_3(\chi_Q) = 2\left|\sum_{\deg p \leq m} W_m(2\deg p)\frac{1}{(\sqrt{q})^{2\deg p}}\sin(2\beta\deg p)\chi_Q(p^2)\right|$$

In section 4 we have used part (2) of lemma 4.1 for this purpose, which we don't have in the real case. However, a simple calculation gives the required bound:

$$\sum_Q |E_3(\chi_Q)|^{2r} \ll$$

$$\ll \sum_Q \left|\sum_{\deg p \leq m/2}\frac{\sin(2\beta\deg p)}{q^{\deg p}} + O\left(\sum_{\substack{\deg p\, \leq\, m/2 \\ p|Q}}\frac{1}{q^{\deg p}}\right) + O\left(\sum_{\deg p = m/2}^{m}\frac{1}{q^{\deg p}}\right)\right|^{2r}$$

Now

$$\sum_{\deg p = m/2}^{m}\frac{1}{q^{\deg p}} \ll \sum_{j = m/2}^{m}\frac{1}{j} = O(1)$$

and

$$\sum_{\deg p \leq m/2}\frac{\sin(2\beta\deg p)}{q^{\deg p}} \ll \sum_{j=1}^{m/2}\frac{\sin(2\beta j)}{j} \ll 1$$

for both scales by lemma A.1. Therefore,

$$\sum_Q |E_3(\chi_Q)|^{2r} \ll \sum_Q \left(\left(\sum_{\substack{\deg p\, \leq\, m/2 \\ p|Q}}\frac{1}{q^{\deg p}}\right) + O(1)\right)^{2r}$$

and by lemma 5.5

$$\sum_Q |E_3(\chi_Q)|^{2r} \ll q^D \qquad \square$$

Applying the exact same reasoning used in the proof of theorem 4.4 , we obtain

Theorem 5.8. *Assume D even, fix $r \geq 0$, and let $0 < \beta < \pi$. Then in both scales:*
(1)

$$\sum_Q S(\beta, \chi_Q)^{2r+1} = O(q^D \log^r(\beta D))$$

(2)

$$\sum_Q S(\beta, \chi_Q)^{2r} = K_s(D)\frac{(2r)!}{r!}\log^r(\beta D) + O(q^D \log^{r-1/2}(\beta D))$$

This implies

Theorem 3. *Let $D \to \infty$ assume even values.*
(1) Fix $0 < \beta < \pi$. The sequence of random variables

$$\frac{S(\beta, \chi_Q)}{\sqrt{2\log D}}$$

has a standard gaussian limiting distribution.
(2) Let $\beta = \beta(D)$ s.t. $0 \neq \beta \to 0$ and $\beta D \to \infty$. Then the sequence of r.v.

$$\frac{S(\beta, \chi_Q)}{\sqrt{2\log(\beta D)}}$$

has a standard gaussian limiting distribution.

A Appendix

Here we prove a technical lemma that was used in various places. It concerns the convergence of certain series in macroscopic and mesoscopic scales.

Lemma A.1. *Let $m \to \infty$. Take β to be either fixed or $\beta = \beta(m) \to 0$ such that $0 < \beta < \pi$ and $m\beta \to \infty$. Then*
(1)

$$\sum_{k=1}^{m} \frac{\sin(2\beta k)}{k} = O(1)$$

(2)

$$\sum_{k=1}^{m} \frac{\sin^2(2\beta k)}{k} = \frac{1}{2}\log(m\beta) + O(1)$$

Proof. (1) By a straighforward calculation,

$$\sum_{j=0}^{n} \sin(2\beta j) = \frac{\sin(n\beta)\sin((n+1)\beta)}{\sin\beta}$$

The case of fixed β is a direct application of Dirichlet's criterion for convergence of sums. Assume now that $\beta \to 0$. Summation by parts implies for all $n \geq 1$

$$\sum_{k=1}^{n} \frac{\sin(2\beta k)}{k} = \sum_{k=1}^{n} \left(\sum_{j=1}^{k} \sin(2\beta j) \right) \left(\frac{1}{k} - \frac{1}{k+1} \right) + \frac{1}{n+1} \sum_{j=1}^{n} \sin(2\beta j) =$$

$$= \frac{1}{\sin\beta} \sum_{k=1}^{n} \frac{\sin(m\beta)\sin((n+1)\beta)}{k(k+1)} + \frac{1}{n+1} \frac{\sin(n\beta)\sin((n+1)\beta)}{\sin\beta}$$

