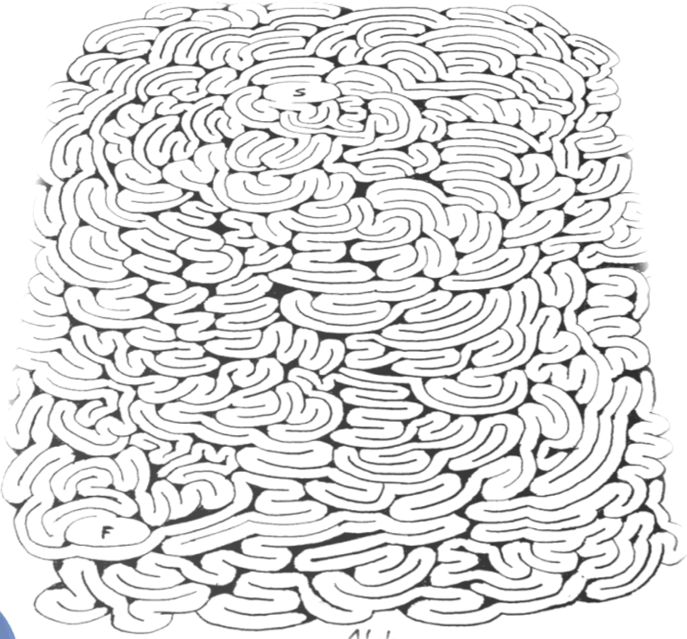

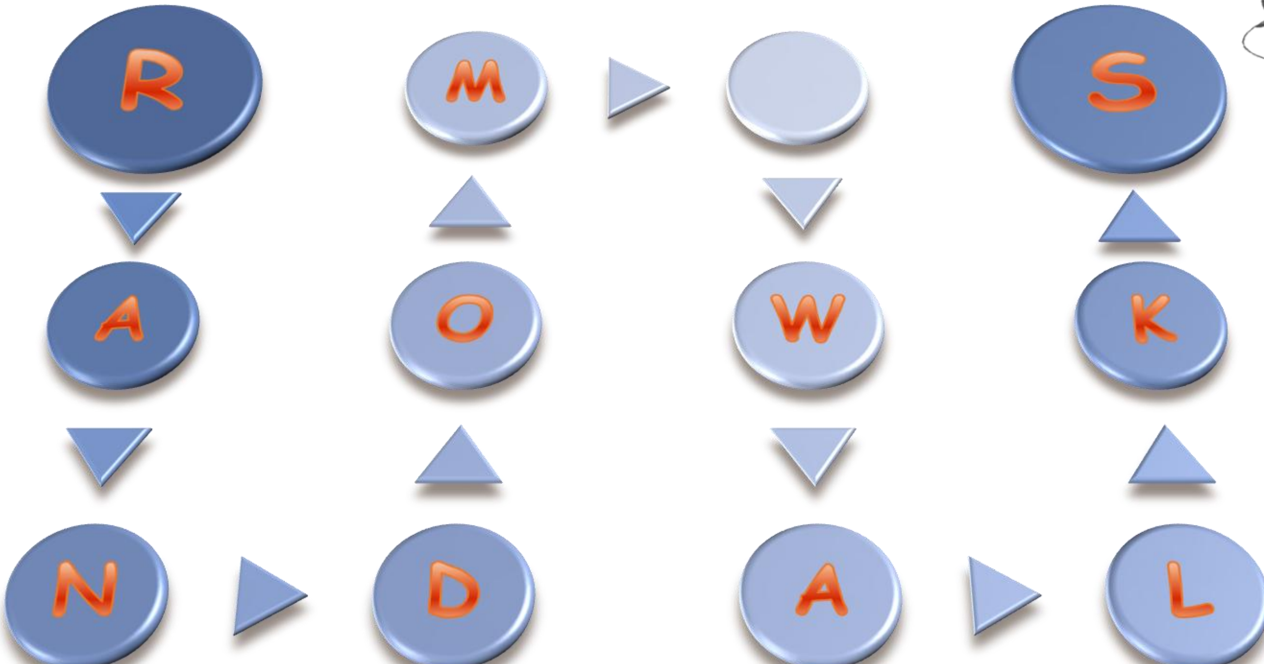
