COMPLEXITY: Exercise No. 2

1 Question 1

For each of the following statements, prove, disprove or show that it is an open problem:

1. If $L_1, L_2 \in \mathbf{coNP}$ then $L_1 \cap L_2 \in \mathbf{coNP}$. (Test 99)
   Answer: True. $L_1^c, L_2^c \in \mathbf{NP}$, so $L_1^c \cup L_2^c \in \mathbf{NP}$, and hence $L_1 \cap L_2 = (L_1^c \cup L_2^c)^c \in \mathbf{coNP}$.

2. If $L \in \mathbf{NP}, L_1 \subsetneq L$ and $L_1 \in \mathbf{coNP}$ then $L - L_1 \in \mathbf{NP}$.
   Answer: True. $L_1 \in \mathbf{coNP}$ so $L_1^c \in \mathbf{NP}$, and since $\mathbf{NP}$ is closed for intersection, $L - L_1 = L \cap L_1^c \in \mathbf{NP}$.

3. If $L \in \mathbf{NPC}$ then $\{xx : x \in L\} \in \mathbf{NPC}$. (Test 94)
   Answer: True. $\{xx : x \in L\}$ is clearly in $\mathbf{NP}$ - check that the input is of the form xx and run the NDTM of L on x. On the other hand, there is a reduction from L, by just duplicating the input, so it is in $\mathbf{NPC}$.

2 Question 3

Consider the following problem:
SUBGRAPH ISOMORPHISM (SGI):
Instance: Two graphs, $G = (V, E), H = (U, F)$.
Question: Is there a 1-1 mapping $\phi : U \to V$, such that if $(u, u') \in F$ then $(\phi(u), \phi(u')) \in E$?

Prove that UHC $\prec$ SGI.

Answer: This is almost reduction by generalization. Observe that UHC is a special case of SGI, with H being just a simple cycle of length $|V|$. For this case, the questions asked by SGI and UHC are equivalent.

The reduction is, therefore, for an instance $G = (V, E)$ of UHC, we produce the instance (G, H), where H is a simple cycle of length $|V|$.

This is clearly a polynomial time reduction.

The reduction is correct, because, by the definitions:
(G, H) is a "yes" instance of $SGI \Leftrightarrow$
G has a subgraph which is a simple cycle of length $|V| \Leftrightarrow$
G has a Hamiltonian cycle $\Leftrightarrow$
G is a "yes" instance of UHC.

3 Question 4

Consider the following problems:
3-SAT: