# The Distribution of Lattice Points in Elliptic Annuli

by

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# ABSTRACT

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We study the distribution of the number of lattice points lying in thin elliptical annuli. It has been conjectured by Bleher and Lebowitz that, if the width of the annuli tend to zero while their area tends to infinity, then the value distribution of this number, normalized to have zero mean and unit variance, is standard Gaussian. This has been proved by Hughes and Rudnick for circular annuli whose width shrink to zero sufficiently slowly.

Our work consists of two main parts. In the first part, we consider the 1-parameter family of for ellipses, whose axes lie on the coordinate axes. We prove the conjecture for a "generic" class of ellipses, whose aspect ratio is transcendental and Diophantine in a strong sense, also assuming the width shrinks slowly to zero.

In the second part, we generalize this result, establishing the central limit theorem for a "generic" ellipse within the 2-parameter family of all the ellipses. One of the obstacles of applying the technique of Hughes-Rudnick in this case, is the existence of so-called close pairs of lattice points. In order to overcome this difficulty, we bound the rate of occurence of this phenomenon by extending some of the work of Eskin-Margulis-Mozes on the quantitative Openheim conjecture. In the case of a

rectangular lattice, it is easy to bound, using properties of the divisor function.

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# TABLE OF CONTENTS

**CHAPTER**

## CHAPTER I

# Introduction

Let $B$ be an open convex domain in the plane containing the origin, with a smooth boundary, and which is strictly convex (the curvature of the boundary never vanishes). Let

$$N_B(t) := \#Z^2 \cap tB,$$

be the number of integral points in the $t$-dilate of $B$. As is well-known, as $t \to \infty$, $N_B(t)$ is approximated by the area of $tB$, that is

$$N_B(t) \sim At^2, \tag{1.1}$$

where $A$ is the area of $B$.

A classical problem is to bound the size of the remainder

$$\Delta_B(t) := N_B(t) - At^2.$$

A simple geometric argument gives

$$\Delta_B(t) = O(t), \tag{1.2}$$

that is a bound in terms of the length of the boundary. It is known that $\Delta_B$ is much smaller than the classical bound, as Sierpinski [22] proved

$$\Delta_B(t) = O(t^{2/3}).$$

Since then, the exponent $2/3$ in the last estimate has been improved due to the works by many different researchers (see [14]). It is conjectured that one could replace the exponent by $1/2 + \epsilon$ for every $\epsilon > 0$.

A different problem is to study the value distribution of the normalized error term, namely, of

$$F_B(t) := \frac{\Delta_B(t)}{\sqrt{t}} = \frac{N_B(t) - At^2}{\sqrt{t}}.$$

Heath-Brown [12] treats this problem for $B = B(0,1)$, the unit circle, and shows that there exists a probability density $p(x)$, such that for every bounded continuous function $g(x)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F_{B(0,1)}(t))dt = \int_{-\infty}^{\infty} g(x)p(x)dx.$$

Somewhat surprisingly, the $p(t)$ is not a Gaussian: it decays as $x \to \infty$ roughly as $\exp(-x^4)$, and it can be extended to an entire function on a complex plane. Bleher [4] establishes an analogue to Heath-Brown's theorem for general ovals.

Motivated in parts by questions coming from mathematical physics, we will concentrate on counting lattice points on annuli, namely, integer points in

$$(t+\rho)B \setminus tB,$$

that is, we study the remainder term of

$$N_B(t,\,\rho) := N_B(t+\rho) - N_B(t),$$

where $\rho = \rho(t)$ is the width of annulus, depending on the inner radius $t$. The "expected" number of points is the area $A(2t\rho + \rho^2)$ of the annulus. Thus the corresponding *normalized* remainder term is:

$$S_B(t,\,\rho) := \frac{N_B(t+\rho) - N_B(t) - A(2t\rho+\rho^2)}{\sqrt{t}}.$$

The statistics of $S_B(t,\,\rho)$ vary depending to the size of $\rho(t)$. Of our particular interest are the following regimes:

(1) The *microscopic regime*, $\rho t$ is constant. It was conjectured by Berry and Tabor [7] that the statistics of $N_B(t,\,\rho)$ are Poissonian. Eskin, Margulis and Mozes [10] proved that the pair correlation function (which is roughly equivalent to the variance of $N_B(t,\,\rho)$), is consistent with the Poisson-random model.

(2) The *"global"*, or *"macroscopic"*, *regime*, $\rho(t) \to \infty$ (but $\rho = o(t)$). In such a case, Bleher and Lebowitz [5] showed that for a wide class of $B$'s, $S_B(t,\,\rho)$ has a limiting distribution with tails which decay roughly as $\exp(-x^4)$.

(3) The *intermediate* or *"mesoscopic"*, *regime*, $\rho \to 0$ (but $\rho t \to \infty$). If $B$ is the inside of a "generic" ellipse

$$\Gamma = \Big\{(x_1,\,x_2):\; x_1^2 + \alpha^2 x_2^2 = 1\Big\}, \tag{1.3}$$

with $\alpha$ is Diophantine, the variance of $S_B(t,\,\rho)$ was computed in [2] to be asymptotic to

$$\sigma^2 := \frac{8\pi}{\alpha}\cdot\rho \tag{1.4}$$

For the circle ($\alpha = 1$), the value is $16\rho\log\frac{1}{\rho}$.

Bleher and Lebowitz [5] conjectured that $S_B(t,\,\rho)/\sigma$ has a standard Gaussian distribution. In 2004, Hughes and Rudnick [13] established the Gaussian distribution for the unit circle, provided that $\rho(t) \gg t^{-\delta}$ for every $\delta > 0$.

In chapter II of this thesis, we prove the Gaussian distribution for the normalized remainder term of "generic" *elliptic* annuli, whose axes are lying on the coordinate axes. Equivalently to counting integral points inside such elliptic annuli, we will count $\Lambda$-points inside $B(0,1)$-annuli, where $\Lambda = \langle 1,\,i\alpha\rangle$ is a *rectangular lattice* (we make the natural identification of $i$ with $(0,\,1)$).

Given a lattice $\Lambda$ as above, with determinant

$$d := \det(\Lambda) = \alpha,$$

we denote the corresponding counting function $N_\Lambda$, that is,

$$N_\Lambda = \#\{\vec{n} \in \Lambda : \ |\vec{n}| \leq t\}.$$

In addition, we define

$$S_\Lambda(t,\, \rho) = \frac{N_\Lambda(t+\rho) - N_\Lambda(t) - \frac{\pi}{d}(2t\rho + \rho^2)}{\sqrt{t}}. \tag{1.5}$$

Obviously, we have

$$S_\Lambda(t,\, \rho) = S_B(t,\, \rho)$$

for an ellipse $B$, which is the inside of $\Gamma$ as in (1.3).

We say that a real number $\alpha$ is *strongly Diophantine*, if for every $n \geq 1$, there is some $K > 0$, such that for integers $a_j$ with $\sum\limits_{j=0}^{n} a_j \alpha^j \neq 0$,

$$\left| \sum_{j=0}^{n} a_j \alpha^j \right| \gg_n \frac{1}{\left( \max\limits_{0 \leq j \leq n} |a_j| \right)^K}.$$

This holds for any algebraic $\alpha$, for $\alpha = e$, and almost every real $\alpha$, see section 2.2.2. We prove:

**Theorem 1.1.1.** *Let $\Lambda = \langle 1,\, i\alpha \rangle$, with $\alpha$ transcendental and strongly Diophantine. Assume that $\rho = \rho(T) \to 0$, but for every $\delta > 0$, $\rho \gg T^{-\delta}$. Then for every interval $\mathcal{A}$,*

$$\lim_{T \to \infty} meas \left\{ t \in [T,\, 2T] : \ \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}} dx, \tag{1.6}$$

*where $\sigma$ is given by* (1.4).

This proves the conjecture of Bleher and Lebowitz in the case of a rectangular lattice (this corresponds to an ellipse with axes lying on the coordinate axes).

**Remarks:** 1. In the formulation of theorem 1.1.1, we assume for technical reasons, that $\rho$ is a function of $T$ and independent of $t \in [T, 2T]$. However one may easily see that since $\rho$ may not decay rapidly, one may refine the result for $\rho = \rho(t)$.

2. We compute statistics of the remainder term when the radius is around $T$. A natural choice is assuming that the radius is uniformly distributed in the interval $[T, 2T]$.

Our case offers some marked differences from that of standard circular annuli treated in [13]. To explain these, we note that there are two main steps in treating these distribution problems: The first step is to compute the moments of a smoothed version of $S_B$, defined in section 2.1. We will show in section 2.2 that the moments of the smooth counting function are Gaussian and that will suffice for establishing a normal distribution for the smooth version of our problem. The second step (section 2.4) is to recover the distribution of the original counting function $S_B$ by estimating the variance of the difference between $S_B$ and its smooth version. The proof of that invokes a truncated Poisson summation formula for the number of points of a general lattice which lie in a disk, stated and proved in section 2.3.

The passage from circular annuli to general elliptical annuli with axes lying on the coordinate axes, gives rise to new problems in both steps. The reason is that to study the counting functions one uses Poisson summation to express the counting functions as a sum over a certain lattice, that is as a sum over closed geodesics of the corresponding flat torus. Unlike the case of the circle, for a generic ellipse the sum is over a lattice where the squared lengths of vectors are no longer integers but of the form $n^2 + m^2\alpha^{-2}$, where $n, m \in \mathbb{Z}$ and $\alpha$ is the aspect ratio of the ellipse.

One new feature present in this case is that these lengths can *cluster* together, or, more generally, one may approximate zero too well by the means of linear combinations of lengths. This causes difficulties both in bounding the variance between the original counting function and its smoothed version, especially in the truncated summation formula of section 2.3, and in showing that the moments of the smooth counting function are given by "diagonal-like" contributions. This clustering can be controlled when $\alpha$ is strongly Diophantine.

Another problem we have to face, in evaluating moments of the smooth counting function, is the possibility of non-trivial correlations in the length spectrum. Their possible existence (e.g. in the case of algebraic aspect ratio) obscures the nature of the main term (the diagonal-like contribution) at this time. If $\alpha$ is transcendental this problem can be overcome, see proposition 2.2.8.

Our next goal is to generalize the result for general elliptical annuli. This is done in chapter III. By a rotation and dilation (which does not effect the counting function), we may assume, with no loss of generality, that $\Lambda$ admits a basis one of whose elements is the vector $(1, 0)$, that is $\Lambda = \left\langle 1,\, \alpha + i\beta \right\rangle$.

Some of the work done in chapter II, extends quite naturally for the 2-parameter family of planar lattices $\left\langle 1,\, \alpha+i\beta \right\rangle$. That is, in this case we will require the algebraic independence of $\alpha$ and $\beta$, as well as a strong Diophantine property of the pair $(\alpha,\, \beta)$ (to be defined), rather than the transcendence and a strong Diophantine property of the aspect ratio of the ellipse, as in theorem 1.1.1.

We say that a pair of numbers $(\alpha,\, \beta)$ is *strongly Diophantine*, if for every *fixed* natural $n$, there exists a number $K_1 > 0$, such that for every integral polynomial $p(x,\, y) = \sum\limits_{i+j \leq n} a_{i,\,j} x^i y^j$ of degree $\leq n$, we have

$$|p(\alpha,\, \beta)| \gg_n \frac{1}{\max\limits_{i+j \leq n} |a_{i,\,j}|^{K_1}},$$

whenever $p(\alpha, \beta) \neq 0$. This holds for almost all real pairs $(\alpha,\, \beta)$, see section 3.1.2.

**Theorem 1.1.2.** *Let* $\Lambda = \left\langle 1,\, \alpha + i\beta \right\rangle$ *where* $(\alpha,\, \beta)$ *is algebraically independent and strongly Diophantine pair of real numbers. Assume that* $\rho = \rho(T) \to 0$*, but for every* $\delta > 0$*,* $\rho \gg T^{-\delta}$*. Then for every interval* $\mathcal{A}$*,*

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{ t \in [T,\, 2T] : \; \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}} dx, \tag{1.7}$$

*where the variance is given by*

$$\sigma^2 := \frac{4\pi}{\beta} \cdot \rho. \tag{1.8}$$

**Remark:** Note that the variance $\sigma^2$ is $\alpha$-independent, since the determinant $\det(\Lambda) = \beta$.

One of the features of a rectangular lattice is that it is quite easy to show that the number of so-called close pairs of lattice points or pairs of points lying within a narrow annulus is bounded by essentially its average (see lemma 2.4.2). This particular feature of the rectangular lattices was exploited while reducing the computation of the moments to the ones of a smooth counting function (we call it "unsmoothing"). In order to prove an analogous bound for a general lattice, we extend a result from Eskin, Margulis and Mozes [9] for our needs to obtain proposition 3.2.1. We believe that this proposition is of independent interest.

## CHAPTER II

# The Rectangular Case

### 2.1 Smoothing

We apply the same smoothing as in [13]: let $\chi$ be the indicator function of the unit disc and $\psi$ a nonnegative, smooth, even function on the real line, of total mass unity, whose Fourier transform, $\hat{\psi}$ is smooth and has compact support [1]. One should notice that

$$N_\Lambda(t) = \sum_{\vec{n}\in\Lambda} \chi\left(\frac{\vec{n}}{t}\right). \tag{2.1}$$

Introduce a rotationally symmetric function $\Psi$ on $\mathbb{R}^2$ by setting $\hat{\Psi}(\vec{y}) = \hat{\psi}(|\vec{y}|)$, where $|\cdot|$ denotes the standard Euclidian norm. For $\epsilon > 0$, set

$$\Psi_\epsilon(\vec{x}) = \frac{1}{\epsilon^2}\Psi\left(\frac{\vec{x}}{\epsilon}\right).$$

Define in analogy with (2.1) a *smooth* counting function

$$\tilde{N}_{\Lambda,M}(t) = \sum_{\vec{n}\in\Lambda} \chi_\epsilon(\frac{\vec{n}}{t}), \tag{2.2}$$

with $\epsilon = \epsilon(M)$, $\chi_\epsilon = \chi * \Psi_\epsilon$, the convolution of $\chi$ with $\Psi_\epsilon$. In what will follow,

$$\epsilon = \frac{1}{t\sqrt{M}}, \tag{2.3}$$

[1] To construct such a function, just take a function $\phi$ with compact support and set $\hat{\psi} = \phi * \phi^*$ where $\phi^*(y) := \overline{\phi(-y)}$. Then $\psi = |\check{\phi}|^2$ is nonnegative.

where $M = M(T)$ is the smoothness parameter, which tends to infinity with $t$.

We are interested in the distribution of

$$\tilde{S}_{\Lambda, M, L}(t) = \frac{\tilde{N}_{\Lambda, M}(t + \frac{1}{L}) - \tilde{N}_{\Lambda, M}(t) - \frac{\pi}{d}(\frac{2t}{L} + \frac{1}{L^2})}{\sqrt{t}}, \tag{2.4}$$

which is the smooth version of $S_\Lambda(t, \rho)$. We assume that for every $\delta > 0$, $L = L(T) = O(T^\delta)$, which corresponds to the assumption of theorem 1.1.1 regarding $\rho := \frac{1}{L}$. However, we will work with a smooth probability space rather than just the Lebesgue measure. For this purpose, introduce $\omega \geq 0$, a smooth function of total mass unity, such that both $\omega$ and $\hat{\omega}$ are rapidly decaying, namely

$$|\omega(t)| \ll \frac{1}{(1+|t|)^A}, \quad |\hat{\omega}(t)| \ll \frac{1}{(1+|t|)^A},$$

for every $A > 0$.

Define the averaging operator

$$\langle f \rangle_T = \frac{1}{T} \int_{-\infty}^{\infty} f(t) \omega(\frac{t}{T}) dt,$$

and let $\mathbb{P}_{\omega, T}$ be the associated probability measure:

$$\mathbb{P}_{\omega, T}(f \in \mathcal{A}) = \frac{1}{T} \int_{-\infty}^{\infty} 1_{\mathcal{A}}(f(t)) \omega(\frac{t}{T}) dt,$$

We will prove the following theorem in section 2.2.

**Theorem 2.1.1.** *Suppose that $M(T)$ and $L(T)$ are increasing to infinity with $T$, such that $M = O(T^\delta)$ for all $\delta > 0$, and $L/\sqrt{M} \to 0$. Then if $\alpha$ is* transcendental *and* strongly Diophantine, *we have for $\Lambda =< 1, i\alpha >$,*

$$\lim_{T \to \infty} \mathbb{P}_{\omega, T} \left\{ \frac{\tilde{S}_{\Lambda, M, L}}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-\frac{x^2}{2}} d$$

*for any interval $\mathcal{A}$, where $\sigma^2 := \frac{8\pi}{dL}$.*

## 2.2 The distribution of $\tilde{S}_{\Lambda,\,M,\,L}$

We start from a well-known definition.

**Definition:** A number $\mu$ is called *Diophantine*, if $\exists K > 0$, such that for a rational $p/q$,

$$\left|\mu - \frac{p}{q}\right| \gg_\mu \frac{1}{q^K}, \tag{2.5}$$

where the constant involved in the " $\gg$ "-notation depends only on $\mu$. Khintchine proved that *almost all* real numbers are Diophantine (see, e.g. [18], pages 60-63).

