On the Complexity of Approximating TSP with Neighborhoods and Related Problems

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Abstract

We prove that various geometric covering problems, related to the Travelling Salesman Problem cannot be efficiently approximated to within any constant factor unless $P = NP$. This includes the Group-Travelling Salesman Problem (TSP with Neighborhoods) in the Euclidean plane, the Group-Steiner-Tree in the Euclidean plane and the Minimum Watchman Tour and the Minimum Watchman Path in 3-D. It resolves three open problems presented in the comprehensive survey of Mitchell [Mit00], improves a previously known approximation hardness factor of $\frac{2041}{2040}$ [GL00, dBGK^{+}02] for the first problem, and it is the first approximation hardness factor for the other problems. Some inapproximability factors are also shown for special cases of the above problems, where the size of the sets is bounded. Group-TSP and Group-Steiner-Tree where each neighbourhood is connected are also considered. It is shown that approximating these variants to within any constant factor smaller than 2, is NP-hard.

For the Group-Travelling Salesman and Group-Steiner-Tree Problems in dimension d, we show an innapproximability factor of $O(\log^{\frac{d-1}{d}} n)$ under a plausible conjecture regarding the hardness of Hyper-Graph Vertex-Cover.

1 Introduction

The Travelling Salesman Problem *(TSP)* is a classical problem in combinatorial optimization, and has been studied extensively in many forms. It is the problem of a travelling salesman who has to visit n locations, returning eventually to the starting point. The goal may be to minimize the total distance traversed, driving time, or money spent on toll roads, where the cost (in terms of length units, time units money or other) is given by an $n \times n$ matrix of non-negative weights. In the geometric TSP, the matrix represents distances in a Euclidean space. In other certain natural instances (e.g. time and money) while weights might not agree with a Euclidean metric, they still obey the triangle inequality, namely the cost of traversing from a to b is not higher than the cost of traversing from a to b, via other points. Formally, the Travelling Salesman Problem can be defined as follows: given a set P of points in a metric space, find a traversal of shortest length visiting each point of P, and return to the starting point.

TSP in the Plane. Finding the optimal solution of a given instance of TSP with triangle inequality is NP-hard, as obtained by a simple reduction from the Hamilton-Cycle problem. Even in the special case where the matrix represents distances between points in the Euclidean plane, it is also proved to be NP-hard [GGJ76, Pap77].

The latter problem has polynomial time approximation scheme (PTAS) - that is, although it is NP-hard to come up with the optimal solution, one can have, in polynomial time, a $1+\varepsilon$ approximation of the optimal solution for any $\varepsilon > 0$, (see [Aro96, Mit96]). This however, is not the case for the non-geometric variants.

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Triangle Inequality. In the general case, approximating TSP to within any factor is NP-hard (again, by a simple reduction from the Hamilton-Cycle problem). When only triangle inequality is assured, the best known algorithm gives a $\frac{3}{2}$ approximation ratio, if weights are symmetric [Chr76]. If weights can be asymmetric (that is, the cost from a to b is not necessarily the same as the cost from b to a), the best known approximation ratio is $O(\log n)$ [FGM82]. Although the asymmetric case may seem unnatural having the Euclidean metric intuition in mind, when weights represent measures other than length, or for example - where the lengths are of one-way roads, the asymmetric formulation is natural. Both the symmetric and the asymmetric variances are conjectured to be efficiently approximable to within $\frac{4}{3}$ (see [CVM00]) .

In regard to the hardness of approximation, [PV00] gave evidence that unless $P = NP$, the symmetric case cannot be efficiently approximated to within a factor smaller than $\frac{234}{233}$, and the asymmetric case to within a factor smaller than $\frac{98}{97}$. For bounded metrics [EK01] showed hardness of approximation factors of $\frac{131}{130}$ and $\frac{174}{173}$ respectively.

