COMPLEXITY: Exercise No. 5

1 Question 1

Show that MAX-CUT is **NP**-complete when the input graph is simple.
Answer: Reduce from MAX CUT on multi graphs as follows;
Let $(G = (V, E), k)$ be an instance of MAX CUT on multi graphs. Construct G' by adding two vertices on each edge. Let $k' = 2|E| + k$. Note that each edge in a cut of G can be translated to 3 edges of a corresponding cut in G', while an edge not in the cut of G, results in 2 edges.

2 Question 2

(Test 2000) Give a polynomial time algorithm that given a graph $G = (V, E)$ which contains an independent set of size $\frac{3}{4}|V|$ finds an independent set of size at least $\frac{1}{2}|V|$.
Answer: G contains an independent set of size $\frac{3}{4}|V|$. Its complement is a vertex cover of size $\frac{1}{4}|V|$. Use the 2-approximation algorithm for vertex cover to find a vertex cover of size at most $\frac{1}{2}|V|$, whose complement is an independent set of size at least $\frac{1}{2}|V|$.

3 Question 3

(Test 98) Consider the following problem:
MAX-3-CUT:
Instance: A simple undirected graph $G = (V, E)$ and a positive integer k.
Question: Is there a partition of V into 3 disjoint sets (V_1, V_2, V_3) such that the number of edges whose endpoints are in different sets is at least k?
a. Prove that MAX-3-CUT is **NP**-complete.
b. Give a polynomial time approximation algorithm to the corresponding optimization problem (i.e. the problem of finding a 3-cut of maximum size). What is the approximation ratio?
Answer:
a. Reduce from 3-coloring by setting $k = |E|$.
b. The greedy algorithm, similar to the one we saw for MAX-CUT, gives a 3-cut of size at least $\frac{2}{3}|E|$. Thus the approximation ratio is $\frac{3}{2}$.

4 Question 4

(Test 98) Consider the TSP problem on a complete undirected graph with a length $l_{i,j} \geq 0$ for each edge (i, j). Suppose that the lengths satisfy $l_{i,j} \leq l_{i,k} + 2l_{k,j}$ for all i, j, k.
Give an approximation algorithm for this problem.
What is the approximation ratio?
Answer: First you are to prove (by induction) that for each path between i and j, the length of

(i, j) is at most twice the length of the path. Now, apply the 2-approximation algorithm shown in class, to get an approximation ratio of 4.

5 Question 5

Consider the TSP problem on a complete undirected graph with a length $l_{i,j} \geq 0$ for each edge (i, j). Suppose that the lengths satisfy $l_{i,j} \leq l_{i,k} + l_{k,j}$ for all i, j, k. Prove that this problem cannot be approximated to within a factor less than $1 + \frac{1}{n}$, whereas n is the number of vertices in the graph.
Answer: Reduce from Hamilton Cycle as follows. Given an instance $G = (V, E)$ for HC, construct G', the complete graph on V, with the following lengths: $\forall (i, j) \in E \; l_{i,j} = 1$, and $\forall (i, j) \notin E \; l_{i,j} = 2$. Clearly, the triangle inequality is satisfied. Let $n = |V|$. If G contains a HC, then this is a HC of length n in G', and if there is no HC in G, then every HC in G' is of length at least $n + 1$. Since HC is NP-hard, so is Gap-TSP$[n, n + 1]$ with the triangle inequality. Thus, it is NP-hard to approximate TSP with the triangle inequality to within any factor less than $1 + \frac{1}{n}$.