It is obvious from the definition, that $\mu$ is Diophantine iff $\frac{1}{\mu}$ is such. For the rest of this section, we will assume that $\Lambda^* = \big\langle 1,\, i\beta \big\rangle$ with a Diophantine $\kappa := \beta^2$, which satisfies (2.5) with

$$K = K_0, \tag{2.6}$$

where $\Lambda^*$ is the *dual* lattice, that is $\beta := \frac{1}{\alpha}$. We may assume that $\kappa$ is Diophantine, since theorem 1.1.1 (and theorem 2.1.1) assume $\alpha$'s being *strongly Diophantine* (see the definition later in this section), which implies, in particular, that $\alpha, \beta$ and $\kappa$ are Diophantine.

We will need a generalization of lemma 3.1 in [13] to a general lattice $\Lambda$ rather than $\mathbb{Z}^2$.

**Lemma 2.2.1.** *As $t \to \infty$,*

$$\tilde{N}_{\Lambda,M}(t) = \frac{\pi t^2}{d} - \frac{\sqrt{t}}{d\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} \cdot \hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \tag{2.7}$$

*where, again $\Lambda^*$ is the dual lattice.*

*Proof.* The proof is essentially the same as the one which obtains the original lemma (see [13], page 642). Using *Poisson summation formula* on (2.2) and estimating $\hat{\chi}(t\vec{k})$ by the well-known asymptotics of the Bessel $J_1$ function, we get:

$$\tilde{N}_{\Lambda,M}(t) = \frac{\pi t^2}{d} - \frac{\sqrt{t}}{d\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \left\{ \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} \cdot \hat{\psi}\left(\epsilon t|\vec{k}|\right) + O\left(\frac{\hat{\psi}(\epsilon t|\vec{k}|)}{t|\vec{k}|^{\frac{5}{2}}}\right)\right\},$$

where we get the main term for $\vec{k} = 0$. Finally, we obtain (2.7) using (2.3). The contribution of the error term is obtained due to the convergence of $\sum\limits_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{|\vec{k}|^{\frac{5}{2}}}$ as well as the fact that $\hat{\psi}(x) \ll 1$. $\square$

Unlike the standard lattice, if $\Lambda = \langle 1,\, i\alpha\rangle$ with an irrational $\alpha^2$, then clearly there are no nontrivial multiplicities, that is

**Lemma 2.2.2.** *Let $\vec{a_i} = (n_i,\, m_i \cdot \alpha) \in \Lambda,\ i = 1, 2,$ with an irrational $\alpha^2$. If $|\vec{a_1}| = |\vec{a_2}|$, then $n_1 = \pm n_2$ and $m_1 = \pm m_2$.*

By the definition of $\tilde{S}_{\Lambda,\,M,\,L}$ in (2.4) and appropriately manipulating the sum in (2.7) we obtain the following

**Corollary 2.2.3.**

$$\tilde{S}_{\Lambda,\,M,\,L}(t) = \frac{2}{d\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k}| + \frac{\pi}{4}\right)\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) + O\left(\frac{1}{\sqrt{t}}\right), \tag{2.8}$$

We used

$$\sqrt{t + \frac{1}{L}} = \sqrt{t} + O\left(\frac{1}{\sqrt{t}L}\right) \tag{2.9}$$

in order to change $\sqrt{t + \frac{1}{L}}$ multiplying the sum in (2.7) for $N_\Lambda(t + \frac{1}{L})$ by $\sqrt{t}$. We use a smooth analogue of the simplest bound (1.2) in order to bound the cost of this change to the error term.

One should note that $\hat{\psi}$'s being compactly supported means that the sum essentially truncates at $|\vec{k}| \approx \sqrt{M}$.

*Proof of theorem 2.1.1.* We will show that the moments of $\tilde{S}_{\Lambda, M, L}$ corresponding to the smooth probability space (i.e. $\langle \tilde{S}^m_{\Lambda, M, L} \rangle_T$, see section 2.1) converge to the moments of the normal distribution with zero mean and variance which is given by theorem 2.1.1. This allows us to deduce that the distribution of $\tilde{S}_{\Lambda, M, L}$ converges to the normal distribution as $T$ approaches infinity, precisely in the sense of theorem 2.1.1.

First, we show that the mean is $O(\frac{1}{\sqrt{T}})$, unconditionally on the Diophantine properties of $\alpha$. Since $\omega$ is real,

$$\left| \left\langle \sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right)\right\rangle_T \right| = \left| \Im m\left\{ \hat{\omega}\big(-T|\vec{k}|\big) e^{i\pi(\frac{|\vec{k}|}{L}+\frac{1}{4}} \right\} \right| \ll \frac{1}{T^A |\vec{k}|^A}$$

for any $A>0$, where we have used the rapid decay of $\hat{\omega}$. Thus

$$\left| \left\langle \tilde{S}_{\Lambda, M, L} \right\rangle_T \right| \ll \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{T^A |\vec{k}|^{A+3/2}} + O\left(\frac{1}{\sqrt{T}}\right) \ll O\left(\frac{1}{\sqrt{T}}\right),$$

due to the convergence of $\sum\limits_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{|\vec{k}|^{A+3/2}}$, for $A > \frac{1}{2}$

Now define

$$(2.10)\quad \mathcal{M}_{\Lambda, m} := \left\langle \left( \frac{2}{d\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi\big(t+\frac{1}{2L}\big)|\vec{k}|+\frac{\pi}{4}\right) \hat{\psi}\big(\frac{|\vec{k}|}{\sqrt{M}}\big) \right)^m \right\rangle_T$$

Then from (2.8), the binomial formula and the Cauchy-Schwartz inequality,

$$\left\langle \big(\tilde{S}_{\Lambda, M, L}\big)^m \right\rangle_T = \mathcal{M}_{\Lambda, m} + O\left( \sum_{j=1}^{m} \binom{m}{j} \frac{\sqrt{\mathcal{M}_{2m-2j}}}{T^{j/2}} \right)$$

Proposition 2.2.4 together with proposition 2.2.7 allow us to deduce the result of theorem 2.1.1 for a *transcendental strongly Diophantine* $\beta^2$. Clearly, $\alpha$'s being transcendental strongly Diophantine is sufficient. $\square$

### 2.2.1 The variance

The variance was first computed by Bleher and Lebowitz [2] and we will give a version suitable for our purpose. This will help the reader to understand our computation of higher moments.

**Proposition 2.2.4.** *Let $\alpha$ be Diophantine and $\Lambda = \langle 1, i\alpha \rangle$. Then if for some fixed $\delta > 0$, $M = O\big(T^{\frac{1}{K_0+1/2+\delta}}\big)$ as $T \to \infty$, then*

$$\left\langle \big(\tilde{S}_{\Lambda, M, L}\big)^2 \right\rangle_T \sim \sigma^2 := \frac{2}{d^2\pi^2} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} r(\vec{k}) \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} \hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right),$$

*where*

$$r(\vec{n}) = \begin{cases} 1, & \vec{n} = (0, 0) \\ 2, & \vec{n} = (x, 0) \text{ or } (0, y) \\ 4, & \text{otherwise} \end{cases}, \tag{2.11}$$

*is the "multiplicity" of $|\vec{n}|$. Moreover, if $L \to \infty$, but $L/\sqrt{M} \to 0$, then*

$$\sigma^2 \sim \frac{8\pi}{dL} \tag{2.12}$$

*Proof.* Expanding out (2.10), we have

$$\begin{aligned} \mathcal{M}_{\Lambda, 2} := \frac{4}{d^2\pi^2} \sum_{\vec{k}, \vec{l} \in \Lambda^* \setminus \{0\}} & \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right) \sin\left(\frac{\pi|\vec{l}|}{L}\right) \hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) \hat{\psi}\left(\frac{|\vec{l}|}{\sqrt{M}}\right)}{|\vec{k}|^{\frac{3}{2}} |\vec{l}|^{\frac{3}{2}}} \\ & \times \left\langle \sin\left(2\pi\left(t + \frac{1}{2L}\right)|\vec{k}| + \frac{\pi}{4}\right) \sin\left(2\pi\left(t + \frac{1}{2L}\right)|\vec{l}| + \frac{\pi}{4}\right) \right\rangle_T \end{aligned} \tag{2.13}$$

Now, it is easy to check that the average of the second line of the previous equation is:

$$
\begin{aligned}
(2.14) \qquad & \frac{1}{4}\Big[\hat{\omega}\big(T(|\vec{k}|-|\vec{l}|)\big)e^{i\pi(1/L)(|\vec{l}|-|\vec{k}|)}+ \\
& \hat{\omega}\big(T(|\vec{l}|-|\vec{k}|)\big)e^{i\pi(1/L)(|\vec{k}|-|\vec{l}|)}+ \\
& \hat{\omega}\big(T(|\vec{k}|+|\vec{l}|)\big)e^{-i\pi(1/2+(1/L)(|\vec{k}|+|\vec{l}|))}- \\
& \hat{\omega}\big(-T(|\vec{k}|+|\vec{l}|)\big)e^{i\pi(1/2+(1/L)(|\vec{k}|+|\vec{l}|))}\Big]
\end{aligned}
$$

Recall that the support condition on $\hat{\psi}$ means that $\vec{k}$ and $\vec{l}$ are both constrained to be of length $O(\sqrt{M})$, and so the off-diagonal contribution (that is for $|\vec{k}| \neq |\vec{l}|$ ) of the first two lines of (2.14) is

$$\ll \sum_{\substack{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|,\,|\vec{k'}|\leq\sqrt{M}}} \frac{M^{A(K_0+1/2)}}{T^A} \ll \frac{M^{A(K_0+1/2)+2}}{T^A} \ll T^{-B},$$

for every $B > 0$, using lemma 2.2.5, the fact that $|\vec{k}|, |\vec{l}| \gg 1$, $|\hat{\psi}| \ll 1$, and the assumption regarding $M$. We may use lemma 2.2.5 since we have assumed in the beginning of this section that $\kappa$ is Diophantine.

Obviously, the contribution to (2.13) of the two last lines of (2.14) is negligible both in the diagonal and off-diagonal cases, and so we are to evaluate the diagonal approximation of (2.13), changing the second line of (2.13) by 1/2, since the first two lines of (2.14) are 2. That proves the first statement of the proposition. To find the asymptotics, we take a large parameter $Y = Y(T) > 0$ (which is to be chosen later), and write:

$$
\begin{aligned}
\sum_{\substack{\vec{k},\,\vec{k'}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|=|\vec{k'}|}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right) &= \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} r(\vec{k})\frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}\hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right) \\
&= \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2\leq Y}} + \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|^2> Y}} := I_1 + I_2,
\end{aligned}
$$

Now for $Y = o(M)$, $\hat{\psi}^2\big(\frac{|\vec{k}|}{\sqrt{M}}\big) \sim 1$ within the constraints of $I_1$, and so

$$I_1 \sim \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} r(\vec{k})\frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3}.$$

Here we may substitute $r(\vec{k}) = 4$, since the contribution of vectors of the form $(x,\,0)$ and $(0,\,y)$ is $O(\frac{1}{L^2})$: representing their contribution as a 1-dimensional Riemann sum.

The sum in

$$4\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} = \frac{4}{L}\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|^2\le Y}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{\left(\frac{|\vec{k}|}{L}\right)^3}\frac{1}{L^2}.$$

is a 2-dimensional Riemann sum of the integral

$$\iint\limits_{1/L^2\ll x^2+\kappa y^2\le Y/L^2} \frac{\sin^2\left(\pi\sqrt{x^2+\kappa y^2}\right)}{|x^2+\kappa y^2|^{3/2}}dxdy \sim \frac{2\pi}{\beta}\int\limits_{\frac{1}{L}}^{\frac{\sqrt{Y}}{L}} \frac{\sin^2(\pi r)}{r^2}dr \to d\pi^3,$$

provided that $Y/L^2\to\infty$, since $\int\limits_0^\infty \frac{\sin^2(\pi r)}{r^2}dr = \frac{\pi^2}{2}$. We have changed the coordinates to the usual elliptic ones. And so,

$$I_1 \sim \frac{4d\pi^3}{L}$$

Next we will bound $I_2$. Since $\hat{\psi} \ll 1$, we may use the same change of variables to obtain:

$$I_2 \ll \frac{1}{L}\iint\limits_{x^2+\kappa y^2\ge Y/L^2} \frac{\sin^2\left(\pi\sqrt{x^2+\kappa y^2}\right)}{|x^2+\kappa y^2|^{3/2}}dxdy \ll \frac{1}{L}\int\limits_{\sqrt{Y}/L}^{\infty}\frac{dr}{r^2} = o\left(\frac{1}{L}\right).$$

This concludes the proposition, provided we have managed to choose $Y$ with $L^2 = o(Y)$ and $Y = o(M)$. Such a choice is possible by the assumption of the proposition regarding $L$.

$\square$

**Lemma 2.2.5.** *Suppose that $\vec{k}$, $\vec{k'} \in \Lambda^*$ with $|\vec{k}|$, $|\vec{k'}| \le \sqrt{M}$. Then if $|\vec{k}| \ne |\vec{k'}|$,*

$$\big||\vec{k}| - |\vec{k'}|\big| \gg M^{-(K_0+1/2)}$$

*Proof.*

$$\big||\vec{k}| - |\vec{k'}|\big| = \frac{\big||\vec{k}|^2 - |\vec{k'}|^2\big|}{|\vec{k}| + |\vec{k'}|} \gg \frac{M^{-K_0}}{\sqrt{M}} = M^{-(K_0+1/2)},$$

by (2.5) and (2.6). □

### 2.2.2 The higher moments

In order to compute the higher moments we will prove that the main contribution comes from the so-called *diagonal* terms (to be explained later). In order to be able to bound the contribution of the *off-diagonal* terms, we restrain ourselves to "generic" numbers, which are given in the following definition:

**Definition:** We call a number $\eta$ strongly Diophantine, if it satisfies the following property: for any fixed $n$, there exists $K_1 \in \mathbb{N}$ such that for an integral polynomial $P(x) = \sum\limits_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$, with $P(\eta) \ne 0$ we have

$$\big|P(\eta)\big| \gg_{\eta,n} h(P)^{-K_1},$$

where $h(P) = \max\limits_{0 \le i \le n} |a_i|$ is the height of $P$.

The fact that the strongly Diophantine numbers are "generic" follows from various classical papers, e.g. [17].

Obviously, the strong Diophantine property implies the Diophantine property. Just as in the case of Diophantine numbers $\eta$ is strongly Diophantine, iff $\frac{1}{\eta}$ is such. Moreover, if $\eta$ is strongly Diophantine, then so is $\eta^2$. As a concrete example of a transcendental strongly Diophantine number, the inequality proven by Baker [1] implies that for any *rational* $r \ne 0$, $\eta = e^r$ satisfies the desired property.

We would like to make some brief comments concerning the number $K_1$, which appears in the definition of a strongly Diophantine number, although the form presented is sufficient for all our purposes.

Let $\eta$ be a *real* number. One defines $\theta_k(\eta)$ to be $\frac{1}{k}$ times the supremum of the real numbers $\omega$, such that $|P(\eta)| < h(P)^{-\omega}$ for infinitely many polynomials $P$ of degree $k$. Clearly,

$$\theta_k(\eta) = \frac{1}{k} \inf\{\omega : \ |P(\eta)| \gg_{\omega,k} h^{-\omega}, \deg P = k\}.$$

It is well known [23], that $\theta_k(\eta) \geq 1$ for all *transcendental* $\eta$. In 1932, Mahler [17] proved that $\theta_k(\eta) \leq 4$ for almost all real $\eta$, and that allows us to take any $K_1 > 4n$. He conjectured that

$$\theta_k(\eta) \leq 1$$

which was proved in 1964 by Sprindẑuk [20], [21], making it legitimate to choose any $K_1 > n$.

Sprindẑuk's result is analogous to Khintchin's theorem which states that almost no $k$-tuple in $\mathbb{R}^k$ is *very well approximable* (see e.g. [18], theorem 3A), for submanifold $M \subset \mathbb{R}^k$, defined by

$$M = \{(x, x^2, \dots, x^k) : \ x \in \mathbb{R}\}.$$

The proof of this conjecture has eventually let to development of a new branch in approximation theory, usually referred to as "Diophantine approximation with dependent quantities" or "Diophantine approximation on manifolds". A number of quite general results were proved for a manifold $M$, see e.g. [15].

We prove the following simple lemma which will eventually allow us to exploit the strong Diophantine property of the aspect ratio of the ellipse.