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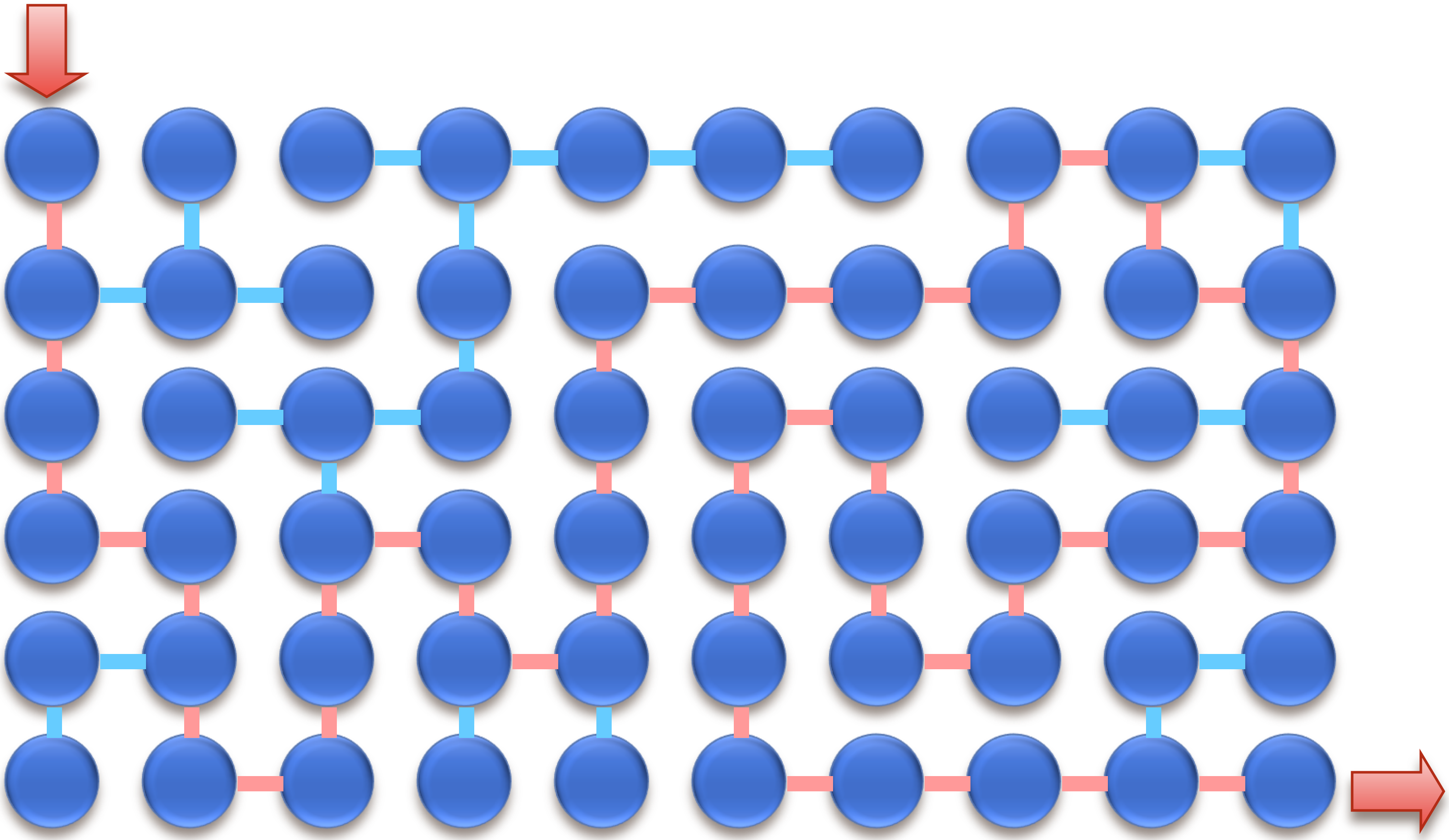
Is there a Solution?