Group-TSP. A natural generalization of this problem is the Group-TSP (*G-TSP*), known also by the names the One-of-a-Set-TSP, TSP with neighborhoods and the Errand Scheduling problem. A travelling salesman has to meet n customers. Each of them is willing to meet the salesman in specified locations (referred to as a region). For instances in which each region contains exactly one point, this becomes the TSP problem. For instances in which all edges are of weight 1, this becomes the Hitting-Sets (or Set-Cover) problem. Another natural illustration of the G-TSP is the Errand Scheduling Problem as described in [Sla97]. A list of n jobs (or errands) to be carried out is given, each of which can be performed in a few locations. The objective is to find a close tour of minimal length, such that all jobs can be performed. That is, for every job on the list, there is at least one location on the tour, at which the job can be performed (it is allowed to perform more than one job in a single location). If every job can be performed in exactly k locations, then we call this problem *k-G-TSP*. k-G-TSP can be approximated to within $\frac{3k}{2}$ [Sla97]. This algorithm generalizes the $\frac{3}{2}$ approximation ratio of [Chr76] for $k \geq 1$.

As G-TSP (with triangle inequality) is a generalization of both TSP and Set-Cover any innapproximability factor for any of those two problems, holds for the G-TSP. Thus, by [LY93, RS97, Fei98], G-TSP is hard to approximate to within a logarithmic factor. However this is not trivially right for the geometric variant of G-TSP.

G-TSP in the Plane. This problem was first studied by [AH94] who gave a constant approximation ratio algorithm for it where the regions (or neighborhoods) are well behaved in some form (e.g. consist of disks, parallel segments of equal length and translates of a convex region). [MM95] and [GL99] showed an $O(\log n)$ approximation ratio for more generalized cases. A constant factor approximation algorithm for the case where neighborhoods are disjoint convex fat objects was suggested by [dBGK$^+$02]. Recently [DM01] gave a constant factor approximation algorithm for the case of arbitrary connected neighborhoods having comparable diameter, and PTAS for the special case of pairwise disjoint unit disk neighborhoods.

The best known approximation hardness result for this problem is of $\frac{2041}{2040} - \varepsilon \approx 1.0005$ [GL00, dBGK$^+$02]. We improve this result to any constant factor, and give better results under a stronger complexity assumption.

Minimum Steiner Spanning Tree. Another related problem is the minimum Steiner spanning tree *(ST)* . A Steiner tree of S is a tree whose nodes contain the given set S. The nodes of the tree that are not the points of S are called Steiner points. Although finding a minimum spanning tree is polynomial, the former problem is NP-hard. In the Euclidean case the problem remains NP-hard [GGJ77], but admits PTAS [Aro96, Mit96].

Group Steiner Tree. The Steiner tree notion can be generalized similarly to the generalization of TSP to G-TSP. In the Group Steiner Tree Problem *(G-ST)* (also known as Class Steiner Problem, the Tree Cover Problem and the One-of-a-Set Steiner Problem) we are given an undirected graph with edge weights and subsets of the vertices. The objective is to find a minimum weighted tree, having at least one vertex of each subset.

As G-ST is another generalization of set cover (even when the weight function obeys triangle inequality) any approximation hardness factor for set-cover applies to G-ST [Ihl92]. Thus, by [LY93, RS97, Fei98], G-ST is hard to approximate within a logarithmic factor. As in G-TSP, this is not trivially right in the geometric domain.

[Sla97] gave an $O(\log n)$ approximation algorithm for a restricted case of this problem and a $2k$ approximation algorithm for the variant in which sets are of size k. For sets of unbounded size, no constant approximation algorithm is known, even under Euclidean constraint [Mit00]. If the weight function obeys the Euclidean metrics in the plane, then, for some restricted variant of the problem, there is a polynomial time algorithm which approximates it within some (large) constant (a corollary of [dBGK$^+$02, DM01]). We show that this problem is NP-hard to approximate to within any constant factor, and give stronger results under a stronger complexity assumption.

Minimum Watchman Tour and Minimum Watchman Path. The Minimum Watchman Tour *(WT)* and Minimum Watchman Path *(WP)* are the problems of a watchman, who must have a view of n objects, while also trying to minimize the length of the tour (or path). These problems were extensively studied, and given some approximation algorithms as well as solving algorithms for special instances of the problem (for example [CN88, NW90, XHHI91, CJN93, MM95, GN98]). We show that these problems are NP-Hard to approximate to within any constant factor, and give stronger results under a stronger complexity assumption.