**Lemma 2.2.6.** *If $\eta > 0$ is strongly Diophantine, then it satisfies the following prop-*

*erty: for any fixed natural $m$, there exists $K \in \mathbb{N}$, such that if $z_j = a_j^2 + \eta b_j^2 \ll M$, and $\epsilon_j = \pm 1$ for $j = 1, \ldots, m$, with integral $a_j$, $b_j$ and if $\sum_{j=1}^{m} \epsilon_j \sqrt{z_j} \neq 0$, then*

$$|\sum_{j=1}^{m} \epsilon_j \sqrt{z_j}| \gg_{\eta, m} M^{-K}. \tag{2.15}$$

*Proof.* Let $m$ be given. We prove that every number $\eta$ that satisfies the property of the definition of a strongly Diophantine number with $n = 2^{m-1}$, satisfies the inequality (2.15) for some K, which will depend on $K_1$.

Let us $\{\sqrt{z_j}\}_{j=1}^{m}$ be given. Suppose first, that there is no $\{\delta_j\}_{j=1}^{m} \in \{\pm 1\}^m$ with $\sum_{j=1}^{m} \delta_j \sqrt{z_j} = 0$. Let us consider

$$Q = Q(z_1, \ldots, z_m) := \prod_{\{\delta_j\}_{j=1}^{m} \in \{\pm 1\}^m} \sum_{j=1}^{m} \delta_j \sqrt{z_j} \neq 0.$$

Now $Q = R\big(\sqrt{z_1}, \ldots, \sqrt{z_m}\big)$, where

$$R\big(x_1, \ldots, x_m\big) := \prod_{\{\delta_j\}_{j=1}^{m} \in \{\pm 1\}^m} \sum_{j=1}^{m} \delta_j x_j.$$

Obviously, $R$ is a polynomial with integral coefficients of degree $2^m$ such that for each vector $\underline{\delta} = (\delta_j) = (\pm 1)$, $R(\delta_1 x_1, \ldots, \delta_m x_m) = R(x_1, \ldots, x_m)$, and thus, $Q(z_1, \ldots, z_m)$ is an integral polynomial in $z_1, \ldots, z_m$ of degree $2^{m-1}$. Therefore, $Q = P(\eta)$, where $P$ is a polynomial of degree $2^{m-1}$, $P = \sum_{j=0}^{2^{m-1}} c_i x^i$, with $c_i \in \mathbb{Z}$, such that $c_i = P_i(a_1, \ldots, a_m, b_1, \ldots, b_m)$, where $P_i$ are polynomials. Thus there exists $K_2$, such that $c_i \ll M^{K_2}$, and so, by the definition of strongly Diophantine numbers, $Q \gg_{\eta, m} M^{-K_2 K_1}$. We conclude the proof of lemma 2.2.6 in this case by

$$|\sum_{j=1}^{m} \epsilon_j \sqrt{z_j}| = \frac{|Q|}{\left| \prod_{\{\delta_j\}_{j=1}^{m} \neq \{\epsilon_j\}_{j=1}^{m}} \sum_{j=1}^{m} \delta_j \sqrt{z_j} \right|} \gg_{\eta, m} M^{-(K_2 K_1 + (2^m - 1)/2)},$$

and so, setting $K := K_2 K_1 + \frac{(2^m - 1)}{2}$, we obtain the result of the current lemma in this case.

Next, suppose that

$$\sum_{i=1}^{m} \delta_j \sqrt{z_j} = 0 \tag{2.16}$$

for some (given) $\{\delta_i\}_{j=1}^{m} \in \{\pm 1\}^m$. Denote $S := \{j : \epsilon_j = \delta_j\}$, $S' = \{1, \ldots, m\} \setminus S$. One should notice that

$$\emptyset \subsetneqq S,\, S' \subsetneqq \{1, \ldots, m\}. \tag{2.17}$$

Writing (2.16) in the new notations, we obtain:

$$\sum_{j \in S} \epsilon_j \sqrt{z_i} - \sum_{j \in S'} \epsilon_j \sqrt{z_i} = 0,$$

Finally,

$$0 \neq \Big| \sum_{j=1}^{m} \epsilon_j \sqrt{z_j} \Big| = 2 \Big| \sum_{j \in S'} \epsilon_j \sqrt{z_j} \Big| \gg_{\eta, m} M^{-K}$$

for some $K$ by induction, due to (2.17). □

**Proposition 2.2.7.** *Let $m \in \mathbb{N}$ be given. Suppose that $\alpha^2$ is* *transcendental* *and* *strongly Diophantine* *which satisfy the property of lemma 2.2.6 for the given $m$, with $K = K_m$. Denote $\Lambda = \langle 1, i\alpha \rangle$. Then if $\mathcal{M} = O\big(T^{\frac{1-\delta}{K_m}}\big)$ for some $\delta > 0$, and if $L \to \infty$ such that $L/\sqrt{M} \to 0$, the following holds:*

$$\frac{\mathcal{M}_{\Lambda, m}}{\sigma^m} = \begin{cases} \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} + O\big(\frac{\log L}{L}\big), & m \text{ is even} \\ O\big(\frac{\log L}{L}\big), & m \text{ is odd} \end{cases}$$

*Proof.* Expanding out (2.10), we have

$$\begin{aligned} \mathcal{M}_{\Lambda, m} = \frac{2^m}{d^m \pi^m} \sum_{\vec{k_1}, \ldots, \vec{k_m} \in \Lambda^* \setminus \{0\}} \prod_{j=1}^{m} \frac{\sin\left(\frac{\pi |\vec{k_j}|}{L}\right) \hat{\psi}\big(\frac{|\vec{k_j}|}{\sqrt{M}}\big)}{|\vec{k_j}|^{\frac{3}{2}}} \\ \times \left\langle \prod_{j=1}^{m} \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k_1}| + \frac{\pi}{4}\right) \right\rangle_T \end{aligned} \tag{2.18}$$

Now,

$$\Big\langle \prod_{j=1}^{m} \sin\Big(2\pi\big(t+\frac{1}{2L}\big)|\vec{k_1}|+\frac{\pi}{4}\Big)\Big\rangle_T$$

$$= \sum_{\epsilon_j=\pm 1} \frac{\prod\limits_{j=1}^{m}\epsilon_j}{2^m i^m}\hat{\omega}\Big(-T\sum_{j=1}^{m}\epsilon_j|\vec{k_j}|\Big)e^{\pi i\sum\limits_{j=1}^{m}\epsilon_j\big((1/L)|\vec{k_j}|+1/4\big)}$$

We call a term of the summation in (2.18) with $\sum\limits_{j=1}^{m}\epsilon_j|\vec{k_j}|=0$ *diagonal*, and *off-diagonal* otherwise. Due to lemma 2.2.6, the contribution of the *off-diagonal* terms is:

$$\ll \sum_{\vec{k_1},\dots,\vec{k_m}\in\Lambda^*\setminus\{0\}} \left(\frac{T}{M^{K_m}}\right)^{-A} \ll M^m T^{-A\delta},$$

for every $A>0$, by the rapid decay of $\hat{\omega}$ and our assumption regarding $M$.

Since $m$ is constant, this allows us to reduce the sum to the *diagonal terms*. The following definition and corollary 2.2.9 will allow us to actually sum over the diagonal terms, making use of $\alpha$'s being transcendental.

**Definition:** We say that a term corresponding to $\{\vec{k_1},\dots,\vec{k_m}\}\in\Big(\Lambda^*\setminus\{0\}\Big)^m$ and $\{\epsilon_j\}\in\{\pm1\}^m$ is a *principal diagonal* term if there is a partition $\{1,\dots,m\}=\bigsqcup\limits_{i=1}^{l}S_i$, such that for each $1\le i\le l$ there exists a primitive $\vec{n_i}\in\Lambda^*\setminus\{0\}$, with non-negative coordinates, that satisfies the following property: for every $j\in S_i$, there exist $f_j\in\mathbb{Z}$ with $|\vec{k_j}|=f_j|\vec{n_i}|$. Moreover, for each $1\le i\le l$, $\sum\limits_{j\in S_i}\epsilon_j f_j=0$.

Obviously, the principal diagonal is contained within the diagonal. However, if $\alpha$ is *transcendental*, the converse is also true. It is easily seen, given the following proposition.

**Proposition 2.2.8.** *Suppose that $\eta\in\mathbb{R}$ is a* <u>*transcendental*</u> *number. Let*

$$z_j = a_j^2+\eta b_j^2$$

*such that* $(a_j, b_j) \in \mathbb{Z}_+^2$ *are all different primitive vectors, for* $1 \leq j \leq m$. *Then* $\{\sqrt{z_j}\}_{j=1}^m$ *are linearly independent over* $\mathbb{Q}$.

The last proposition is an analogue of a well-known theorem due to Besicovitch [6] about incommensurability of square roots of integers. A proof of a much more general statement may be found e.g. in [3] (see lemma 2.3 and the appendix).

Thus we have

**Corollary 2.2.9.** *Every* <u>*diagonal*</u> *term is a* <u>*principle diagonal*</u> *term whenether* $\alpha$ *is* <u>*transcendendal*</u>.

By corollary 2.2.9, summing over diagonal terms is the same as summing over *principal* diagonal terms. Thus:

$$
\begin{aligned}
\frac{\mathcal{M}_{\Lambda,m}}{\sigma^m} \sim & \sum_{l=1}^{m} \sum_{\{1,\dots,m\}=\bigsqcup_{i=1}^{l} S_i} \left( \frac{1}{\sigma^{|S_1|}} \sum_{\vec{n}_1 \in \Lambda^* \setminus \{0\}} {}' D_{\vec{n}_1}(S_1) \right) \\
& \times \left( \frac{1}{\sigma^{|S_2|}} \sum_{\Lambda^* \setminus \{0\} \ni \vec{n}_2 \neq \vec{n}_1} {}' D_{(\vec{n}_2)}(S_2) \right) \cdots \left( \frac{1}{\sigma^{|S_l|}} \sum_{\Lambda^* \setminus \{0\} \ni \vec{n}_l \neq \vec{n}_2, \dots, \vec{n}_{l-1}} {}' D_{\vec{n}_l}(S_l) \right),
\end{aligned} \tag{2.19}
$$

where the inner summations are over primitive 1st-quadrant vectors of $\Lambda^* \setminus \{0\}$, and

$$
D_{\vec{n}}(S) = \frac{r(\vec{n})}{|\vec{n}|^{3|S|/2}} \sum_{\substack{f_j \geq 1 \\ \epsilon_j = \pm 1 \\ \sum_{j \in S} \epsilon_j f_j = 0}} \prod_{j \in S} \frac{-i\epsilon_j}{d\pi f_j^{3/2}} \sin\left( \frac{\pi}{L} f_j |\vec{n}| \right) \hat{\psi}\left( \frac{|\vec{n}|}{\sqrt{M}} \right) e^{i\pi\epsilon_j/4},
$$

with $r(\vec{n})$ given by (2.11).

Lemma 2.2.10 allows us to deduce that the contribution to (2.19) of a partition is $O(\frac{\log(L)}{L})$, unless $|S_i| = 2$ for every $i = 1, \dots, l$. In the latter case the contribution is 1 by the 2nd case of the same lemma. This is impossible for an odd $m$, and so, it finishes the proof of the current proposition in that case. Otherwise, suppose $m$ is even. Then the number of partitions $\{1, \dots, m\} = \bigsqcup_{i=1}^{l} S_i$ with $|S_i| = 2$ for every

$1 \le i \le l$ is

$$\frac{1}{\left(\frac{m}{2}\right)!}\binom{m}{2}\binom{m-2}{2}\cdot\ldots\cdot\binom{2}{2} = \frac{1}{\left(\frac{m}{2}\right)!}\,\frac{m!}{2!\,(m-2)!}\,\frac{(m-2)!}{2!\,(m-4)!}\cdot\ldots\cdot\frac{2!}{2!}$$
$$= \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!}$$

That concludes the proof of proposition 2.2.7.

□

**Lemma 2.2.10.** *If $L \to \infty$ such that $L/\sqrt{M} \to 0$, then*

$$\frac{1}{\sigma^m}\left|\sum_{\vec{n}\in\Lambda^*\setminus\{0\}}{}' D_{\vec{n}}(S)\right| = \begin{cases} 0, & |S| = 1 \\ 1, & |S| = 2 \\ O\left(\frac{\log L}{L}\right), & |S| \ge 3 \end{cases}$$

*where the ' in the summation means that it is over primitive vectors $(a, b)$.*

*Proof.* Without loss of generality, we may assume that $S = \{1, 2, \ldots, |S|\}$, and we assume that $k := |S| \ge 3$. Now,

$$\left|\sum_{\vec{n}\in\Lambda^*\setminus\{0\}}{}' D_{\vec{n}}(S)\right| \ll \sum_{\vec{n}\in\Lambda^*\setminus\{0\}} \frac{1}{|\vec{n}|^{3k/2}} Q(|\vec{n}|), \tag{2.20}$$

where

$$Q(z) := \sum_{\{\epsilon_j\}\in\{\pm1\}^k} \sum_{\substack{f_j\ge 1 \\ \sum_{j=1}^{k} \epsilon_j f_j = 0}} \prod_{j=1}^{k} \frac{|\sin(\frac{\pi}{L} f_j z)|}{f_j^{3/2}}.$$

Note that $Q(z) \ll 1$ for all $z$. We would like to establish a sharper result for $z \ll L$. In order to have $\sum_{j=1}^{k} \epsilon_j f_j = 0$, at least two of the $\epsilon_j$ must have different signs, and so, with no loss of generality, we may assume, $\epsilon_k = -1$ and $\epsilon_{k-1} = +1$. We notice that the last sum is, in fact, a Riemann sum, and so

$$Q(z) \ll \frac{L^{k-1}}{L^{3k/2}} \int_{1/L}^{\infty} \cdots \int_{1/L}^{\infty} dx_1 \cdots dx_{k-2} \sum_{\{\epsilon_j\}_{j=1}^{k-2} \in \{\pm 1\}^{k-2}} \int_{\frac{1}{L}+\max(0,-\sum_{j=1}^{k-2} \epsilon_j f_j)}^{\infty} dx_{k-1}$$

$$\times \left( \prod_{j=1}^{k-1} \frac{\left|\sin(\pi x_j z)\right|}{x_j^{3/2}} \right) \frac{\left| \sin\left( \pi z \cdot \left( x_{k-1} + \sum_{j=1}^{k-2} \epsilon_j x_j \right) \right) \right|}{\left( x_{k-1} + \sum_{j=1}^{k-1} \epsilon_j x_j \right)^{3/2}}$$

By changing variables $y_i = z \cdot x_i$ of the last integral, we obtain:

$$Q(z) \ll \frac{z^{k/2+1}}{L^{k/2+1}} \int_{1/L}^{\infty} \cdots \int_{1/L}^{\infty} dy_1 \cdots dy_{k-2} \sum_{\{\epsilon_j\}_{j=1}^{k-2} \in \{\pm 1\}^{k-2}} \int_{\frac{z}{L}+\max(0,-\sum_{j=1}^{k-2} \epsilon_j f_j)}^{\infty} dy_{k-1}$$

$$\times \left( \prod_{j=1}^{k-1} \frac{\left|\sin(\pi y_j)\right|}{y_j^{3/2}} \right) \frac{\left| \sin\left( \pi \cdot \left( y_{k-1} + \sum_{j=1}^{k-2} \epsilon_j y_j \right) \right) \right|}{\left( y_{k-1} + \sum_{j=1}^{k-1} \epsilon_j y_j \right)^{3/2}},$$

and since the last multiple integral is bounded, we may conclude that

$$Q(z) \ll \begin{cases} \frac{z^{k/2+1}}{L^{k/2+1}}, & z < L \\ 1, & z \geq L \end{cases}$$

Thus, by (2.20),

$$\left| \sum_{\vec{n} \in \Lambda^* \setminus \{0\}} {}' D_{\vec{n}}(S) \right| \ll \sum_{\substack{\vec{n} \in \Lambda^* \setminus \{0\} \\ |\vec{n}| \leq L}} \frac{1}{|\vec{n}|^{3k/2}} \cdot \frac{|\vec{n}|^{k/2+1}}{L^{k/2+1}} + \sum_{\substack{\vec{n} \in \Lambda^* \setminus \{0\} \\ |\vec{n}| > L}} \frac{1}{|\vec{n}|^{3k/2}} =: S_1 + S_2.$$

Now, considering $S_1$ and $S_2$ as Riemann sums, and computing the corresponding integrals in the usual elliptic coordinates we get:

$$S_1 \ll \frac{1}{L^{k/2+1}} \sum_{\substack{\vec{n} \in \Lambda^* \setminus \{0\} \\ |\vec{n}| \leq L}} \frac{1}{|\vec{n}|^{k-1}} \ll \frac{1}{L^{k/2+1}} \int_{1}^{L} \frac{dr}{r^{k-2}} \ll \frac{\log L}{L^{k/2+1}},$$

since $k \geq 3$.

Similarly,

$$S_2 \ll \int_L^\infty \frac{dr}{r^{3k/2-1}} \ll \frac{1}{L^{3k/2-2}} \ll \frac{1}{L^{k/2+(k-2)}} \ll \frac{1}{L^{k/2+1}},$$

again since $k \geq 3$.

And so, returning to the original statement of the lemma, if $k = |S| \geq 3$,

$$\frac{1}{\sigma^m}\left|\sum_{\vec{n}\in\Lambda^*\setminus\{0\}}{}' D_{\vec{n}}(S)\right| \ll L^{k/2}\left(\frac{\log L}{L^{k/2+1}}\right) \ll \frac{\log L}{L},$$

by (2.12).