Goal

- Undirected Connectivity
- In Random LOGSPACE

Plan

- Introduce Random Walks

Undirected Connectivity

Instance:

- An undirected graph $G=(V,E)$ and two vertices $s,t \in V$

Decision Problem:

- Is there a path in G from s to t

Theorem:

- CONN $\in$ NL

Proof:

- Nondeterm. walk

What shall we do for an undirected graph?

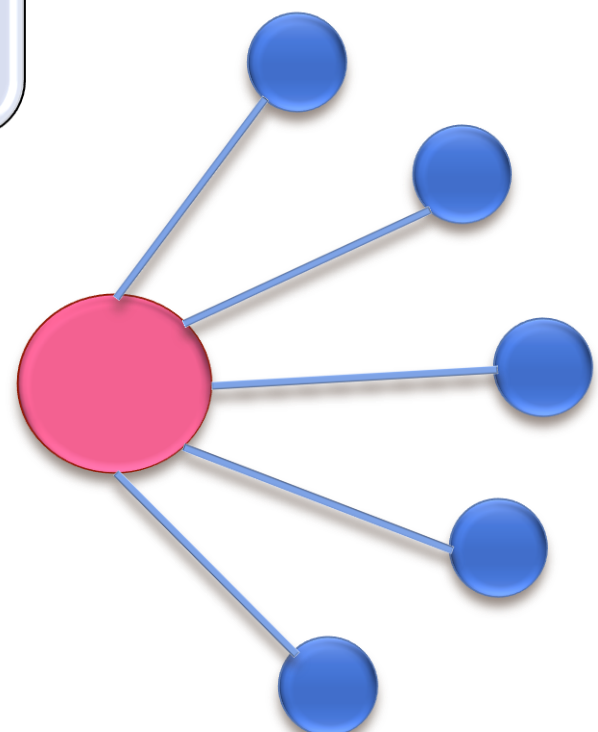
Nondeterministic vs. Random

All-powerful **guesses**

- which **neighbor** to go to **next**

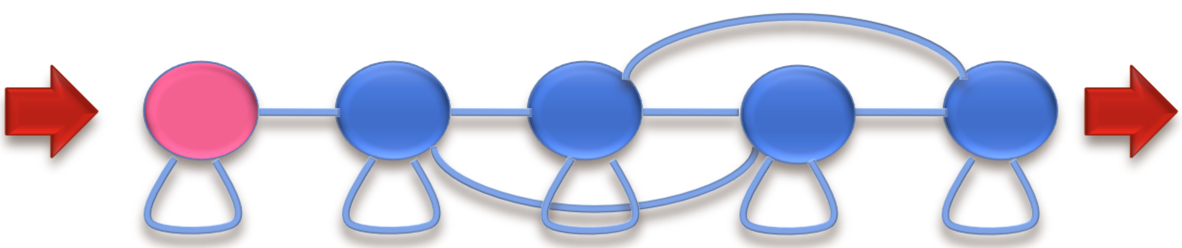
Randomly **guess**

- which **neighbor** to go to **next**



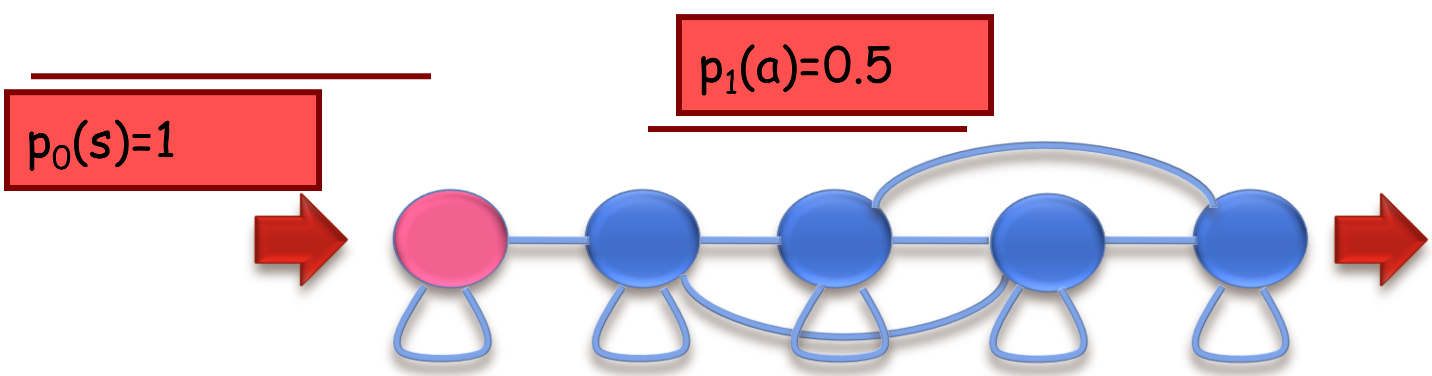
Random Walks

0	• Add a self loop to each vertex
Start	• at s