In particular, for $N < n \leq \infty$

$$\left| \sum_{k=N}^{n} \frac{\sin(2\beta k)}{k} \right| \leq \frac{1}{\sin\beta} \sum_{k=N}^{n} \frac{1}{k(k+1)} + \frac{2}{N\sin\beta} \leq \frac{3}{N\sin\beta}$$

Now for every $c > 0$

$$\left| \sum_{k=1}^{\infty} \frac{\sin(2\beta k)}{k} - \sum_{k=1}^{c/\beta} \frac{\sin(2\beta k)}{2\beta k} 2\beta \right| \leq \frac{3\beta}{c\sin\beta} \leq \frac{4}{c}$$

As $\beta \to 0$

$$\sum_{k=1}^{c/\beta} \frac{\sin(2\beta k)}{2\beta k} 2\beta \to \int_0^{2c} \frac{\sin t}{t} dt$$

This implies by letting $c \to \infty$

$$\lim_{\beta \to 0} \sum_{k=1}^{\infty} \frac{\sin(2\beta k)}{k} = \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

and finally

$$\sum_{k=1}^{m} \frac{\sin(2\beta k)}{k} = \sum_{k=1}^{\infty} \frac{\sin(2\beta k)}{k} + O\left(\frac{1}{m\sin\beta}\right) = O(1)$$

(2)

$$\sum_{k=1}^{m} \frac{\sin^2(2\beta k)}{k} = \frac{1}{2}\sum_{k=1}^{m}\frac{1}{k} - \frac{1}{2}\sum_{k=1}^{m}\frac{\cos(2\beta k)}{k} = \frac{1}{2}\log m + O(1) - \frac{1}{2}\sum_{k=1}^{m}\frac{\cos(2\beta k)}{k}$$

We have the identity

$$\sum_{j=0}^{m}\cos(2\beta j) = \frac{\cos(m\beta)\sin((m+1)\beta)}{\sin\beta}$$

And so the case of fixed β is again proved by Dirichlet's criterion. Assume $\beta \to 0$. The integral

$$\int_1^{\infty}\frac{\cos t}{t}dt$$

converges, and so by exactly the same reasoning of part (1)

$$\sum_{k=1/2\beta}^{m}\frac{\cos(2\beta k)}{k} = O(1)$$

Now for $t \in (0,1]$

$$\left(\frac{\cos t}{t}\right)' \ll \frac{1}{t^2}$$

This implies that

$$\sum_{k=1}^{1/2\beta}\frac{\cos(2\beta k)}{2\beta k}2\beta = \int_{2\beta}^{1}\frac{\cos t}{t}dt + O\left(\beta\sum_{k=1}^{1/2\beta}\frac{1}{4\beta^2k^2}2\beta\right)$$

The error term is $O(1)$, while $\cos t = 1 + O(t^2)$ and so

$$\int_{2\beta}^{1}\frac{\cos t}{t}dt = \int_{2\beta}^{1}\frac{dt}{t} + O(1) = -\log\beta + O(1)$$

Putting it together,

$$\sum_{k=1}^{m}\frac{\sin^2(2\beta k)}{k} = \frac{1}{2}\log m + \frac{1}{2}\log\beta + O(1) = \frac{1}{2}\log(m\beta) + O(1) \qquad \square$$

References

[Ro1] Michael Rosen, *Number Theory in Function Fields*, Springer-Verlag, 2002, ISBN 0-387-95335-3.

[Se1] Atle Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip* $0 < t < T$, Avhandlinger utgitt av Det Norske Videnskaps-Akademi i Oslo I. Mat.-Naturv. Klasse (1944), No.1, 1-27.

[Se2] Atle Selberg, *Contributions to the theory of the Riemann zeta-function*, Archiv for Mathematik og Naturvidenskab B.48 (1946), No.5, 89-155.

[Se3] Atle Selberg, *Contributions to the theory of the Dirichlet's L-functions*, Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo I.Mat.-Naturv.Klasse (1946), No.3, 1-62.

[We1] André Weil *Sur les Courbes Algébriques et les Variétés qui s'en Déduisent*, Hermann, Paris (1948).

[We2] André Weil, *Basic Number Theory*, Springer-Verlag New York• Heidelberg• Berlin, 1973, ISBN 0-387-06177-0.