In the case $|S| = 2$, by the definition of $D_{\vec{n}}$ and $\sigma^2$, we see that

$$\sum_{\vec{n}\in\Lambda^*\setminus\{0\}}{}' D_{\vec{n}}(S) = \sigma^2.$$

This completes the proof of the lemma.

□

## 2.3 An asymptotical formula for $N_\Lambda$

We need an asymptotical formula for the *sharp* counting function $N_\Lambda$. Unlike the case of the standard lattice, $\mathbb{Z}^2$, in order to have a good control over the error terms we should use some Diophantine properties of the lattice we are working with. We adapt the following notations:

Let $\Lambda$ be a lattice and $t > 0$ a real variable. Denote the set of squared norms of $\Lambda$ by

$$SN_\Lambda = \{|\vec{n}|^2 : \; n \in \Lambda\}.$$

Suppose we have a function $\delta_\Lambda : SN_\Lambda \to \mathbb{R}$, such that given $\vec{k} \in \Lambda$, there are <u>no</u> vectors $\vec{n} \in \Lambda$ with $0 < ||\vec{n}|^2 - |\vec{k}|^2| < \delta_\Lambda(|\vec{k}|^2)$. That is,

$$\Lambda \cap \{\vec{n} \in \Lambda : \; |\vec{k}|^2 - \delta_\Lambda(|\vec{k}|^2) < |\vec{n}|^2 < |\vec{k}|^2 + \delta_\Lambda(|\vec{k}|^2)\} = A_{|\vec{k}|},$$

where

$$A_y := \{\vec{n} \in \Lambda : \; |\vec{n}| = y\}.$$

Extend $\delta_\Lambda$ to $\mathbb{R}$ by defining $\delta_\Lambda(x) := \delta_\Lambda(|\vec{k}|^2)$, where $\vec{k} \in \Lambda$ minimizes $|x - |\vec{k}|^2|$ (in the case there is any ambiguity, that is if $x = \frac{|\vec{n_1}|^2 + |\vec{n_2}|^2}{2}$ for vectors $\vec{n_1}, \vec{n_2} \in \Lambda$ with consecutive increasing norms, choose $\vec{k} := \vec{n_1}$). We have the following lemma:

**Lemma 2.3.1.** *For every $a > 0$, $c > 1$,*

$$\begin{aligned} N_\Lambda(t) = \frac{\pi}{d}t^2 - \frac{\sqrt{t}}{d\pi} \sum_{\substack{\vec{k} \in \Lambda^* \setminus \{0\} \\ |\vec{k}| \leq \sqrt{N}}} \frac{\cos\left(2\pi t |\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} + O(N^a) \\ + O\left(\frac{t^{2c-1}}{\sqrt{N}}\right) + O\left(\frac{t}{\sqrt{N}} \cdot \left(\log t + \log(\delta_\Lambda(t^2))\right)\right) \\ + O\left(\log N + \log(\delta_{\Lambda^*}(t^2))\right) \end{aligned}$$

As a typical example of such a function, $\delta_\Lambda$, for $\Lambda = \langle 1, i\alpha \rangle$, with a Diophantine $\gamma := \alpha^2$, we may choose $\delta_\Lambda(y) = \frac{c}{y^{K_0}}$, where $c$ is a constant. In this example, if $\Lambda \ni \vec{k} = (a, b)$, then by lemma 2.2.2, $A_{|\vec{k}|} = (\pm a, \pm b)$, provided that $\gamma$ is irrational.

Our ultimate goal in this section is to prove lemma 2.3.1. However, it would be more convenient to work with $x = t^2$, and by abuse of notations we will call the counting function $N_\Lambda$. Moreover, we will redefine

$$N_\Lambda(x) := \begin{cases} \#\{\vec{k} : \; |\vec{k}|^2 \leq x\}, & x \neq |\vec{k}|^2 \text{ for every } \vec{k} \in \Lambda \\ \#\{\vec{k} : \; |\vec{k}|^2 < x\} + 2, & \text{otherwise} \end{cases}$$

(recall that every norm of a $\Lambda$-vector is of multiplicity 4). We are repeating the argument of Titchmarsh [24] that establishes the corresponding result for the remainder of the arithmetic function, which counts the number of different ways to write $m$ as a multiplication of a fixed number of natural numbers.

Let $\Lambda = \big\langle 1, \, i\alpha \big\rangle$. For $\gamma := \alpha^2$, introduce a function $\mathcal{Z}_\gamma(s)$ (this is a special value of an Eisenstein series) where $s = \sigma + it$ is a complex variable. For $\sigma > 1$, $\mathcal{Z}_\gamma(s)$ is defined by the following converging series:

$$\mathcal{Z}_\gamma(s) := \frac{1}{4} \sum_{\vec{k} \in \Lambda \setminus 0} \frac{1}{|\vec{k}|^{2s}}. \tag{2.21}$$

Then $\mathcal{Z}_\gamma$ has an analytic continuation to the whole complex plane, except for a single pole at $s = 1$, defined by the formula

$$\Gamma(s)\pi^{-s}\mathcal{Z}_\gamma(s) = \int_1^\infty x^{s-1}\psi_\gamma(x)dx + \frac{1}{\sqrt{\gamma}} \int_1^\infty x^{-s}\psi_{1/\gamma}(x)dx - \frac{s - \sqrt{\gamma}(s-1)}{4\sqrt{\gamma}s(1-s)},$$

where

$$\psi_\gamma(x) := \frac{1}{4} \sum_{\vec{k} \in \Lambda \setminus 0} e^{-\pi |\vec{k}|^2 x}.$$

This enables us to compute the residue of $\mathcal{Z}_\gamma$ at $s = 1$:

$$Res(\mathcal{Z}_\gamma, 1) = \frac{\pi}{4\sqrt{\gamma}}.$$

Moreover, $\mathcal{Z}_\gamma$ satisfies the following functional equation:

$$\mathcal{Z}_\gamma(s) = \frac{1}{\sqrt{\gamma}} \chi(s) \mathcal{Z}_{1/\gamma}(1-s), \tag{2.22}$$

with

$$\chi(s) = \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)}. \tag{2.23}$$

We will adapt the notation

$$\chi_\gamma(s) := \frac{1}{\sqrt{\gamma}} \chi(s).$$

The connection between $N_\Lambda$ and $\mathcal{Z}_\gamma$ is given in the following formula, which is satisfied for every $c > 1$:

$$\frac{1}{4} N_\Lambda(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{Z}_\gamma(s) \frac{x^s}{s} ds,$$

To prove it, just write $\mathcal{Z}_\gamma$ explicitly as the converging series, and use

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{y^s}{s}ds=\nu(y),$$

where

$$\nu(y):=\begin{cases}1, & y>1\\ \frac{1}{2} & y=1\\ 0; & 0<y<1\end{cases},$$

see [8], lemma on page 105, for example. One should bear in mind that the infinite integral above is not converging, and so we consider it in the symmetrical sense (that is, $\lim\limits_{T\to\infty}\int\limits_{c-iT}^{c+iT}$ ).

The following lemma will convert the infinite vertical integral in the last equation into a finite one, accumulating the corresponding error term. It will make use of the Diophantine properties of $\gamma$.

**Lemma 2.3.2.** *In the notations of lemma 2.3.1, for any constant $c>1$,*

$$\frac{1}{4}N_\Lambda(x)=\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\mathcal{Z}_\gamma(s)\frac{x^s}{s}ds+O\Big(\frac{x^c}{T}\Big)+O\Big(\frac{x}{T}\big(\log x+\log\delta_\Lambda(x)\big)\Big) \tag{2.24}$$

*as $x$, $T\to\infty$.*

*Proof.* Lemma on page 105 of [8] asserts moreover that for $y\neq 1$

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{y^s}{s}ds=\nu(y)+O\Big(y^c\min\big(1,\ \frac{1}{T|\log y|}\big)\Big), \tag{2.25}$$

whereas for $y=1$,

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{ds}{s}=\frac{1}{2}+O\Big(\frac{1}{T}\Big) \tag{2.26}$$

Suppose first that $x \neq |\vec{k}|^2$ for every $\vec{k} \in \Lambda$. Summing (2.25) for $y = \frac{x}{|\vec{k}|^2}$, where $\vec{k} \in \Lambda \setminus \{0\}$ gives (dividing both sides by 4):

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \mathcal{Z}_\gamma(s)\frac{x^s}{s}ds = \frac{1}{4}N_\Lambda(x) + O\left(x^c \sum_{\vec{k}\in\Lambda\setminus\{0\}} \frac{\min\left(1, \frac{1}{T\log\frac{x}{|\vec{k}|^2}}\right)}{|\vec{k}|^{2c}}\right).$$

The contribution to the error term of the right hand side of the last equality of $\vec{k} \in \Lambda$ with $|\vec{k}|^2 > 2x$ or $|\vec{k}|^2 < \frac{1}{2}x$ is

$$\ll \frac{x^c}{T} \sum_{|\vec{k}|\geq 2x \text{ or } |\vec{k}|\leq \frac{1}{2}x} \frac{1}{|\vec{k}|^{2c}} \leq \frac{x^c}{T}\mathcal{Z}_\gamma(c) \ll \frac{x^c}{T}.$$

For vectors $\vec{k_0} \in \Lambda$, which minimize $\left||\vec{k}|^2 - x\right|$ (in the case of ambiguity we choose $\vec{k_0}$ the same way we did in lemma 2.3.1 while extending $\delta_\Lambda$), the corresponding contribution is

$$\frac{x^c}{|\vec{k}|^{2c}} \ll \frac{x^c}{x^c} = 1.$$

Finally, we bound the contribution of vectors $\vec{k} \in \Lambda \setminus \{0\}$ with $|\vec{k_0}|^2 < |\vec{k}|^2 < 2x$, and similarly, of vectors with $\frac{1}{2}x < |\vec{k}|^2 < |\vec{k_0}|^2$. Now, by the definition of $\delta_\Lambda$, every such $\vec{k}$ satisfies:

$$|\vec{k}|^2 \geq |\vec{k_0}|^2 + \delta_\Lambda(x) \geq x + \frac{1}{2}\delta_\Lambda(x).$$

Moreover, $\log\frac{|\vec{k}|^2}{x} \gg \frac{|\vec{k}|^2 - |\vec{k_0}|^2}{x}$, and so the contribution is:

$$\ll \frac{x^c}{x^c T}x \sum_{x+\frac{1}{2}\delta_\Lambda(x)\leq|\vec{k}|^2<2x} \frac{1}{|\vec{k}|^2 - |\vec{k_0}|^2} \ll \frac{x}{T}\int_{\sqrt{|\vec{k_0}|^2+\delta_\Lambda(x)}}^{\sqrt{2x}} \frac{r}{r^2 - |\vec{k_0}|^2}dr$$

$$= \frac{x}{2T}\int_{|\vec{k_0}|^2+\delta_\Lambda(x)}^{2x} \frac{du}{u - |\vec{k_0}|^2} \ll \frac{x}{T}\log\left(u - |\vec{k_0}|^2\right)\Big|_{|\vec{k_0}|^2+\delta_\Lambda(x)}^{2x}$$

$$\ll \frac{x}{T}\big(\log x + \log\delta_\Lambda(x)\big)$$

If $x = |\vec{k_0}|^2$ for some $\vec{k_0} \in \Lambda$, the proof is the same except that we should invoke (2.26) rather than (2.25) for $|\vec{k}| = |\vec{k_0}|$.

That concludes the proof of lemma 2.3.2. □

*Proof of lemma 2.3.1.* We use lemma 2.3.2 and would like to move the contour of the integral in (2.24) from $\sigma = c$, $-T \le t \le T$ left to $\sigma = -a$ for some $a > 0$. Now, for $\sigma \ge c$,

$$\big|\mathcal{Z}_\gamma(s)\big| = O\big(1\big),$$

and by the functional equation (2.22) and the Stirling approximation formula,

$$\big|\mathcal{Z}_\gamma(s)\big| \ll t^{1+2a}$$

for $\sigma = -a$. Thus by the Phragmén-Lindelöf argument

$$\big|\mathcal{Z}_\gamma(s)\big| \ll t^{(1+2a)(c-\sigma)/(a+c)}$$

in the rectangle $-a-iT$, $c-iT$, $c+iT$, $-a+iT$. Using this bound, we obtain

$$\Big|\int_{-a+iT}^{c+iT} Z_\gamma(s)\frac{x^s}{s}ds\Big| \ll \frac{T^{2a}}{x^a} + \frac{x^c}{T},$$

and so is $\big|\int_{-a-iT}^{c-iT}\big|$. Collecting the residues at $s=1$ with residue being the main term of the asymptotics,

$$Res\big(Z_\gamma(s)\frac{x^s}{s},\, 1\big) = \frac{\pi}{4\sqrt{\gamma}}x$$

and at $s=0$ with

$$Res\big(Z_\gamma(s)\frac{x^s}{s},\, 0\big) = Z_\gamma(0) = O\big(1\big),$$

we get:

$$\Delta_\Lambda(x) := \frac{1}{4}N_\Lambda(x) - \frac{\pi}{4\sqrt{\gamma}}x = \frac{1}{2\pi i}\int_{-a-iT}^{-a+iT} \mathcal{Z}_\gamma(s)\frac{x^s}{s}ds$$
$$+ O\left(\frac{x^c}{T}\right) + O\left(\frac{x}{T}\big(\log x + \log\delta_\Lambda(x)\big)\right) + O(1) + O\big(\frac{T^{2a}}{x^a}\big).$$

Denote the integral in the last equality by $I$ and let $\kappa := \frac{1}{\gamma}$. Using the functional equation of $\mathcal{Z}_\gamma$ (2.22) again, and using the definition of $\mathcal{Z}_\kappa$ for $\sigma > 1$, (2.21), we get:

$$I = \frac{1}{2\pi i} \int\limits_{-a-iT}^{-a+iT} \chi_\gamma(s)\mathcal{Z}_\kappa(1-s)\frac{x^s}{s}ds = \frac{1}{2\pi i} \sum_{\vec{k}\in\Lambda^*}{}' \int\limits_{-a-iT}^{-a+iT} \frac{\chi_\gamma(s)}{|\vec{k}|^{2-2s}}\frac{x^s}{s}ds, \tag{2.27}$$

where the $'$ means that the summation is over vectors in the 1st quadrant. Put

$$\frac{T^2}{\pi^2 x} := N + \frac{1}{2}\delta_{\Lambda^*}(N), \tag{2.28}$$

where $N = |\vec{k_0}|^2$ for some $\vec{k_0} \in \Lambda^*$ and consider separately vectors $\vec{k} \in \Lambda^*$ with $|\vec{k}|^2 > N$ and ones with $|\vec{k}|^2 \le N$.

First we bound the contribution of vectors $\vec{k} \in \Lambda^*$ with $|\vec{k}|^2 > N$. Write the integral in (2.27) as $\int\limits_{-a-iT}^{-a+iT} = \int\limits_{-a-iT}^{-a-i} + \int\limits_{-a-i}^{-a+i} + \int\limits_{-a+i}^{-a+iT}$. Then

$$\Bigg|\sum_{\vec{k}\in\Lambda^*}{}' \int\limits_{-a-i}^{-a+i} \frac{\chi_\gamma(s)}{|\vec{k}|^{2-2s}}\frac{x^s}{s}ds\Bigg| \ll x^{-a} \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2>N}}{}' \frac{1}{|\vec{k}|^{2+2a}} \le x^{-a}\mathcal{Z}_\kappa(1+a) \ll x^{-a}.$$

Now,

$$\begin{aligned}|J| = \Bigg|\int\limits_{-a+i}^{-a+iT} \frac{\chi_\gamma(s)}{|\vec{k}|^{2-2s}}\frac{x^s}{s}ds\Bigg| &= \frac{x^{-a}\pi^{-2a-1}}{\sqrt{\gamma}|\vec{k}|^{2+2a}}\Bigg|\int\limits_1^T i\frac{\Gamma(1-s)}{\Gamma(s)}\frac{\left(|\vec{k}|^2 x\right)^{ti}}{ti}\pi^{2ti}dt\Bigg| \\ &\ll \frac{x^{-a}}{|\vec{k}|^{2+2a}}\Bigg|\int\limits_1^T e^{iF(t)}\Big(t^{2a} + O\big(t^{2a-1}\big)\Big)dt\Bigg|,\end{aligned}$$

with

$$F(t) = 2t\big(-\log t + \log\pi + 1\big) + t\log\left(|\vec{k}|^2 x\right) = t\log\frac{\pi^2 e^2 |\vec{k}|^2 x}{t^2},$$

due to the Stirling approximation formula.

One should notice that the contribution of the error term in the last bound is

$$\ll \frac{T^{2a}}{x^a}\sum_{\vec{k}\in\Lambda^*}{}' \frac{1}{|\vec{k}|^{2+2a}} = \frac{T^{2a}}{x^a}\mathcal{Z}_\kappa(1+a) \ll N^a.$$

We would like to invoke lemma 4.3 of [24] in order to bound the integral above. For this purpose we compute the derivative:

$$F'(t) = \log\left(\frac{|\vec{k}|^2 x\pi^2}{t^2}\right) \geq \log\left(\frac{|\vec{k}|^2}{N + \frac{1}{2}\delta_{\Lambda^*}(N)}\right),$$

by the definition of $N$, (2.28). Thus in the notations of lemma 4.3 of [24],

$$\frac{F'(t)}{G(t)} = \frac{\log\left(\frac{|\vec{k}|^2 x\pi^2}{t^2}\right)}{t^{2a}} \geq \frac{\log\left(\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_{\Lambda^*}(N)}\right)}{T^{2a}}.$$

We would also like to check that $\frac{G(t)}{F'(t)}$ is monotonic. Differentiating that function and leaving only the numerator, we get:

$$-t^{2a-1}\big(2a\log\frac{|\vec{k}|^2 x\pi^2}{t^2} + 2\big) < -2a\cdot t^{2a-1}\log\frac{|\vec{k}|^2}{N + \frac{1}{2}\delta_{\Lambda^*}(N)} < 0,$$

since $|\vec{k}^2| > N$. Thus

$$|J| \ll \frac{x^{-a}}{|\vec{k}|^{2+2a}} \frac{T^{2a}}{\log\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_{\Lambda^*}(N)}},$$

getting the same bound for $|\int\limits_{-a-iT}^{-a-i}|$, and therefore we are estimating

$$\sum_{\vec{k}\in\Lambda^*}{}' \frac{T^{2a}}{\log\frac{|\vec{k}|^2}{N+\frac{1}{2}\delta_{\Lambda^*}(N)}}.$$

For $|\vec{k}|^2 \geq 2N$, the contribution of the sum in (2.27) is:

$$\ll \frac{T^{2a}}{x^a}\sum_{\substack{\vec{k}\in\Lambda^*\\ |\vec{k}|^2\geq 2N}} \frac{1}{|\vec{k}|^{2+2a}} \leq \frac{T^{2a}}{x^a}\mathcal{Z}_\kappa(1+a) \ll N^a$$

As for vectors $\vec{k}\in\Lambda^*$ with $N + \delta_\kappa(N) \leq |\vec{k}|^2 < 2N$,

$$\log\frac{|\vec{k}|^2}{N + \frac{1}{2}\delta_\kappa(N)} \gg \frac{|\vec{k}|^2 - N}{N},$$

which implies that the corresponding contribution to the sum in (2.27) is:

$$\ll \frac{T^{2a}}{x^a N^{1+a}} \sum_{\substack{\vec{k}\in\Lambda^*\\ N+\delta_\kappa(N)\leq|\vec{k}|^2<2N}}{}' \frac{N}{|\vec{k}|^2 - N} \ll \int\limits_{\sqrt{N+\delta_\kappa(N)}}^{\sqrt{2N}} \frac{r}{r^2 - N - \frac{1}{2}\delta_\kappa(N)}$$

$$\ll \log\big(\delta_\kappa(N)\big) + \log N$$

The main term of I comes from $|\vec{k}|^2 \le N$. For such a $\vec{k}$, we write

$$\int\limits_{-a-iT}^{-a+iT} = \int\limits_{-i\infty}^{i\infty} - \Bigg( \int\limits_{iT}^{i\infty} + \int\limits_{-i\infty}^{-iT} + \int\limits_{-iT}^{-a-iT} + \int\limits_{-a+iT}^{iT} \Bigg), \tag{2.29}$$

that is, we are moving the contour of the integration to the imaginary axis.

Consider the first integral in the brackets. It is a constant multiple of

$$\int\limits_{T}^{\infty} e^{iF(t)} dt \ll \frac{1}{\log\left(\frac{N+\frac{1}{2}\delta_\kappa(N)}{|\vec{k}|^2}\right)},$$

and so the contribution of the corresponding sum is

$$\ll \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2 \log\left(\frac{N+\frac{1}{2}\delta_\kappa(N)}{|\vec{k}|^2}\right)} \ll N \int\limits_{1}^{\sqrt{N}} \frac{dr}{r\big(N+\frac{1}{2}\delta_\kappa(N)-r^2\big)}$$
$$\ll \log N + \log \delta_\kappa(N),$$

by lemma 4.2 of [24], and similarly for the second integral in the brackets in (2.29).

The last two give

$$\ll \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2} \int\limits_{-a}^{0} \left(\frac{|\vec{k}|^2 x}{T^2}\right)^{\sigma} d\sigma \ll \sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2} \left(\frac{T^2}{|\vec{k}|^2 x}\right)^{a}$$
$$\ll \frac{T^{2a}}{x^a} \int\limits_{1}^{\sqrt{N}} \frac{dr}{r^{2a+1}} \ll N^a.$$

Altogether we have now proved:

$$\begin{aligned} \Delta_\Lambda(x) = {} & \frac{1}{2\pi^2 di} \sideset{}{'}\sum_{\substack{\vec{k}\in\Lambda^* \\ |\vec{k}|^2\le N}} \frac{1}{|\vec{k}|^2} \int\limits_{-i\infty}^{i\infty} \pi^{2s} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{\big(|\vec{k}|^2 x\big)^s}{s} ds + O(N^a) \\ & + O\bigg(\frac{x^{c-1/2}}{\sqrt{N}}\bigg) + O\bigg(\frac{\sqrt{x}}{\sqrt{N}} \cdot \big(\log x + \log(\delta_\Lambda(x))\big)\bigg) \\ & + O\big(\log N + \log(\delta_{\Lambda^*}(x))\big) \end{aligned} \tag{2.30}$$

Recall the integral $\int\limits_{-i\infty}^{i\infty} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{y^s}{s} ds$ is a principal value, that is $\lim\limits_{T\to\infty} \int_{-iT}^{iT}$. We have

$$\lim_{T\to\infty} \int_{-iT}^{iT} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{y^s}{s} ds = -\sqrt{y} J_{-1}(2\sqrt{y})$$

as can be seen by shifting contours. Note that the analogous Barnes-Mellin formula

$$J_\nu(x) = \frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} \Gamma(-s)[\Gamma(\nu+s+1)]^{-1}(x/2)^{\nu+2s} ds$$

valid for $Re(\nu) > 0$ (see [11], (36), page 83), which deals with convergent integrals, is proved in this manner.

The well-known asymptotics of the bessel $J$-function,

$$J_{-1}(y) = \sqrt{\frac{2}{\pi y}} \cos\left(y + \frac{\pi}{4}\right) + O(y^{-3/2})$$

as $y \to \infty$, allow us to estimate the integral involved in (2.30) in terms of x and $\vec{k}$. Collecting all the constants and the error terms, we obtain the result of lemma 2.3.1. $\square$

## 2.4 Unsmoothing

**Proposition 2.4.1.** *Let a lattice $\Lambda = \langle 1, i\alpha \rangle$ with a Diophantine $\gamma := \alpha^2$ be given. Suppose that $L \to \infty$ as $T \to \infty$ and choose $M$, such that $L/\sqrt{M} \to 0$, but $M = O(T^\delta)$ for every $\delta > 0$ as $T \to \infty$. Suppose furthermore, that $M = O(L^{s_0})$ for some (fixed) $s_0 > 0$. Then*

$$\left\langle \left| S_\Lambda(t, \rho) - \tilde{S}_{\Lambda, M, L}(t) \right|^2 \right\rangle_T \ll \frac{1}{\sqrt{M}}$$

*Proof.* Since $\gamma$ is Diophantine, we may invoke lemma 2.3.1 with $\delta_\Lambda(y) = \frac{c_1}{y^{K_0}}$ and $\delta_{\Lambda^*}(y) = \frac{c_2}{y^{K_0}}$, where $c_1$, $c_2$ are constants. Choosing $a = \delta'$ and $c = 1 + \delta'/2$ for $\delta' > 0$ arbitrarily small and using essentially the same manipulation we used in order to obtain (2.8), and using (2.9) again, we get the following asymptotical formula:

$$S_\Lambda(t,\,\rho) = \frac{2}{d\pi}\sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}}\sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k}|+\frac{\pi}{4}\right)+R_\Lambda(N,\,t), \tag{2.31}$$

where

$$|R_\Lambda(N,\,t)| \ll \frac{N^{\delta'}}{\sqrt{|t|}}+\frac{|t|^{1/2+\delta'}}{\sqrt{N}}+\frac{1}{|t|^{1/2-\delta'}}.$$

Set $N = T^3$. Since M is small, the infinite sum in (2.8) is truncated before $n = T^3$. Thus (2.8) together with (2.31) implies:

$$\begin{aligned} &S_\Lambda(t,\,\rho)-\tilde{S}_{\Lambda,\,M,\,L}(t) = \\ &\frac{2}{d\pi}\sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\}\\ |\vec{k}|\le T^{3/2}}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}}\sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k}|+\frac{\pi}{4}\right)\left(1-\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right)\right) \\ &\qquad + R_\Lambda(T^3,\,t). \end{aligned} \tag{2.32}$$

Let $P_\Lambda(N,\,t)$ denote the sum in (2.32). Then the Cauchy-Schwartz inequality gives:

$$\begin{aligned} \left\langle\left|S_\Lambda(t,\,\rho)-\tilde{S}_{\Lambda,\,M,\,L}(t)\right|^2\right\rangle_T =& \left\langle P_\Lambda^2\right\rangle_T+\left\langle R_\Lambda(N,\,t)^2\right\rangle_T + \\ & O\left(\sqrt{\left\langle P_\Lambda^2\right\rangle_T}\sqrt{\left\langle R_\Lambda(N,\,t)^2\right\rangle_T}\right). \end{aligned} \tag{2.33}$$

Observe that for the chosen $N$,

$$\left\langle R_\Lambda(N,\,t)^2\right\rangle_T = O\left(T^{-1+\delta'}\right)$$

for arbitrary small $\delta' > 0$, since the above equality is satisfied pointwise.

Next we would like to bound $\left\langle P_\Lambda^2\right\rangle_T$. Just as we did while computing the variance of the smoothed variable, $\tilde{S}_{\Lambda,\,M,\,L}$, we divide all the terms of the expanded sum into the diagonal terms and the off-diagonal ones (see section 2.2.1). Namely,

$$
\begin{aligned}
\langle P_\Lambda^2 \rangle_T = & \frac{2}{d^2\pi^2} \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\le T^{3/2}}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} \left(1-\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right)\right)^2 \\
& + O\left( \sum_{\substack{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\ne|\vec{l}|\le T^{3/2}}} \frac{1}{|\vec{k}|^{3/2}|\vec{l}|^{3/2}} \hat{\omega}\Big(T\big(|\vec{k}|-|\vec{l}|\big)\Big)\right)
\end{aligned}
\tag{2.34}
$$

We will evaluate the diagonal contribution now. For $|\vec{k}| \le \sqrt{M}$,

$$\hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) = 1 + O\left(\frac{|\vec{k}|}{\sqrt{M}}\right),$$

and so the diagonal contribution is:

$$\frac{1}{M} \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ 1\ll|\vec{k}|\le\sqrt{M}}} \frac{1}{|\vec{k}|} + \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ \sqrt{M}\le|\vec{k}|\le T^{3/2}}} \frac{1}{|\vec{k}|^3} \ll \frac{1}{\sqrt{M}},$$

converting the sums into corresponding integrals and evaluating these integrals in the elliptic variables.

Finally, we are evaluating the off-diagonal contribution to (2.34) (that is, the second sum in the right-hand side of (2.34)). Set $0 < \delta_0 < 1$. With no loss of generality, we may assume that $|\vec{k}| < |\vec{l}|$. Evaluating the contribution of pairs $\vec{k}$, $\vec{l}$ with

$$|\vec{l}|^2 - |\vec{k}|^2 \ge \frac{|\vec{k}|}{T^{1-\delta_0}}$$

gives:

$$\ll \sum_{\substack{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|<|\vec{l}|\le T^{3/2}}} \frac{1}{|\vec{k}|^{3/2}|\vec{l}|^{3/2}} \hat{\omega}\Big(T\big(|\vec{k}|-|\vec{l}|\big)\Big) \ll T^{-A\delta_0+6}$$

for every $A > 0$, since

$$T\big(|\vec{l}|-|\vec{k}|\big) = T\frac{|\vec{l}|^2-|\vec{k}|^2}{|\vec{k}|+|\vec{l}|} \gg T^{\delta_0}\frac{|\vec{k}|}{|\vec{k}|+|\vec{l}|} \ge T^{\delta_0}\frac{|\vec{k}|}{|\vec{k}|+2|\vec{k}|} \gg T^{\delta_0},$$

as otherwise,

$$T\big(|\vec{l}| - |\vec{k}|\big) \geq T\big(|\vec{k}|\big) \gg T \gg T^{\delta_0}.$$

Thus the contribution of such terms is negligible.

In order to bound the contribution of pairs of $\Lambda^*$-vectors with

$$|\vec{l}|^2 - |\vec{k}|^2 \leq \frac{|\vec{k}|}{T^{1-\delta_0}}$$

we use the Diophantine assumption on $\beta$ again. Recall that we chose $\delta_{\Lambda^*}(y) = \frac{c_2}{y^{K_0}}$ with a constant $c_2$ in the beginning of the current proof. Choose a constant $R_0 > 0$ and assume that $|\vec{l}|^2 \leq cL^{R_0}$, for a constant $c$. Then

$$|\vec{l}|^2 - |\vec{k}|^2 \geq \delta_{\Lambda^*}(L^{R_0}) \gg \frac{1}{L^{K_0 R_0}} \gg \frac{1}{M^{K_0 R_0/2}} \gg \frac{|\vec{k}|}{T^{1-\delta_0}}.$$

Therefore, for an appropriate choice of $c$, there are no such pairs. Denote

$$S_n := \left\{(\vec{k}, \vec{l}) \in (\Lambda^*)^2 :\ 2^n \leq |\vec{k}|^2 \leq 2^{n+1},\ |\vec{k}|^2 \leq |\vec{l}|^2 \leq |\vec{k}|^2 + \frac{2^{n/2}}{T^{1-\delta_0}}\right\}$$

Thus, by dyadic partition, the contribution is:

$$\begin{aligned} &\ll \sum_{n=\lfloor R_0 \log L\rfloor}^{\lceil 3\log T\rceil} \sum_{\substack{2^n \leq |\vec{k}|^2 \leq 2^{n+1} \\ |\vec{k}|^2 \leq |\vec{l}|^2 \leq |\vec{k}|^2 + \frac{2^{n/2}}{T^{1-\delta_0}}}} \frac{1}{|\vec{k}|^{3/2}|\vec{l}|^{3/2}} \hat{\omega}\Big(T\big(|\vec{k}| - |\vec{l}|\big)\Big) \\ &\ll \sum_{n=\lfloor R_0 \log L\rfloor}^{\lceil 3\log T\rceil} \frac{\#S_n}{2^{3n/2}}, \end{aligned}$$

using $|\hat{\omega}| \ll 1$ everywhere. In order to bound the size of $S_n$, we use the following lemma, which is just a restatement of lemma 3.1 from [2]. We will prove it immediately after we finish proving proposition 2.4.1.

**Lemma 2.4.2.** *Let $\Lambda = \big\langle 1, i\eta\big\rangle$ be a rectangular lattice. Denote*

$$A(R, \delta) := \{(\vec{k}, \vec{l}) \in \Lambda :\ R \leq |\vec{k}|^2 \leq 2R,\ |\vec{k}|^2 \leq |\vec{l}|^2 \leq |\vec{k}|^2 + \delta\}.$$

*Then if $\delta > 1$, we have for every $\epsilon > 0$,*

$$\#A(R, \delta) \ll_\epsilon R^\epsilon \cdot R\delta$$

Thus, lemma 2.4.2 implies

$$\#S_n \ll 2^{n+\epsilon(n/2)} \max\left(1, \frac{2^{n/2}}{T^{1-\delta_0}}\right),$$

for every $\epsilon > 0$. Thus the contribution is:

$$\ll \sum_{n=R_0 \log L-1}^{C\log T+1} \frac{1}{2^{3n/2}} \cdot 2^{n+\epsilon n/2} \cdot 1 + \sum_{n=C\log T-1}^{3\log T+1} \frac{1}{2^{3n/2}} \cdot 2^{n+\epsilon n/2} \cdot \frac{2^{n/2}}{T^{1-\delta_0}}$$
$$\ll L^{-R_0(1-\epsilon)/2} + \frac{\log T}{T^{1-\delta_1}} \ll L^{-R_0(1-\epsilon)/2},$$

since $L$ is much smaller than $T$. Since $R_0$ is arbitrary, and we have assumed $M = O(L^{s_0})$, that implies

$$\left\langle P_\Lambda^2 \right\rangle_T \ll \frac{1}{\sqrt{M}}.$$

Collecting all our results, and using them on (2.33) we obtain

$$\left\langle \left| S_\Lambda(t,\, \rho) - \tilde{S}_{\Lambda,\, M,\, L}(t) \right|^2 \right\rangle \ll \frac{1}{\sqrt{M}} + \frac{1}{T^{1-\delta'}} + \frac{\sqrt{\log M}}{M^{1/4} T^{1/2-\delta'/2}} \ll \frac{1}{\sqrt{M}},$$

again, since $M$ is much smaller than $T$. □

*Proof of lemma 2.4.2.* Let $\vec{k} = (k_1,\, i\eta k_2)$ and $\vec{l} = (l_1,\, i\eta l_2)$. Denote $\mu := \eta^2$, $n := l_1^2 - k_1^2$ and $m := k_2^2 - l_2^2$. The number of 4-tuples $(k_1,\, k_2,\, l_1,\, l_2)$ with $m \neq 0$ is

$$\#A(\delta,\, T) \ll \sum_{\substack{0 \le n-\mu m \le \delta \\ 1 \le m \le 4R}} d(n) d(m) \ll \delta \sum_{1 \le m \le 4R} d(m)^2 \ll R^{1+\epsilon}\delta$$

Next, we bound the number of 4-tuples with $m = 0$, $n \neq 0$:

$$\sum_{k_2=0}^{\sqrt{2R}} \sum_{0<n<\delta} d(n) \ll R^{1/2+\epsilon}\delta,$$

and similarly we bound the number of 4-tuples with $n = 0$, $m \neq 0$.