Our Results

We show that G-TSP in 2-D, G-ST in 2-D, WT in 3-D and WP 3-D are all NP-hard to approximate to within any constant factor. This resolves three open problems presented by Mitchell (see [Mit00], open problems 21, 27 and 30). For dimension d, and under a plausible conjecture regarding the hardness of approximation of hyper-graph vertex-cover, the hardness of approximation factor for G-TSP and G-ST becomes $O(\log^{\frac{d-1}{d}} n)$.

These problems can be categorized according to three important parameters. One is the dimension of the domain; the second is whether each subset (region, neighbourhood) is connected and the third is whether sets are pairwise disjoint. For the G-TSP and G-ST problems in 2-D domain our results hold only if sets are allowed to be unconnected (but hold even for pairwise disjoint sets). If each set is connected, (but sets are allowed to coincide) we show an innapproximability factor of $2-\varepsilon$ for all these problems, using an adaptation of a technique from [GL00, dBGK$^+$02]. In the 3-D domain our results hold for all parameter settings, that is, even when each set is connected and all sets are pairwise disjoint. The following table summarizes these results for G-TSP and G-ST:

G-TSP and G-ST				
Dimension	2-D		3-D or more	
Pairwise Disjoint Sets	Yes	No	Yes	No
connected sets	-	$2-\varepsilon$	$\forall c$	$\forall c$
unconnected sets	$\forall c$	$\forall c$	$\forall c$	$\forall c$

We also show the following innapproximability factors for the G-ST and G-TSP, where every region is a set of a constant number of points.

2-D		
Sets Size	G-TSP	G-ST
$k > 4$	$\frac{\sqrt{k-1}}{\sqrt{2}\sqrt[4]{3}} - \varepsilon$	$\frac{\sqrt{k-1}}{\sqrt[4]{3}} - \varepsilon$
$k = 4$	$\frac{4}{\sqrt{3}+2} - \varepsilon > 1.07$	$\frac{8}{\sqrt{3}+4} - \varepsilon > 1.39$

1.1 Paper Organization

We first give some required preliminaries (section 2). The first proof shown concerns the approximation hardness factor for G-ST (section 3.1). The same factor for G-TSP is then deduced (section 3.2). We next provide the proof regarding these problems where each region is connected (section 3.3). Finally, the problems of WT and WP are examined (section 3.4, appendix 5.5). In the appendix we also address the variants of pairwise disjoint regions (appendix 5.1), connected regions (appendix 5.2 and appendix 5.3) and regions of bounded size (appendix 5.4).

2 Preliminaries

A *hyper-graph* $G = (V, E)$ is a set of vertices V, and a family E of subsets of V, called edges. It is called *k-uniform*, if all edges $e \in E$ are of size k, namely $E \subseteq \binom{V}{k}$. A *vertex-cover* of a hyper-graph $G = (V, E)$ is a subset $U \subseteq V$, that "hits" every edge in G, namely, for all $e \in E$, $e \cap U \neq \emptyset$.

Definition 1 (Ek-Vertex-Cover) *The Ek-Vertex-Cover problem is, given a k-uniform graph $G = (V, E)$, to find a minimum size vertex-cover U.*

For $k = 2$ this is the vertex-cover problem on conventional graphs *(VC)*.

Definition 2 (Gap-Ek-Vertex-Cover) *Let* Gap-Ek-Vertex-Cover-$[\alpha_{yes}, \alpha_{no}]$ *be the problem of distinguishing whether a given k-uniform hyper-graph $G = (V, E)$, with $|V| = n$, has a vertex-cover of size at most $\alpha_{yes} \cdot n$, or whether every vertex-cover of G is at least of size $\alpha_{no} \cdot n$.*

To prove the approximation hardness result of G-ST (for any constant factor) we use the following approximation hardness of hyper-graph vertex-cover:

Theorem 1 *[Tre01] For a sufficiently large k, Gap-Ek-Vertex-Cover-$[\frac{1}{k^{\frac{1}{19}}}, 1 - \varepsilon]$ is NP-hard.*

For the variants of these problems, where each region is a set of points of bounded size k, we use the recently improved approximation hardness result:

Theorem 2 *[DGKR02] For $k > 4$, Gap-Ek-Vertex-Cover-$[\frac{1}{k-1-\varepsilon}, 1 - \varepsilon]$ is NP-Hard.*

3 G-ST, G-TSP and Minimum Watchman Tour and Path

We start by proving the results for G-ST.