Let	• d_i be the degree of the current node.
Jump	• to each neighbor with probability $1/d_i$
Stop	• if reach t



Notation:

- Let v_t denote the vertex visited at time t (v_0=s)
- Let $p_t(i) = Pr[v_t=i]$



Stationary Distribution

Lemma:

- Let $G=(V,E)$ be a connected graph, then for any $i \in V$

$$\lim_{t \to \infty} p_t(i) = \frac{d_i}{2|E|}$$

Weaker Claim

Lemma:

- If for some t, for all $i \in V$

$$p_t(i) = \frac{d_i}{2|E|}$$

then for all $i \in V$

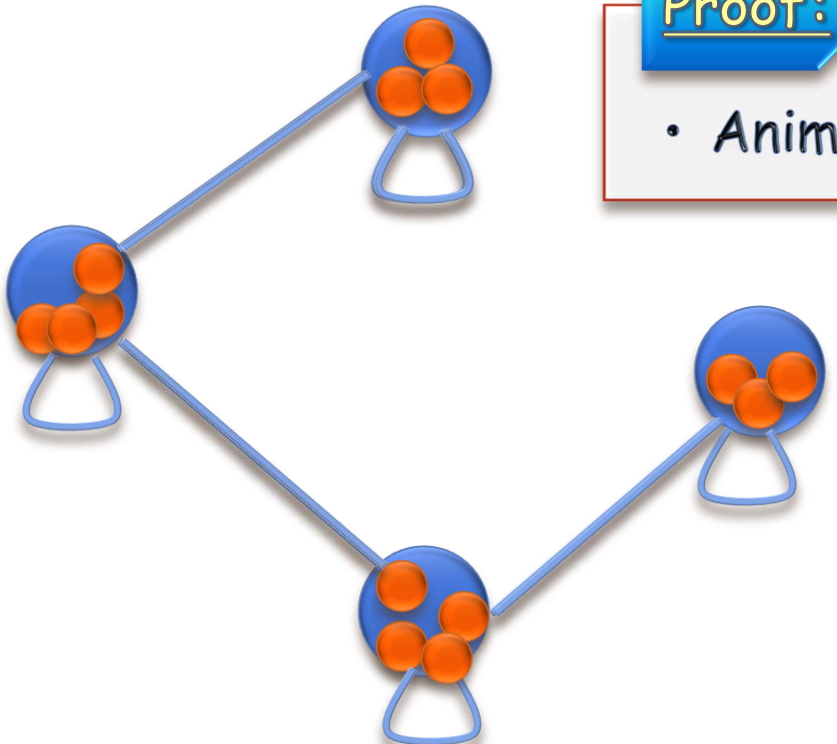
$$p_{t+1}(i) = \frac{d_i}{2|E|}$$

FixPoint

Proof:

- $\Sigma d_i = 2|E|$

  Lemma: Vertex i has probability d_i at time t ⇒ Vertex i has the same probability at time $t+1$ ■



Proof:

- Animated

Proof:

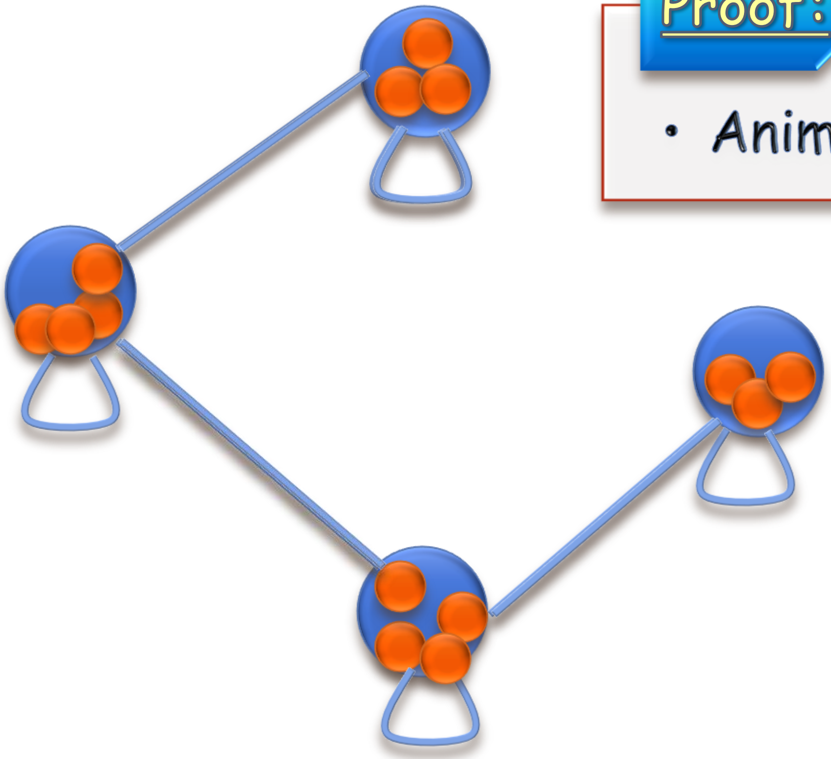
FixPoint

- $\Sigma d_i = 2|E|$

  Lemma: Vertex i has probability d_i at time t ⇒ Vertex i has the same probability at time $t+1$ ■

Proof:

- Animated



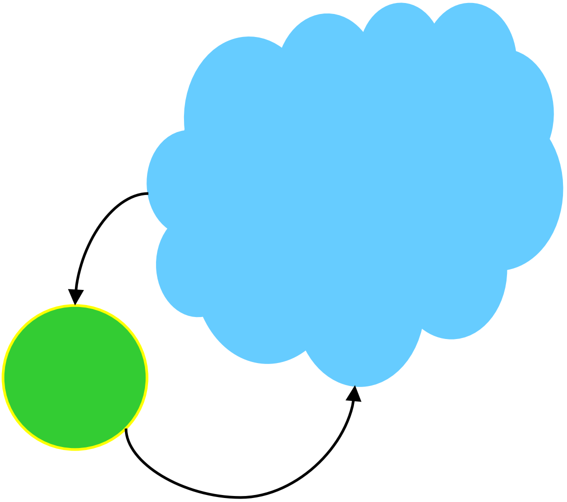
Using the Asymptotic Estimate

Corollary:

- Starting from any vertex i, the expected time till revisit i is $2|E|/d_i$

Proof:

- If expected time were longer, we could not have seen i as many times as we should ■



One-Sided Error

- Note that if the right answer is 'NO', we clearly answer 'NO'.
- Hence, this random walk algorithm has one-sided error.
- Such algorithms are called "Monte-Carlo" algorithms.