All in all, we have proved that

$$\#A(\delta,\, T) \ll R^{1+\epsilon}\delta$$

$\square$

¿From now on we will assume that $\Lambda = \big\langle 1,\, i\alpha \big\rangle$ with a Diophantine $\gamma := \alpha^2$, and so the use of proposition 2.4.1 is justified.

**Lemma 2.4.3.** *Under the conditions of proposition 2.4.1, for all fixed $\xi > 0$,*

$$\mathbb{P}_{\omega,\, T}\left\{ \left| \frac{S_\Lambda(t,\, \rho)}{\sigma} - \frac{\tilde{S}_{\Lambda,\, M,\, L}(t)}{\sigma} \right| > \xi \right\} \to 0,$$

*as $T \to \infty$, where $\sigma^2 = \frac{8\pi}{dL}$.*

*Proof.* Use Chebychev's inequality and proposition 2.4.1. $\square$

**Corollary 2.4.4.** *For a number $\alpha \in \mathbb{R}$, suppose that $\alpha^2$ is strongly Diophantine and denote $\Lambda = \big\langle 1,\, \alpha \big\rangle$. Then if $L \to \infty$, but $L = O\big(T^\delta\big)$ for all $\delta > 0$ as $T \to \infty$, then for any interval $\mathcal{A}$,*

$$\mathbb{P}_{\omega,\, T}\left\{ \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A} \right\} \to \frac{1}{\sqrt{2\pi}} \int\limits_{A} e^{-\frac{x^2}{2}}\, dx,$$

*where $\sigma^2 = \frac{8\pi}{dL}$.*

*Proof.* Set $M = L^3$, then, obviously, $L$, $M$ satisfy the conditions of lemma 2.4.3 and theorem 2.1.1. Denote $X(t) := \frac{S_\Lambda(t,\rho)}{\sigma}$ and $Y(t) := \frac{\tilde{S}_{\Lambda,\, M}(t)}{\sigma}$. In the new notations lemma 2.4.3 states that for any $\xi > 0$,

$$\mathbb{P}_{\omega,\, T}\big\{ |X(t) - Y(t)| > \xi \big\} \to 0, \tag{2.35}$$

as $T \to \infty$. Now, for every $\epsilon > 0$,

$$\big\{ a \le X \le b \big\} \subseteq \big\{ a - \epsilon \le Y \le b + \epsilon \big\} \cup \big\{ |X - Y| > \epsilon \big\},$$

and so, taking $\limsup_{T\to\infty} \mathbb{P}_{\omega,T}$ of both of the sides, we obtain:

$$\limsup_{T\to\infty} \mathbb{P}_{\omega,T}\big\{a \le X \le b\big\} \le \lim_{T\to\infty} \mathbb{P}_{\omega,T}\big\{a-\epsilon \le Y \le b+\epsilon\big\}$$
$$= \frac{1}{\sqrt{2\pi}} \int\limits_{a-\epsilon}^{b+\epsilon} e^{-\frac{x^2}{2}} dx,$$

due to (2.35) and theorem 2.1.1. Starting from

$$\big\{a+\epsilon \le Y \le b-\epsilon\big\} \subseteq \big\{a \le X \le b\big\} \cup \big\{|X-Y| > \epsilon\big\},$$

and doing the same manipulations as before, we get the converse inequality, and thus this implies the result of the present corollary.

□

We are now in a position to prove our main result, namely, theorem 1.1.1. It states that the result of corollary 2.4.4 holds for $\omega = \mathbf{1}_{[1,2]}$, the indicator function. We are unable to substitute it directly because of the rapid decay assumption on $\hat{\omega}$. Nonetheless, we are able to prove the validity of the result by the means of *approximating* the indicator function with functions which will obey the rapid decay assumption. The proof is essentially the same as of theorem 1.1 in [13], pages 655-656, and we repeat it in this paper for the sake of the completeness.

*Proof of theorem 1.1.1.* Fix $\epsilon > 0$ and approximate the indicator function $\mathbf{1}_{[1,2]}$ above and below by smooth functions $\chi_\pm \ge 0$ so that $\chi_- \le \mathbf{1}_{[1,2]} \le \chi_+$, where both $\chi_\pm$ and their Fourier transforms are smooth and of rapid decay, and so that their total masses are within $\epsilon$ of unity $\big|\int \chi_\pm(x)dx - 1\big| < \epsilon$. Now, set $\omega_\pm := \chi_\pm / \int \chi_\pm$. Then $\omega_\pm$ are "admissible", and for all $t$,

$$(1-\epsilon)\omega_-(t) \le \mathbf{1}_{[1,2]}(t) \le (1+\epsilon)\omega_+(t). \tag{2.36}$$

Now,

$$meas\left\{t \in [T,\, 2T] : \; \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A}\right\} = \int\limits_{-\infty}^{\infty} \mathbf{1}_{\mathcal{A}}\left(\frac{S_\Lambda(t,\, \rho)}{\sigma}\right)\mathbf{1}_{[1,\,2]}\big(\frac{t}{T}\big)dt,$$

and since (2.36) holds, we find that

$$(1-\epsilon)\mathbb{P}_{\omega_-,\,T}\left\{\frac{S_{\Lambda,\,M,\,L}}{\sigma} \in \mathcal{A}\right\} \le \frac{1}{T}meas\left\{t \in [T,\, 2T] : \; \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A}\right\}$$
$$\le (1+\epsilon)\mathbb{P}_{\omega_+,\,T}\left\{\frac{S_{\Lambda,\,M,\,L}}{\sigma} \in \mathcal{A}\right\}.$$

As it was mentioned immediately after the definition of the strong Diophantine property, $\alpha$'s being strongly Diophantine implies the same for $\alpha^2$, making a use of corollary 2.4.4 legitimate. Now by corollary 2.4.4, the two extreme sides of the last inequality have a limit, as $T \to \infty$, of

$$(1 \pm \epsilon)\frac{1}{\sqrt{2\pi}}\int\limits_A e^{-\frac{x^2}{2}}dx,$$

and so we get that

$$(1-\epsilon)\int\limits_A e^{-\frac{x^2}{2}}dx \le \liminf_{T\to\infty}\frac{1}{T}meas\left\{t \in [T,\, 2T] : \; \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A}\right\}$$

with a similar statement for lim sup; since $\epsilon > 0$ is arbitrary, this shows that the limit exists and equals

$$\lim_{T\to\infty}\frac{1}{T}meas\left\{t \in [T,\, 2T] : \; \frac{S_\Lambda(t,\, \rho)}{\sigma} \in \mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}}\int\limits_A e^{-\frac{x^2}{2}}dx,$$

which is the Gaussian law. $\square$

## CHAPTER III

# The General Case

### 3.1 The distribution of $\tilde{S}_{\Lambda,\,M,\,L}$

We apply the same smoothing as in chapter II (see section 2.1). That is, we consider

$$\tilde{S}_{\Lambda,\,M,\,L}(t) = \frac{\tilde{N}_{\Lambda,\,M}(t+\frac{1}{L}) - \tilde{N}_{\Lambda,\,M}(t) - \frac{\pi}{d}(\frac{2t}{L}+\frac{1}{L^2})}{\sqrt{t}}, \tag{3.1}$$

where $\tilde{N}_{\Lambda,\,M}$ is given by (2.2), under the same notations for $\chi$ and $\chi_\epsilon$ as in that chapter.

Recall that we assume that for every $\delta > 0$, $L = L(T) = O(T^\delta)$, which corresponds to the assumption of theorem 1.1.2 regarding $\rho := \frac{1}{L}$. Moreover, rather than drawing $t$ at random from $[T,\,2T]$ with a uniform distribution, we prefer to work with a smooth sample space, defined by $\omega$. The function $\omega$ and its Fourier transform, $\hat{\omega}$, are rapidly decaying at infinity.

The analogue of theorem 2.1.1 in the case of general lattice, is:

**Theorem 3.1.1.** *Suppose that $M(T)$ and $L(T)$ are increasing to infinity with $T$, such that $M = O(T^\delta)$ for all $\delta > 0$, and $L/\sqrt{M} \to 0$. Then if $(\alpha,\,\beta)$ is an* algebraically independent strongly Diophantine *pair, we have for $\Lambda = \langle 1,\,\alpha + i\beta\rangle$,*

$$\lim_{T\to\infty} \mathbb{P}_{\omega,\,T}\left\{\frac{\tilde{S}_{\Lambda,\,M,\,L}}{\sigma} \in \mathcal{A}\right\} = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathcal{A}} e^{-\frac{x^2}{2}}\,dx,$$

*for any interval $\mathcal{A}$, where*

$$\sigma^2 := \frac{4\pi}{\beta L}. \tag{3.2}$$

We generalize the Diophantine property (see section 2.2) to a tuple of real numbers:

**Definition:** A tuple of real numbers $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is called *Diophantine*, if there exists a number $K > 0$, such that for every integer tuple $\{a_i\}_{i=0}^n$,

$$\left| a_0 + \sum_{i=1}^n a_i \alpha_i \right| \gg \frac{1}{q^K}, \tag{3.3}$$

with $q = \max\limits_{0 \le i \le n} |a_i|$, whenever the LHS of the inequality doesn't vanish. Khintchine proved that *almost all* tuples in $\mathbb{R}^n$ are Diophantine (see, e.g. [18], pages 60-63,99).

Denote the dual lattice

$$\Lambda^* = \left\langle 1,\, \gamma + i\delta \right\rangle$$

with $\gamma = -\frac{\alpha}{\beta}$ and $\delta = \frac{1}{\beta}$. In the rest of the current section, we assume, that, unless specified otherwise, the set of the squared lengths of vectors in $\Lambda^*$ satisfy the Diophantine property. That means, that $(\alpha^2, \alpha\beta, \beta^2)$ is a Diophantine triple of real numbers. We may assume $(\alpha^2, \alpha\beta, \beta^2)$ being Diophantine, since theorem 1.1.2 (and theorem 3.1.1) assume $(\alpha, \beta)$ is *strongly Diophantine*, which is, obviously, a stronger assumption.

The analogue of corollary 2.2.3 is:

**Lemma 3.1.2.**

$$\begin{aligned}\tilde{S}_{\Lambda, M, L}(t) = \frac{2}{\beta\pi} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi(t + \frac{1}{2L})|\vec{k}| + \frac{\pi}{4}\right) \hat{\psi}\left(\frac{|\vec{k}|}{\sqrt{M}}\right) \\ + O\left(\frac{1}{\sqrt{t}}\right).\end{aligned} \tag{3.4}$$

The proof is literally the same as the one of corollary 2.2.3.

Recall that $\hat{\psi}$ being compactly supported means that the sum in (3.4) essentially truncates at $|\vec{k}| \approx \sqrt{M}$.

In the current case, there are even less multiplicities than in case of rectangular lattice (compare with lemma 2.2.2):

**Lemma 3.1.3.** *Let $\vec{a_j} = m_j + n_j(\alpha + i\beta) \in \Lambda$, $j = 1, 2$, with an irrational $\alpha$ such that $\beta \notin \mathbb{Q}(\alpha)$. Then if $|\vec{a_1}| = |\vec{a_2}|$, either $n_1 = n_2$ and $m_1 = m_2$ or $n_1 = -n_2$ and $n_2 = -m_2$.*

*Proof of theorem 3.1.1.* We will show that the moments of $\tilde{S}_{\Lambda, M, L}$ corresponding to the smooth probability space converge to the moments of the normal distribution with zero mean and variance which is given by theorem 3.1.1. This allows us to deduce that the distribution of $\tilde{S}_{\Lambda, M, L}$ converges to the normal distribution as $T \to \infty$, precisely in the sense of theorem 3.1.1.

First, we show that the mean is $O(\frac{1}{\sqrt{T}})$. Since $\omega$ is real,

$$\left|\left\langle \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k}| + \frac{\pi}{4}\right)\right\rangle\right| = \left|\Im m\left\{\hat{\omega}\big(-T|\vec{k}|\big)e^{i\pi(\frac{|\vec{k}|}{L}+\frac{1}{4}}\right\}\right| \ll \frac{1}{T^A|\vec{k}|^A}$$

for any $A > 0$, where we have used the rapid decay of $\hat{\omega}$. Thus

$$\left|\left\langle \tilde{S}_{\Lambda, M, L}\right\rangle\right| \ll \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{T^A|\vec{k}|^{A+3/2}} + O\left(\frac{1}{\sqrt{T}}\right) \ll O\left(\frac{1}{\sqrt{T}}\right),$$

due to the convergence of $\sum\limits_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{1}{|\vec{k}|^{A+3/2}}$, for $A > \frac{1}{2}$

Now define

$$\mathcal{M}_{\Lambda, m} := \left\langle \left(\frac{2}{\beta\pi} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^{\frac{3}{2}}} \sin\left(2\pi\big(t + \frac{1}{2L}\big)|\vec{k}| + \frac{\pi}{4}\right)\hat{\psi}\big(\frac{|\vec{k}|}{\sqrt{M}}\big)\right)^m\right\rangle \tag{3.5}$$

Then from (3.4), the binomial formula and the Cauchy-Schwartz inequality,

$$\left\langle \big(\tilde{S}_{\Lambda, M, L}\big)^m\right\rangle = \mathcal{M}_{\Lambda, m} + O\left(\sum_{j=1}^{m} \binom{m}{j} \frac{\sqrt{\mathcal{M}_{2m-2j}}}{T^{j/2}}\right)$$

Proposition 3.1.4 together with proposition 3.1.7 allow us to deduce the result of theorem 3.1.1 for an algebraically independent strongly Diophantine $(\xi, \eta) := (-\frac{\alpha}{\beta}, \frac{1}{\beta})$. Clearly, $(\alpha, \beta)$ being algebraically independent and strongly Diophantine is sufficient. $\square$

### 3.1.1 The variance

The computation of the variance is done in two steps. First, we reduce the main contribution to the *diagonal* terms, using the assumption on the pair $(\alpha, \beta)$ (i.e. $(\alpha^2, \alpha\beta, \beta^2)$ is *Diophantine*). Then we compute the contribution of the *diagonal* terms. Both these steps are very close to the corresponding ones in chapter II.

Suppose that the triple $(\alpha^2, \alpha\beta, \beta^2)$ satisfies (3.3).

**Proposition 3.1.4.** *If $M = O\big(T^{1/(K+1/2+\delta)}\big)$ for fixed $\delta > 0$, then the variance of $\tilde{S}_{\Lambda, M, L}$ is asymptotic to*

$$\sigma^2 := \frac{4}{\beta^2\pi^2} \sum_{\vec{k}\in\Lambda^*\setminus\{0\}} \frac{\sin^2\left(\frac{\pi|\vec{k}|}{L}\right)}{|\vec{k}|^3} \hat{\psi}^2\left(\frac{|\vec{k}|}{\sqrt{M}}\right)$$

*If $L \to \infty$, but $L/\sqrt{M} \to 0$, then*

$$\sigma^2 \sim \frac{4\pi}{\beta L} \tag{3.6}$$

**Remark:** In the formulation of proposition 3.1.4, $K$ is implicitly given by (3.3).

*Proof.* Expanding out (3.5), we have

$$\begin{aligned} \mathcal{M}_{\Lambda,2} = \frac{4}{\beta^2\pi^2} \sum_{\vec{k},\vec{l}\in\Lambda^*\setminus\{0\}} & \frac{\sin\left(\frac{\pi|\vec{k}|}{L}\right)\sin\left(\frac{\pi|\vec{l}|}{L}\right)\hat{\psi}(\frac{|\vec{k}|}{\sqrt{M}})\hat{\psi}(\frac{|\vec{l}|}{\sqrt{M}})}{|\vec{k}|^{\frac{3}{2}}|\vec{l}|^{\frac{3}{2}}} \\ & \times \left\langle \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k}| + \frac{\pi}{4}\right) \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{l}| + \frac{\pi}{4}\right)\right\rangle \end{aligned} \tag{3.7}$$

It is easy to check that the average of the second line of the previous equation is:

$$\frac{1}{4}\Big[\hat\omega\big(T(|\vec k| - |\vec l|)\big)e^{i\pi(1/L)(|\vec l|-|\vec k|)}+ \hat\omega\big(T(|\vec l| - |\vec k|)\big)e^{i\pi(1/L)(|\vec k|-|\vec l|)}+ \hat\omega\big(T(|\vec k| + |\vec l|)\big)e^{-i\pi(1/2+(1/L)(|\vec k|+|\vec l|))}- \hat\omega\big(-T(|\vec k| + |\vec l|)\big)e^{i\pi(1/2+(1/L)(|\vec k|+|\vec l|))}\Big] \tag{3.8}$$

Recall that the support condition on $\hat\psi$ means that $\vec k$ and $\vec l$ are both constrained to be of length $O(\sqrt{M})$. Thus the off-diagonal contribution (that is for $|\vec k| \neq |\vec l|$ ) of the first two lines of (3.8) is

$$\ll \sum_{\substack{\vec k,\vec l\in\Lambda^*\backslash\{0\}\\ |\vec k|,|\vec k'|\le\sqrt{M}}} \frac{M^{A(K+1/2)}}{T^A} \ll \frac{M^{A(K+1/2)+2}}{T^A} \ll T^{-B},$$

for every $B > 0$, since $(\alpha,\ \alpha\beta,\ \beta^2)$ is Diophantine.