3.1 Group Steiner Tree in the Plane

Definition 3 (G-ST) *We are given a set P of n points in the plane, and a family X of subsets of P. A solution to the G-ST is a tree T, such that every set $r \in X$ has at least one point in the tree, that is*

$$\forall r \in X, r \cap T \neq \emptyset$$

The size (length) of a solution T is the sum of length of all its edges. The objective is to find a solution T of minimal length.

Definition 4 (Gap-G-ST) *Let* Gap-G-ST-$[I_{yes}, I_{no}]$ *be the problem of distinguishing whether an input instance for G-ST has a solution of size at most I_{yes}, or whether every solution to it is of size at least I_{no}.*

Let us now prove the main result:

Theorem 3 *G-ST is NP-hard to approximate to within any constant factor.*

Proof: The proof is by reduction from vertex-cover in hyper-graphs to G-ST. The reduction generates an instance X of G-ST, such that the size of its minimal tree T, is related to the size of the minimal vertex-cover U of the input graph $G = (V, E)$. Therefore, an approximation for T would yield an approximation for U, and hence the inapproximability factor known for Gap-Ek-Vertex-Cover, yields an inapproximability factor for G-ST.

The Construction. Given a k-uniform hyper-graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges (assume w.l.o.g. that $\sqrt{n}$ and $\sqrt{\frac{n}{k}}$ are integers), we embed it in the plane to construct a set X of regions. All the regions are subsets of points of a single square of $\sqrt{n} \times \sqrt{n}$ section of the grid. Each point represents an arbitrary vertex of G and each region stands for an edge of G. Formally,

$$P = \{p_{v_i} | v_i \in V\},\ p_{v_i} = (i \ \ mod \sqrt{n}, \lfloor \frac{i}{\sqrt{n}} \rfloor)$$

We now define the set of regions X. For every $e \in E$ we define the region r_e to be the union of the k points on the grid, of the vertices in the edge e, namely

$$X = \{r_e | e \in E\},\ r_e = \{p_v |\ v \in e\}$$

Claim 4 (Soundness) *If every vertex cover U of G is of size at least $(1-\varepsilon)n$ then every solution T for X is of size at least $(1-\varepsilon)n - 1$.*

Proof: At least $(1-\varepsilon)n$ of the grid points have to be covered, hence any tree cover contains at least $(1-\varepsilon)n - 1$ segments, each of length at least 1. ■

Lemma 5 (Completeness) *If there is a vertex cover U of G of size at most $\frac{n}{t}$ then there is a solution T for X of size at most $\frac{3n}{\sqrt{t}}$.*

Proof: We define $T_N(U)$, the natural tree according to a vertex-cover U of G, as follows (see figure 1). A vertical segment on the first column of points, and horizontal segments every d$^{\text{th}}$ row. For every point p_v of $v \in U$ which is not already covered by the tree, we add a segment from it to the closest point q_v on any of the vertical segments. Formally,

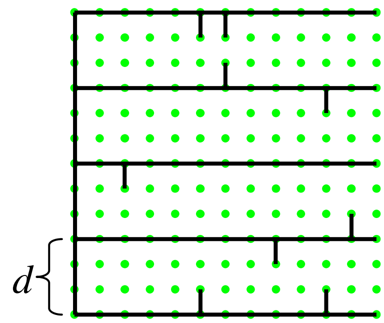

Figure 1: The Natural Tree

Definition 5 (Natural Tree) *The natural tree $T_N(U)$ of a subset $U \subseteq V$ is the polygon, consisting of the following segments:*

$$T_N(U) = \{((0,0),(0,\sqrt{n}))\} \cup \{((0,(i-1)\cdot d),(\sqrt{n},(i-1)\cdot d))\}_{i\in[\frac{\sqrt{n}}{d}]} \cup \{(p_v,q_v)\}_{v\in U}$$

where

$$q_{v_i} = (i \ mod \sqrt{n}, d \cdot \lfloor \frac{1}{d} \lfloor \frac{i}{\sqrt{n}} \rfloor + \frac{1}{2} \rfloor)$$