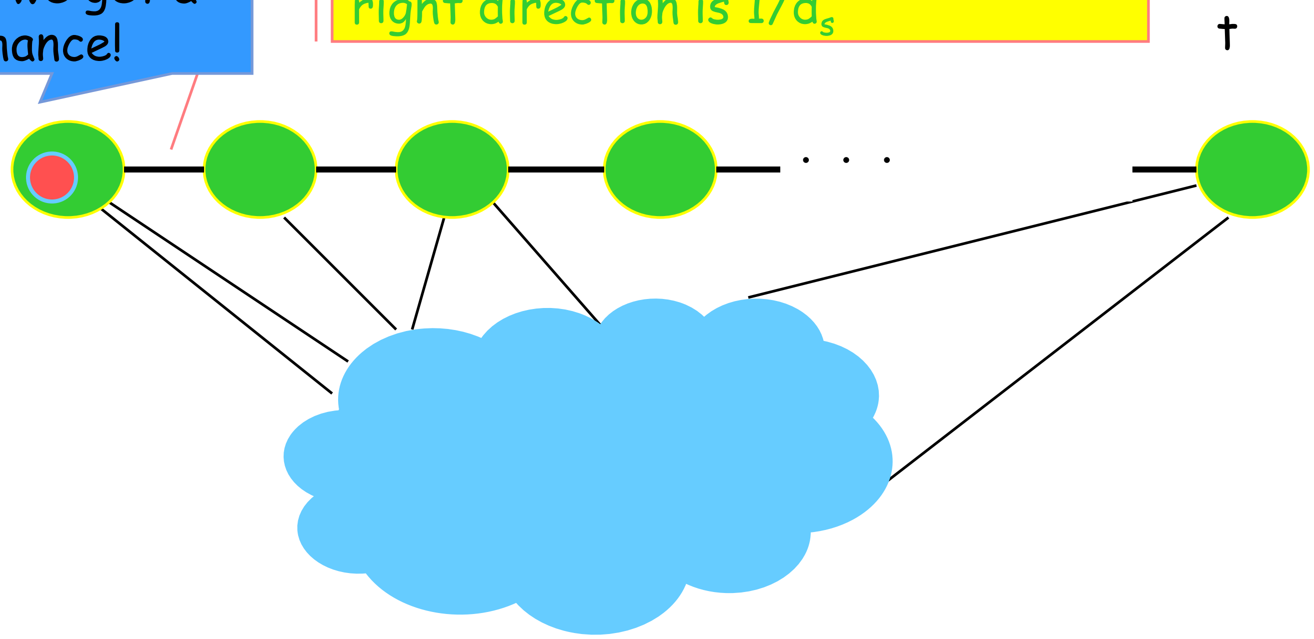
Number of Steps Required?

Note:

- If there is a path, in how many steps do we expect to arrive at t?

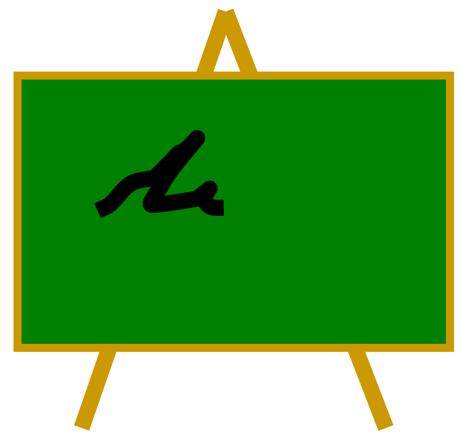
But every time we get here, we get a second chance!

The probability we head in the right direction is $1/d_s$



How Many Steps Are Needed?

- Since expectedly we return to each vertex within $|E|/d_i$ steps
- The walk expectedly heads in the right direction within $2|E|$ steps
- By linearity of the expectation, it is expected to reach t within $d(s,t) \cdot 2|E| \leq 2|V| \cdot |E|$ steps.



Randomized Algorithm for Undirected Connectivity

1. Run the random walk from s for $2|V|\cdot|E|$ steps.
2. If node t is ever visited, answer "there is a path from s to t".
3. Otherwise, reply "there is probably no path from s to t".

Main Theorem

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Theorem: The above algorithm

- uses logarithmic space
- always right for 'NO' instances.
- errs with probability at most $\frac{1}{2}$ for

To maintain the current position we only need log|V| space

Markov: Pr(X>2E[X])<$\frac{1}{2}$

Summary

- We explored the undirected connectivity problem.
- We saw a log-space randomized algorithm for this problem.
- We used an important technique called random walks.