Obviously, the contribution to (3.7) of the two last lines of (3.8) is negligible both in the diagonal and off-diagonal cases, justifying the diagonal approximation of (3.7) in the first statement of the proposition. The computation of the asymptotics is done literally the same way as in chapter II (see proposition 2.2.4).

□

### 3.1.2 The higher moments

In order to compute the higher moments we will prove that the main contribution comes from the so-called *diagonal* terms (to be explained later). Our bound for the contribution of the *off-diagonal* terms holds for a *strongly Diophantine* pair of real numbers, which is defined below. In order to show that the strongly Diophantine pairs are "generic", we use theorem 3.1.5 below, which is a consequence of the work of Kleinbock and Margulis [15]. The contribution of the diagonal terms is computed exactly in the same manner it was done in chapter II, and so we will omit it here.

**Definition:** We call the pair $(\xi,\,\eta)$ *strongly Diophantine*, if for all natural $n$ there exists a number $K_1 = K_1(\xi,\,\eta,\,n) \in \mathbb{N}$ such that for every integral polynomial of 2 variables $p(x,\,y) = \sum_{i+j\le n} a_{i,\,j}x^iy^j$ of degree $\le n$, we have

$$\big|p(\xi,\,\eta)\big| \gg h^{-K_1}, \tag{3.9}$$

where $h = \max_{i+j\le n} |a_{i,\,j}|$ is the height of $p$. The constant involved in the " $\gg$ " notation may depend only on $\xi$, $\eta$, $n$ and $K_1$.

**Theorem 3.1.5.** *Let an integer $n$ be given. Then almost all pairs of real numbers $(\xi,\,\eta) \in \mathbb{R}^2$ satisfy the following property: there exists a number $K_1 = K_1(n) \in \mathbb{N}$ such that for every integer polynomial of* 2 *variables $p(x,\,y) = \sum_{i+j\le n} a_{i,\,j}x^iy^j$ of degree $\le n$,* (3.9) *is satisfied.*

Theorem 3.1.5 states that almost all real pairs of numbers are strongly Diophantine.

**Remark:** Theorem A in [15] is much more general than the result we are using. As a matter of fact, we have the inequality

$$\big|b_0 + b_1f_1(x) + \ldots + b_nf_n(x)\big| \gg_\epsilon \frac{1}{h^{n+\epsilon}}$$

with $b_i \in \mathbb{Z}$ and

$$h := \max_{0\le i\le n} |b_i|.$$

The inequality above holds for every $\epsilon > 0$ for a wide class of functions $f_i : U \to \mathbb{R}$, for almost all $x \in U$, where $U \subset \mathbb{R}^m$ is an open subset. Here we use this inequality for the monomials.

**Remark:** Simon Kristensen [16] has recently shown, that the set of all pairs $(\xi,\,\eta) \in \mathbb{R}^2$ which fail to be strongly Diophantine has Hausdorff dimension 1.

Obviously, if $(\xi,\,\eta)$ is strongly Diophantine, then any $n$-tuple of real numbers, which consists of a set of monomials in $\xi$ and $\eta$, is Diophantine. Moreover, $(\xi,\,\eta)$ is strongly Diophantine iff $(-\frac{\xi}{\eta},\,\frac{1}{\eta})$ is such.

We have the following analogue of lemma 2.2.6, which will eventually allow us to exploit the strong Diophantine assumption of $(\alpha,\,\beta)$.

**Lemma 3.1.6.** *If $(\xi,\,\eta)$ is strongly Diophantine, then it satisfies the following property: for any fixed natural $m$, there exists $K \in \mathbb{N}$, such that if*

$$z_j = a_j^2 + b_j^2\xi^2 + 2a_jb_j\xi + b_j^2\eta^2 \ll M,$$

*and $\epsilon_j = \pm 1$ for $j = 1, \ldots, m$, with integral $a_j$, $b_j$ and if $\sum\limits_{j=1}^{m} \epsilon_j\sqrt{z_j} \neq 0$, then*

$$\Big|\sum_{j=1}^{m} \epsilon_j\sqrt{z_j}\Big| \gg M^{-K}, \tag{3.10}$$

*where the constant involved in the " $\gg$ " notation depends only on $\eta$ and $m$.*

The proof is essentially the same as the one of lemma 2.2.6, considering the product $Q$ of numbers of the form $\sum\limits_{j=1}^{m} \delta_j\sqrt{z_j}$ over all possible signs $\delta_j$. Here we use the Diophantine condition of the real tuple $(\xi,\,\eta)$ rather than of a single real number.

**Proposition 3.1.7.** *Let $m \in \mathbb{N}$ be given. Suppose that $\Lambda = \langle 1,\,\alpha + i\beta\rangle$, such that the pair $(\xi,\,\eta) := (-\frac{\alpha}{\beta},\,\frac{1}{\beta})$ is algebraically independent strongly Diophantine, which satisfy the property of lemma 3.1.6 for the given $m$, with $K = K_m$. Then if $M = O\big(T^{\frac{1-\delta}{K_m}}\big)$ for some $\delta > 0$, and if $L \to \infty$ such that $L/\sqrt{M} \to 0$, the following holds:*

$$\frac{\mathcal{M}_{\Lambda,\,m}}{\sigma^m} = \begin{cases} \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} + O\big(\frac{\log L}{L}\big), & m \text{ is even} \\ O\big(\frac{\log L}{L}\big), & m \text{ is odd} \end{cases}$$

*Proof.* Expanding out (3.5), we have

$$\mathcal{M}_{\Lambda,m} = \frac{2^m}{\beta^m \pi^m} \sum_{\vec{k_1},\dots,\vec{k_m} \in \Lambda^* \setminus \{0\}} \prod_{j=1}^{m} \frac{\sin\left(\frac{\pi|\vec{k_j}|}{L}\right) \hat{\psi}\left(\frac{|\vec{k_j}|}{\sqrt{M}}\right)}{|\vec{k_j}|^{\frac{3}{2}}} \times \left\langle \prod_{j=1}^{m} \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k_1}| + \frac{\pi}{4}\right) \right\rangle \tag{3.11}$$

Now,

$$\left\langle \prod_{j=1}^{m} \sin\left(2\pi\left(t+\frac{1}{2L}\right)|\vec{k_1}| + \frac{\pi}{4}\right) \right\rangle = \sum_{\epsilon_j=\pm 1} \frac{\prod_{j=1}^{m} \epsilon_j}{2^m i^m} \hat{\omega}\left(-T\sum_{j=1}^{m} \epsilon_j |\vec{k_j}|\right) e^{\pi i \sum_{j=1}^{m} \epsilon_j \left((1/L)|\vec{k_j}|+1/4\right)}$$

We call a term of the summation in (3.11) with $\sum_{j=1}^{m} \epsilon_j |\vec{k_j}| = 0$ *diagonal*, and *off-diagonal* otherwise. Due to lemma 3.1.6, the contribution of the *off-diagonal* terms is:

$$\ll \sum_{\substack{\vec{k_1},\dots,\vec{k_m} \in \Lambda^* \setminus \{0\} \\ |\vec{k_1}|,\dots,|\vec{k_m}| \le \sqrt{M}}} \left(\frac{T}{M^{K_m}}\right)^{-A} \ll M^m T^{-A\delta},$$

for every $A > 0$, by the rapid decay of $\hat{\omega}$ and our assumption regarding $M$.

Since $m$ is constant, this allows us to reduce the sum to the *diagonal terms*. The analogue of Besicovich's theorem holds in this case too:

**Proposition 3.1.8.** *Suppose that $\xi$ and $\eta$ are algebraically independent, and*

$$z_j = a_j^2 + 2a_j b_j \xi + b_j^2(\xi^2 + \eta), \tag{3.12}$$

*such that $(a_j, b_j) \in \mathbb{Z}_+^2$ are all different primitive vectors, for $1 \le j \le m$. Then $\{\sqrt{z_j}\}_{j=1}^m$ are linearly independent over $\mathbb{Q}$.*

The last proposition is an immediate consequence of a theorem proved in the appendix of [3].

Recall the definition of the *principal diagonal* terms in section 2.2.2. The analogue of corollary 2.2.9 in this case is:

**Corollary 3.1.9.** *Every* <u>*diagonal*</u> *term is a* <u>*principle diagonal*</u> *term whenether* $\xi$ *and* $\eta$ *are algebraically independent.*

Having corollary 3.1.9 in our hands, the computation of the contribution of the principal diagonal terms is done literally the same way it was done in chapter II, proposition 2.2.7

□

## 3.2 Bounding the number of close pairs of lattice points

Roughly speaking, we say that a pair of lattice points, $n$ and $n'$ is *close*, if $\big||n|-|n'|\big|$ is *small*. We would like to show that this phenomenon is *rare*. This is closely related to the Oppenheim conjecture, as $|n|^2 - |n'|^2$ is a quadratic form on the coefficients of $n$ and $n'$.

In order to establish a quantative result, we use a technique developed in a paper by Eskin, Margulis and Mozes [9]. Note that the proof is unconditional on any Diophantine assumptions.

### 3.2.1 Statement of the results

The ultimate goal of this section is to establish the following

**Proposition 3.2.1.** *Let* $\Lambda$ *be a lattice and denote*

$$A(R,\delta) := \{(\vec{k},\vec{l}) \in \Lambda\times\Lambda : \; R \le |\vec{k}|^2 \le 2R,\; |\vec{k}|^2 \le |\vec{l}|^2 \le |\vec{k}|^2 + \delta\}. \tag{3.13}$$

*Then if* $\delta > 1$, *such that* $\delta = o(R)$, *we have*

$$\#A(R,\delta) \ll R\delta\cdot\log R$$

In order to prove this result, we note that evaluating the size of $A(R,\, \delta)$ is equivalent to counting integer points $\vec{v} \in \mathbb{R}^4$ with $T \leq \|\vec{v}\| \leq 2T$ such that

$$0 \leq Q_1(v) \leq \delta,$$

where $Q_1$ is a quadratic form of signature (2, 2), given explicitly by

$$Q_1(\vec{v}) = (v_1 + v_2\alpha)^2 + (v_2\beta)^2 - (v_3 + v_4\alpha)^2 - (v_4\beta)^2. \tag{3.14}$$

For a fixed $\delta > 0$ and a large $R$, this situation was considered extensively by Eskin, Margulis and Mozes [9]. The authors give an asymptotical upper bound in this situation. We will examine how the constants involved in their bound depend on $\delta$, and find out that there is a linear dependency, which is what we essentially need. The author wishes to thank Alex Eskin for his assistance with this matter.

**Remarks:** 1. In a more recent paper, Eskin Margulis and Mozes [10] prove that for "generic" lattice $\Lambda$, there is a constant $c > 0$, such that for any *fixed* $\delta > 0$, as $R \to \infty$, $\#A(R,\, \delta)$ is asymptotic to $c\delta R$.

2. For our purposes we need a weaker result:

$$\#A(R, \delta) \ll_\epsilon R\delta \cdot R^\epsilon,$$

for every $\epsilon > 0$, just as in the case of a rectangular lattice (see lemma 2.4.2).

Theorem 2.3 in [9] considers a more general setting than proposition 3.2.1. We state here theorem 2.3 from [9] (see theorem 3.2.2). It follows from theorem 3.3 from [9], which will be stated as well (see theorem 3.2.3). Then we give an outline of the proof of theorem 2.3 of [9], and inspect the dependency on $\delta$ of the constants involved.

### 3.2.2 Theorems 2.3 and 3.3 from [9]

Let $\Delta$ be a lattice in $\mathbb{R}^n$. We say that a subspace $L \subset \mathbb{R}^n$ is $\Delta$-rational, if $L \cap \Delta$ is a lattice in $L$. We need the following definitions:

**Definitions:**

$$\alpha_i(\Delta) := \sup \left\{ \frac{1}{d_\Delta(L)} \middle| \, L \text{ is a } \Delta - \text{rational subspace of dimension } i \right\},$$

where

$$d_\Delta(L) := vol(L/(L \cap \Delta)).$$

Also

$$\alpha(\Delta) := \max_{0 \le i \le n} \alpha_i(\Delta).$$

Since the space of unimodular lattices is canonically isomorphic to $SL(n,\, \mathbb{R})/SL(n,\, \mathbb{Z})$, the notation $\alpha(g)$ makes sense for $g \in G := SL(n,\, \mathbb{R})$.

For a bounded function $f : \mathbb{R}^n \to \mathbb{R}$, with $|f| \le M$, which vanishes outside a ball $B(0,\, R)$, define $\tilde{f} : SL(n,\, \mathbb{R}) \to \mathbb{R}$ by the following formula:

$$\tilde{f}(g) := \sum_{v \in \mathbb{Z}^n} f(gv).$$

Lemma 3.1 in [19] implies that

$$\tilde{f}(g) < c\alpha(g), \tag{3.15}$$

where $c = c(f)$ is an explicit constant constant

$$c(f) = c_0 M \max(1, R^n),$$

for some constant $c_0 = c_0(n)$, independent on f. In section 3.2.4 we prove a stronger result, assuming some additional information about the support of $f$.

Let $Q_0$ be a quadratic form defined by

$$Q_0(\vec{v}) = 2v_1v_n + \sum_{i=2}^{p} v_i^2 - \sum_{i=p+1}^{n-1} v_i^2.$$

Since

$$v_1v_n = \frac{(v_1+v_n)^2-(v_1-v_n)^2}{2},$$

$Q_0$ is of signature $p$, $q$. Obviously, $G := SL(n,\mathbb{R})$ acts on the space of quadratic forms of signature $(p,\, q)$, and discriminant $\pm 1$, $\mathcal{O} = \mathcal{O}(p,\, q)$ by:

$$Q^g(v) := Q(gv).$$

Moreover, by the well known classification of quadratic forms, $\mathcal{O}$ is the orbit of $Q_0$ under this action.

In our case the signature is $(p,\, q) = (2,\, 2)$ and $n = 4$. We fix an element $h_1 \in G$ with $Q^{h_1} = Q_1$, where $Q_1$ is given by (3.14). There exists a constant $\tau > 0$, such that for every $v \in \mathbb{R}^4$,

$$\tau^{-1}\|v\| \le \|h_1 v\| \le \tau\|v\|. \tag{3.16}$$

We may assume, with no loss of generality that $\tau \ge 1$.

Let $H := Stab_{Q_0}(G)$. Then the natural mophism $H\backslash G \to \mathcal{O}(p,q)$ is a homeomorphism. Define a 1-parameter family $a_t \in G$ by:

$$a_t e_i = \begin{cases} e^{-t}e_1, & i = 1 \\ e_i, & i = 2, \ldots, n-1 \\ e^t e_n, & i = n \end{cases} \cdot$$

Clearly, $a_t \in H$. Furthermore, let $\hat{K}$ be the subgroup of $G$ consisting of orthogonal matrices, and denote $K := H \cap \hat{K}$.