Setting $d = \sqrt{t}$, the natural tree contains $\sqrt{\frac{n}{t}} + 1$ horizontal segments of length $\sqrt{n}$ each, a vertical segment of length $\sqrt{n}$ and at most $\frac{n}{t}$ segments, each of length not more than $\frac{\sqrt{t}}{2}$. Therefore

$$|T_N(U)| \leq (\sqrt{\frac{n}{t}} + 2)\sqrt{n} + \frac{n}{t} \cdot \frac{\sqrt{t}}{2} < \frac{3n}{\sqrt{t}}$$

■

Applying claim 4 and lemma 5 to theorem 1 we get that for sufficiently large k, Gap-G-ST-$[\frac{3n}{\sqrt{k^{\frac{1}{19}}}}, (1-\varepsilon)n]$ is NP-Hard. Thus, as k can be arbitrary large, it is NP-hard to approximate G-ST to within any constant factor. ■

The above reduction can be amended to generate instances of G-ST in 2-D, where regions are pairwise disjoint. It can also be amended so that regions are both pairwise disjoint and connected, but in 3-D. For details see appendix 5.1,5.2.

3.2 Group TSP in the Plane

We next show that the hardness factor shown for G-ST holds for G-TSP with the same parameter settings.

Definition 6 (G-TSP) *In G-TSP in the plane we are given a set P of points in the plane, and a family X of n subsets of P. A solution to the G-TSP is a traversal T, such that every set (region) $r \in X$ has at least one point in the traversal, that is*

$$\forall r \in X, r \cap T \neq \emptyset$$

The objective is to find a solution T of minimal length.

Claim 6 *If there exists an efficient α-approximation for G-TSP then an efficient 2α-approximation for G-ST also exists.*

Proof: The size of a minimal tree of G-ST, T^*_{G-ST}, is smaller than the size of a minimal tour of G-TSP, T^*_{G-TSP}, of the same given instance (as one can take a tour and transform it to a tree by removing one of its edges). On the other hand, T^*_{G-ST} is at most $2T^*_{G-TSP}$ (as one can take two copies of the same tree to have a tour). Thus

$$T^*_{G-ST} < T^*_{G-TSP} \leq 2T^*_{G-ST}$$

■

Therefore, by this claim and theorem 3 we get the same results for G-TSP:

Corollary 7 *G-TSP in the plane is NP-hard to approximate to within any constant factor.*

This holds for the same parameter settings as in G-ST (i.e. unconnected sets for 2-D, and both connected and unconnected for 3-D and higher dimension).

3.3 Group ST and Group TSP in the Plane - The Case of Connected Regions.

In the following proof, we show that G-ST and G-TSP in the plane where *every region is connected* are NP-hard to approximate to within any constant factor smaller than 2. This extends a known factor of $\frac{2041}{2040} \approx 1.0005$ by [GL00, dBGK$^+$02] for G-TSP. These reductions are similar to theirs but differ in the problem we reduce from (vertex cover on hyper graph instead of vertex cover on graphs of bounded degree) and in some parts of the gadgets constructed.

Group ST - Connected Regions in the plane

Theorem 8 *G-ST in the plane with connected regions is NP-hard to approximate to within $2 - \varepsilon$ for any constant $\varepsilon > 0$.*

Proof: Given a hyper-graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges, we construct a set X of regions in the plane. All the regions are subsets of points of two circles in the plane, of perimeter approximately 1. Some of the regions represent edges of G (one region for each edge). Other regions represent vertices of G (l region for each vertex). Let us first describe the set of points of interest P.

The set P is composed of two sets of points, each of which is equally spread on one of two circles. The two circles are concentric, the second one having a slightly larger radius than the first. They are thus referred to as the *inner circle* and the *outer circle*. We will later add to the construction a third circle named the *outmost circle*.

P contains a set P_{inner} of nl points on the inner circle (l to be set later) and a set P_{outer} of n points on the outer circle, one point for each vertex. We set the radius of the inner circle $\rho \approx \frac{1}{2\pi}$, so that the distance between consecutive points on the inner circle is $\varepsilon = \frac{1}{nl}$. Let us define, formally, the set of points $P = P_{inner} \cup P_{outer}$, which, for the sake of simplicity , would be specified using polar coordinates - namely specifying (radius, angle) of each.

Formally,

$$\theta_\varepsilon = \frac{2\pi}{nl} \qquad \text{and} \qquad \rho = \frac{\frac{\varepsilon}{2}}{sin(\frac{\theta_\varepsilon}{2})} \quad \approx \frac{1}{2\pi}$$

$$P_{inner} = \{p_{v,j} | \ \ v \in V, j \in [l]\}, \ p_{v_i,j} = (\rho, \ \ (i \cdot l + j) \cdot \theta_\varepsilon)$$

$$P_{outer} = \{q_v | \ \ v \in V\}, \ q_{v_i} = (\rho + \frac{1}{n}, i \cdot \ l \cdot \theta_\varepsilon)$$

We now define the set of regions $X = X_V \cup X_E$ where X_V contains a region for each edge, and X_E contains l regions for each vertex:

$$X_V = \{r^{in}_{v,j} |\ v \in V, j \in [l]\},\ r^{in}_{v,j} = \{p_{v,j}\}$$

$$X_E = \{r^{out}_e |\ e \in E\},\ r^{out}_e = \{q_v |\ v \in e\}$$

One can easily amend each of the unconnected regions (which are all in X_E) to be connected without changing the correctness of the following proof. For details see appendix 5.3.

Proof's Idea. We are next going to show, that the most efficient way to cover X is by taking all segments on the inner circle (except for one arbitrary segment), and for each vertex in the minimal vertex-cover U of G, connect the corresponding point on the outer circle to the closest point on the inner circle (see figure 2).

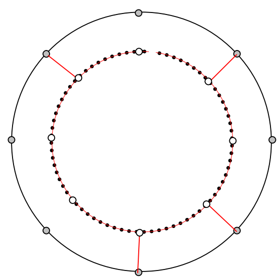

Figure 2: A Natural Tree

We call such a solution the natural tree $T_N(U)$. More formally,

Definition 7 (Natural Tree) *The natural tree $T_N(U)$ of a subset $U \subseteq V$ is the following set of segments:*

$$T_N(U) = T_{in} \cup T_{out}$$

where

$$T_{in} = \{(p_{v,j}, p_{v,j+1}) \mid v \in V, j \in [l-1]\} \quad \cup \quad \{(p_{v_i,l}, p_{v_i+1,1}) \mid i \in [n-1]\}$$
$$T_{out} = \{(p_{v,1}, q_v) \mid v \in U\}$$

Claim 9 *The length of the natural tour $|T_N(U)| = 1 + \frac{|U|}{n} - \varepsilon$*

Proof: $T_N(U)$ contains $nl - 1$ segments of length $\varepsilon = \frac{1}{nl}$ and $|U|$ segments of length $\frac{1}{n}$. ∎

Claim 10 *A natural tree $T_N(U)$ is a solution of X if and only if U is a vertex-cover of G.*

Proof: If U is a vertex-cover of G then for every $e \in E$ there is at least one vertex $v \in e \cap U$, thus every region r^{out}_e, is covered at least once, at the point q_v (and trivially, in any natural tree, all regions of X_{in} are covered).

If U is not a vertex-cover of G then there is an edge $e \in E$ such that $e \cap U = \emptyset$, thus no point of the region r^{out}_e is covered, and $T_N(U)$ is not a solution of X. ∎

We therefore get:

Claim 11 *[Completeness] If there is a vertex-cover U of G of size bn, then there is a solution of X of length at most $1+b-\varepsilon$.*

Claim 12 *The shortest solution of X is a natural tree $T_N(U)$ of a minimal vertex-cover U of G.*

Proof: Let T be a solution of G, and $U = T \cap P_{outer}$ the set of vertices on the outer circle, covered by T. The closest point to a point in P_{outer} is at distance $\frac{1}{n}$. Hence T contains at least one segment of length $\geq \frac{1}{n}$ for any vertex $v \in U$, and for all (but one) of the points of $T \cap P_{inner}$, it contains a segment of length at least $\varepsilon = \frac{1}{nl}$. Therefore

$$|T| \geq (nl-1) \cdot \varepsilon + |U| \cdot \frac{1}{n}$$

which is the exact total length of the natural tree $T_N(U)$ (claim 9). Thus, by claim 10, any solution to X is of length at least $|T_N(U)|$. ■