Let $(a,\, b) \in \mathbb{R}^2$ be given and let $Q : \mathbb{R}^n \to \mathbb{R}$ be any quadratic form. The object of our interest is:

$$V_{(a,\, b)}(\mathbb{Z}) = V^Q_{(a,\, b)}(\mathbb{Z}) = \{x \in \mathbb{Z}^n :\ a < Q(x) < b\}.$$

Theorem 2.3 states, in our case:

**Theorem 3.2.2** (Theorem 2.3 from [9])**.** *Let $\Omega = \{v \in \mathbb{R}^4 |\, \|v\| < \nu(v/\|v\|)\}$, where $\nu$ is a nonnegative continuous function on $S^3$. Then we have:*

$$\# V^{Q_1}_{(a,\, b)}(\mathbb{Z}) \cap T\Omega < cT^2 \log T,$$

*where the constant $c$ depends only on $(a,\, b)$.*

The proof of theorem 3.2.2 relies on theorem 3.3 from [9], and we give here a particular case of this theorem

**Theorem 3.2.3** (Theorem 3.3 from [9])**.** *For any (fixed) lattice $\Delta$ in $\mathbb{R}^4$,*

$$\sup_{t>1} \frac{1}{t} \int\limits_K \alpha(a_t k \Delta) dm(k) < \infty,$$

*where the upper bound is universal.*

### 3.2.3 Outline of the proof of theorem 3.2.2:

**Step 1:** Define

$$J_f(r, \zeta) = \frac{1}{r^2} \int\limits_{\mathbb{R}^2} f(r, x_2,\, x_3,\, x_4) dx_2 dx_3, \tag{3.17}$$

where

$$x_4 = \frac{\zeta - x_2^2 + x_3^2}{2r}$$

Lemma 3.6 in [9] states that $J_f$ is approximable by means of an integral over the compact subgroup K. More precisely, there is some constant $C > 0$, such that for every $\epsilon > 0$,

$$\left| C \cdot e^{2t} \int_K f(a_t k v) \nu(k^{-1} e_1) dm(k) - J_f\big(\|v\| e^{-t},\, Q_0(v)\big) \nu\big(\frac{v}{\|v\|}\big) \right| < \epsilon \tag{3.18}$$

with $e^t$, $\|v\| > T_0$ for some $T_0 > 0$.

**Step 2:** Choose a continuous nonnegative function $f$ on $\mathbb{R}^4_+ = \{x_1 > 0\}$ which vanishes outside a compact set so that

$$J_f(r, \zeta) \geq 1 + \epsilon$$

on $[\tau^{-1},\, 2\tau] \times [a,\, b]$. We will show later, how one can choose $f$.

**Step 3:** Denote $T = e^t$, and suppose that $T \leq \|v\| \leq 2T$ and $a \leq Q_0(h_1 v) \leq b$. Then by (3.16), $J_f\big(\|h_1 v\| T^{-1}, Q_0(h_1 v)\big) \geq 1 + \epsilon$, and by (3.18), for a sufficiently large $t$,

$$C \cdot T^2 \int_K f(a_t k h_1 v) dm(k) \geq 1, \tag{3.19}$$

for $T \leq \|v\| \leq 2T$ and

$$a \leq Q_0^x(v) \leq b. \tag{3.20}$$

**Step 4:** Summing (3.19) over all $v \in \mathbb{Z}^4$ with (3.20) and $T \leq \|v\| \leq 2T$, we obtain:

$$\begin{aligned} \# V_{(a,\, b)}(\mathbb{Z}) \cap [T,\, 2T] S^3 &\leq \sum_{v \in \mathbb{Z}^n} C \cdot T^2 \int_K f(a_t k h_1 v) dm(k) \\ &= C \cdot T^2 \int_K \tilde{f}(a_t k h_1) dm(k) \end{aligned} \tag{3.21}$$

using the nonnegativity of $f$.

**Step 5:** By (3.15), (3.21) is

$$\leq C \cdot c(f) \cdot T^2 \int_K \alpha(a_t k h_1) dm(k).$$

**Step 6:** The result of theorem 2.3 is obtained by using theorem 3.2.3 on the last expression.

### 3.2.4 $\delta$-dependency:

In this section we assume that $(a,\, b) = (0,\, \delta)$, which suits the definition of the set $A(R,\, \delta)$, (3.13). One should notice that there only 3 $\delta$-dependent steps:

• Choosing $f$ in step 2, such that $J_f \geq 1+\epsilon$ on $[\tau^{-1},\, 2\tau] \times [0,\, \delta]$. We will construct a family of functions $f_\delta$ with an universal bound $|f_\delta| \leq M$, such that $f_\delta$ vanishes outside of a compact set which is only slightly larger than

$$V(\delta) = [\tau^{-1},\, 2\tau] \times [-1,\, -1]^2 \times [0,\, \frac{\delta\tau}{2}]. \tag{3.22}$$

This is done in section 3.2.4.

• The dependency of $T_0$ of step 3, so that the usage of lemma 3.6 in [9] is legitimate. For this purpose we will have to examine the proof of this lemma. This is done in section 3.2.4.

• The constant $c$ in (3.15). We would like to establish a *linear* dependency on $\delta$. This is straightforward, once we are able to control the number of integral points in a domain defined by (3.22). This is done in section 3.2.4.

**Choosing $f_\delta$:**

**Notation:** For a set $U \subset \mathbb{R}^n$, and $\epsilon > 0$, denote

$$U_\epsilon := \{x \in \mathbb{R}^n : \max_{1\leq i\leq n} |x_i - y_i| \leq \epsilon, \text{ for some } y \in U\}.$$

Choose a nonnegative continuous function $f_0$, on $\mathbb{R}^4_+$, which vanishes outside a compact set, such that its support, $E_{f_0}$, slightly exceeds the set $V(1)$. More precisely, $V(1) \subset E_{f_0} \subset V(1)_{\delta_0}$ for some $\delta_0 > 0$. By the uniform continuity of $f$, there are $\epsilon_0$, $\delta_0 > 0$, such that if $\max\limits_{1\le i\le 4} |x_i - x_i^0| \le \delta_0$, then $f(x) > \epsilon_0$, for every $x^0 = (x_1^0, 0, 0, x_4^0) \in V(1)$.

Thus for $(r, \zeta) \in [\tau^{-1}, 2\tau] \times [0, \delta]$, the contribution of $[-\delta_0, \delta_0]^2$ to $J_{f_0}$ is $\ge \epsilon_0 \cdot (2\delta_0)^2$. Multiplying $f_0$ by a suitable factor, and by the linearity of $J_{f_0}$, we may assume that this contribution is at least $1 + \epsilon$.

Now define $f_\delta(x_1, \ldots, x_4) := f_0(x_1, x_2, x_3, \frac{x_4}{\delta})$. We have for $\delta \ge 1$

$$\frac{\zeta - x_2^2 + x_3^2}{2r\delta} = \frac{\zeta/2r}{\delta} - \frac{(x_2/\sqrt{\delta})^2}{2r} + \frac{(x_3/\sqrt{\delta})^2}{2r}.$$

Thus for $\delta \ge 1$, if $(r, \zeta) \in [\tau^{-1}, 2\tau] \times [0, \delta]$ and for $i = 2, 3$, $|x_i| < \delta_0$, $f_\delta$ satisfies:

$$f_\delta(r, x_2, x_3, x_4) > \epsilon_0,$$

and therefore the contribution of this domain to $J_{f_\delta}$ is

$$\ge \epsilon_0(2\delta)^2 \ge 1 + \epsilon$$

by our assumption.

By the construction, the family $\{f_\delta\}$ has a universal upper bound $M$ which is the one of $f_0$.

**How large is $T_0$**

The proof of lemma 3.6 from [9] works well along the same lines, as long as

$$f(a_t x) \ne 0 \tag{3.23}$$

implies that for $t \to \infty$, $x/\|x\|$ converges to $e_1 = (1, 0, 0, 0)$. Now, since $a_t$ preserves $x_1x_4$, (3.23) implies for the particular choice of $f = f_\delta$ in section 3.2.4:

$$|x_1x_4| = O(\delta); \quad x_1 \gg T.$$

Thus

$$\|x\| = x_1 + O\left(\frac{\delta}{T}\right) + O(1),$$

and so, as long as $\delta = O(T)$, $x/\|x\|$ indeed converges to $e_1$.

**Bounding integral points in $V_\delta$:**

**Lemma 3.2.4.** *Let $V(\delta)$ defined by*

$$V(\delta) = [\tau^{-1},\, 2\tau] \times [-1,\, -1]^{n-2} \times [0,\, \frac{\delta\beta}{2}]. \tag{3.24}$$

*for some constant $\tau$ and $n \geq 3$. Let $g \in SL(n, \mathbb{R})$ and denote*

$$N(g,\, \delta) := \#V(\delta) \cap g\mathbb{Z}^n.$$

*Then for $\delta \geq 1$,*

$$\left| N(g,\, \delta) - \frac{2^{n-2}(2\tau - \tau^{-1})\delta}{\det g} \right| \leq c_5 \delta \sum_{i=1}^{n-1} \frac{1}{vol(L_i/(g\mathbb{Z}^n \cap L_i)}$$

*for some $g$-rational subspaces $L_i$ of $\mathbb{R}^4$ of dimension $i$, where $c_5 = c_5(n)$ depends only on $n$.*

A direct consequence of lemma 3.2.4 is the following

**Corollary 3.2.5.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function which vanishes outside a compact set $E$. Suppose that $E \subset V_\epsilon(\delta)$ for some $\epsilon > 0$. Then for $\delta \geq 1$,* (3.15) *is satisfied with*

$$c(f) = c_3 \cdot M\delta,$$

*where the constant $c_3$ depends on $n$ only.*

In order to prove lemma 3.2.4, we shall need the following:

**Lemma 3.2.6.** *Let $\Lambda \subset \mathbb{R}^n$ be a $m$-dimensional lattice, and let*

$$A_t = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & t \end{pmatrix} \tag{3.25}$$

*an $n$-dimensional linear transformation. Then for $t > 0$ we have*

$$\det A_t\Lambda \leq t \det \Lambda. \tag{3.26}$$

*Proof.* We may assume that $m < n$, since if $m = n$, we obviously have an equality. Let $v_1, \ldots, v_m$ the basis of $\Lambda$ and denote for every $i$, $u_i \in \mathbb{R}^{n-1}$ the vector, which consists of first $n-1$ coordinates of $v_i$. Also, let $x_i \in \mathbb{R}$ be the last coordinate of $v_i$. By switching vectors, if necessary, we may assume $x_1 \neq 0$. We consider the function

$$f(t) := (\det A_t\Lambda)^2,$$

as a function of $t \in \mathbb{R}$.

Obviously,

$$f(t) = \det\left(\langle u_i, u_j\rangle + x_i x_j t^2\right)_{1 \leq i, j \leq m}.$$

Substracting $\frac{x_i}{x_1}$ times the first row from any other, we obtain:

$$f(t) = \begin{vmatrix} \langle u_1, u_j\rangle + x_1 x_j t^2 \\ \langle u_2, u_j\rangle - \frac{x_2}{x_1}\langle u_1, u_j\rangle \\ \vdots \\ \langle u_m, u_j\rangle - \frac{x_m}{x_1}\langle u_1, u_j\rangle \end{vmatrix},$$

and by the multilinearity property of the determinant, $f$ is a linear function of $t^2$. Write

$$f(t) = a(t^2 - 1) + bt^2.$$

Thus

$$b = f(1); \quad a = -f(0),$$

and so $b = \det \Lambda$, and $a = -\det\big(\langle u_i,\, u_j\rangle\big) \le 0$, being minus the determinant of a Gram matrix. Therefore,

$$(\det A_t\Lambda)^2 - t^2 \det \Lambda = a(t^2 - 1) \le 0$$

for $t \ge 1$, implying (3.25). $\square$

*Proof of lemma 3.2.4.* We will prove the lemma, assuming $\beta = 2$. However, it implies the result of the lemma for any $\beta$, affecting only $c_5$. Let $\delta > 0$. Trivially,

$$N(g,\, \delta) = N(g_0,\, 1),$$

where $g_0 = A_\delta^{-1} g$ with $A_\delta$ given by (3.25). Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be the successive minima of $g_0$, and pick linearly independent lattice points $v_1, \ldots, v_n$ with $\|v_i\| = \lambda_i$. Denote $M_i$ the linear space spanned by $v_1, \ldots, v_i$ and the lattice $\Lambda_i = g_0\mathbb{Z}^n \cap M_i$.

First, assume that $\lambda_n \le \sqrt{\tau^2 + (n-1)} =: r$. Now, by Gauss' argument,

$$\left| N(g_0,\, 1) - \frac{2^{n-1}(2\tau - \tau^{-1})\delta}{\det g} \right| \le \frac{1}{\det g_0} vol(\Sigma),$$

where

$$\Sigma := \{x :\ dist(x,\, \partial V(1)) \le n\lambda_n\}.$$

Now, for $\lambda_n \le r$,

$$vol(\Sigma) \ll \lambda_n,$$

where the constant implied in the " $\ll$ "-notation depends on $n$ only (this is obvious for $\lambda_n \le \frac{1}{2n}$, and trivial otherwise, since for $\lambda_n \le r$, $vol(\Sigma) = O(1)$). Thus,

$$\begin{aligned}\left| N(g_0,\, 1) - \frac{2^{n-1}(2\tau - \tau^{-1})\delta}{\det g} \right| &\ll \frac{\lambda_n}{\det g_0} \ll \frac{1}{\det \Lambda_{n-1}} \\ = \frac{1}{vol(M_{n-1}/M_{n-1} \cap g_0\mathbb{Z}^n)} &\le \frac{\delta}{vol(A_\delta M_{n-1}/A_\delta M_{n-1} \cap g\mathbb{Z}^n)}\end{aligned}$$

Next, suppose that $\lambda_n > r$. Then,

$$V(\delta) \cap g_0\mathbb{Z}^n \subset V(\delta) \cap \Lambda_{n-1}.$$

Thus, by the induction hypothesis, the number of such points is:

$$\begin{aligned}
\leq & c_4 \sum_{i=0}^{k-1} \frac{1}{\det(\Lambda_i)} = \sum_{i=0}^{k-1} \frac{1}{vol(M_i/M_i \cap g_0\mathbb{Z}^n)} \\
\leq & \,\delta \sum_{i=0}^{k-1} \frac{1}{vol(A_\delta M_i/A_\delta \mathcal{M}_i \cap g\mathbb{Z}^n)}.
\end{aligned}$$

Since $\lambda_n > r$, we have

$$\frac{1}{\det g} = \frac{1}{\lambda_n} \frac{1}{\det g/\lambda_n} \ll \frac{1}{\det g/\lambda_n} \ll \frac{1}{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}},$$

and we're done by defining $L_i := A_\delta M_i$. $\square$

### 3.3 Unsmoothing

We have the following analogue of lemma 2.3.1:

**Lemma 3.3.1.** *Under the notations of lemma 2.3.1, for every $a > 0$, $c > 1$ we have*

$$\begin{aligned}
N_\Lambda(t) = & \frac{\pi}{\beta} t^2 - \frac{\sqrt{t}}{\beta\pi} \sum_{\substack{\vec{k} \in \Lambda^* \setminus \{0\} \\ |\vec{k}| \leq \sqrt{N}}} \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} + O(N^a) \\
& + O\left(\frac{t^{2c-1}}{\sqrt{N}}\right) + O\left(\frac{t}{\sqrt{N}} \cdot \left(\log t + \log(\delta_\Lambda(t^2))\right)\right) \\
& + O\left(\log N + \log(\delta_{\Lambda^*}(t^2))\right)
\end{aligned}$$

As in section 2.3, a typical example of the function $\delta_\Lambda$ for $\Lambda = \langle 1, \alpha + i\beta \rangle$, with a Diophantine $(\alpha, \alpha^2, \beta^2)$, we may choose $\delta_\Lambda(y) = \frac{c}{y^K}$, where $c$ is a constant. In this case, if $\Lambda \ni \vec{k} = (a, b)$, then $A_{|\vec{k}|} = \pm(a, b)$, due to lemma 3.1.3, provided that $\beta \notin \mathbb{Q}(\alpha)$.

*Sketch of proof.* The proof of this lemma is literally the same as the one of lemma 2.3.1, starting from

$$\mathcal{Z}_\Lambda(s) := \frac{1}{2}\sum_{\vec{k}\in\Lambda\setminus 0}\frac{1}{|\vec{k}|^{2s}} = \sum_{(m,\,n)\in\mathbb{Z}^2_+\setminus 0}\frac{1}{\big((m+n\alpha)^2+(\beta n)^2\big)^s},$$

where the series is convergent for $\Re s > 1$. □

**Proposition 3.3.2.** *Let a lattice $\Lambda = \langle 1,\, \alpha + i\beta\rangle$ with a Diophantine triple of numbers $(\alpha^2, \alpha\beta,\, \beta^2)$ be given. Suppose that $L \to \infty$ as $T \to \infty$ and choose $M$, such that $L/\sqrt{M} \to 0$, but $M = O\big(T^\delta\big)$ for every $\delta > 0$ as $T \to \infty$. Suppose furthermore, that $M = O(L^{s_0})$ for some (fixed) $s_0 > 0$. Then*

$$\left\langle \left| S_\Lambda(t,\, \rho) - \tilde{S}_{\Lambda,\, M,\, L}(t)\right|^2 \right\rangle \ll \frac{1}{\sqrt{M}}$$

The proof of proposition 3.3.2 proceeds along the same lines as the one of proposition 2.4.1, using again an asymptotic formula for the sharp counting function, given by lemma 3.3.1. The only difference is that here we use proposition 3.2.1 rather than lemma 2.4.2.

Once we have proposition 3.3.2 in our hands, the proof of our main result, namely, theorem 1.1.2 proceeds along the same lines as the one of theorem 1.1.1.

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