We therefore get

Claim 13 *[Soundness] If any vertex-cover U of G is of size at least an, then any solution of X is of length at least $1+a-\varepsilon$.*

Hence by the soundness (claim 13) and completeness (claim 11) we have the following:

Lemma 14 *If Gap-Ek-Vertex-Cover-$[b,a]$ is NP-hard then for any $\varepsilon > 0$, it is NP-hard to approximate G-TSP in the plane with connected regions, to within $\frac{1+a}{1+b} - \varepsilon$.*

Plugging in the known gap for vertex-cover in hyper-graphs (theorem 1) we get that G-ST is NP-hard to approximate to within $\frac{1+1-\varepsilon}{1+\frac{1}{k^{\frac{1}{19}}}} - \varepsilon$, hence, for arbitrary small $\varepsilon > 0$ and for a sufficiently large k, G-ST is NP-hard to approximate to within $2-\varepsilon$, even if each region is connected. ■

Group TSP - Connected Regions in the Plane

We consider the 2-D variant of G-TSP, in which each set is connected.

Theorem 15 *G-TSP in the plane with connected regions is NP-hard to approximate to within $2-\varepsilon$ for any constant $\varepsilon > 0$.*

Proof:

The Construction The reduction is from hyper-graph vertex-cover, and is essentially the same as the reduction for G-ST. The only change is the radius of the outer circle; the gap between the radiuses of the two circles is $\frac{1}{2n}$ (instead of $\frac{1}{n}$). Hence we redefine the set P_{outer}, namely,

$$P_{outer} = \bigcup_{v_i \in V} q_{v_i} \quad \text{where} \quad q_{v_i} = (\rho + \frac{1}{2n}, i \cdot l \cdot \theta_\varepsilon)$$

The rest of the construction remains unchanged.

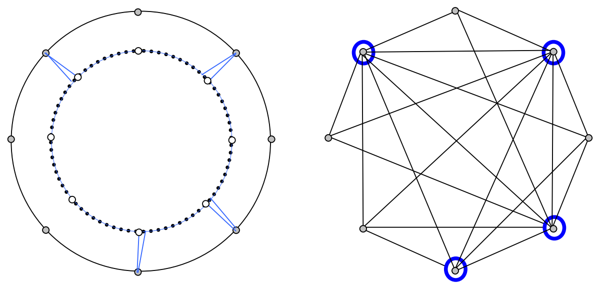

Figure 3: A vertex-cover and a natural tour

Proof's Idea. We are next going to show, that the most efficient way to traverse X is by traversing all points on the inner circle (say counterclockwise), detouring to visit the closest points on the outer circle, for every point that corresponds to a vertex in the minimal vertex-cover of G (see figure 3).

More formally,

Definition 8 (Natural Tour) *The natural tour $T_N(U)$ of a subset $U \subseteq V$ is the closed polygon, consisting of the following segments:*

$$T_N(U) = T_{in} \cup T_{out}$$

where

$$T_{in} = \{(p_{v,j}, p_{v,j+1}) \mid v \in V, j \in [l-1]\} \quad \cup \quad \{(p_{v_i,l}, p_{v_{i+1 \ mod \ n},1}) | i \in [n]\}$$
$$T_{out} = \{(p_{v,i}, q_v) \mid v \in U, i \in [2]\}$$

Let us consider the length of this tour $|T_N(U)|$.

The natural tour $T_N(U)$ consists of $nl - |U|$ segments of size $\varepsilon = \frac{1}{nl}$ (on the inner circle), $|U|$ segments for the detourings of size $\frac{1}{2n}$ and $|U|$ segments of size in the range $(\frac{1}{2n}, \ \frac{n}{2n} + \varepsilon)$. Thus $n + \frac{|U|}{n} - \delta \leq T_N(U) \leq 1 + \frac{|U|}{n}$ (for some $0 < \delta < \varepsilon$).

The exact length of $T_N(U)$ can be computed, but is not important for our purpose. We therefore have:

Claim 16 *[Completeness] If there is a vertex-cover U of G of size bn, then there is a solution of X of length at most $1 + b - \varepsilon$.*

We next show

Claim 17 *[Soundness] If any vertex-cover U of G is of size at least $a \cdot n$, then any solution of X is of length at least $1 + a - \varepsilon'$ (where $\varepsilon' = \varepsilon'(\varepsilon) > 0$ is an arbitrary small constant).*

Proof:

Let T be a solution of X. Clearly T covers all points of P_{inner} (otherwise it is not a solution for X). Let U be a set of vertices that correspond to points on the outer circle, visited by T, namely

$$U = \{v \mid q_v \in T \cap P_{outer}\}$$

and denote $s = |U|$. Clearly T is a solution only if U is a vertex cover of G, hence $s \geq a$. W.l.o.g. T doesn't traverse any point twice and consists of $nl + 2s$ segments . If T has no direct path between two points on the outer circle, then it has $2s$ segments of length $\geq \frac{1}{2n}$ (those going between a point on the inner circle and a point on the outer circle) and $nl-2s$ segments of length $\geq \frac{1}{nl}$. Therefore T is of length $\geq 1+\frac{a}{n}(1-2\varepsilon)$

A direct path between two points in P_{outer} is of length $D=2(\rho+\frac{1}{2n})\sin\frac{\theta_\varepsilon l}{2}$ if the points are consecutive, and is greater otherwise. If there are c direct paths between points in P_{outer}, then the total length of T is at least

$$1+\frac{s}{n}(1-2\varepsilon)+c(D-\frac{1}{2n})$$

which suffices as half the distance between consecutive points on the outer circle is approximately the distance between a point on the outer circle to the closest point on the inner circle, that is,

$$D-\frac{1}{2n}=(\rho+\frac{1}{2n})\sin\frac{\theta_\varepsilon l}{2}-\frac{1}{2n}\geq -\varepsilon'$$

for arbitrary small constant ε'. This holds as $\rho=\frac{\frac{\varepsilon}{2}}{sin(\frac{\theta_\varepsilon}{2})}$, $\theta_\varepsilon=\frac{2\pi}{nl}$ and $\varepsilon=\frac{1}{nl}$, and:

$$\frac{1}{2}\left(\frac{\frac{1}{nl}}{\sin\frac{\pi}{nl}}+\frac{1}{n}\right)\sin\frac{\pi}{n}\geq\frac{1}{2n}-\varepsilon'$$

for sufficiently large n.

■

The rest of the proof of hardness of G-TSP in the plain with connected regions (theorem 15) precisely follows the proof of hardness of G-ST (theorem 8). ■

3.4 Minimum Watchman Tour and Path

Definition 9 (WT and WP) *In WT (WP) we are given a set of n objects P (polygons in the 2-D variant, or polyhedrals in the 3-D case). A solution to the WT (WP) is a traversal (path) T, such that any point of every object of P is seen by some point of the cycle (path). The objective is to find a solution T of minimal length.*

As noted in [Mit00] these problems seem closely related to G-TSP, since it can be thought of as the shortest cycle (path) problem, in which we have the constraint that the cycle (path) must visit the visibility region associated with each point of the domain. We show that this is correct, by a reduction from G-TSP in the plane, and obtain the following factors for WT and WP in 3-D domain; For details see appendix 5.5

Corollary 18 *WT and WP in 3-D cannot be approximates to within any constant factor unless P=NP.*

4 Discussion

We have shown that G-TSP, G-ST, WP and WT cannot be efficiently approximated to within any constant factor unless $P=NP$. In this aspect Group-TSP and Group-ST seem to behave more like the Set-Cover problems, rather than the geometric-TSP and geometric-Steiner tree problems.

These reductions illustrate *the importance of gap location*; the approximation hardness results for hyper-graph vertex-cover of [Tre01, DGKR02, Hol02] are weaker than that of [Fei98], in the sense that the gap ratio is smaller. However, their gap location, namely, their almost perfect soundness, is a powerful tool (see [Pet94]). In the reductions shown here this aspect plays an essential role. We conjecture that the two properties can be joint, namely that,

Conjecture 19 *Gap-Hyper-Graph-Vertex-Cover-*$[O(\frac{1}{\log n}), 1-\varepsilon]$ *is intractable.*

Using the exact same reductions, this will extend the known approximation hardness factors of G-TSP, G-ST, WT and WP, as follows:

Corollary 20 *If conjecture 19 is correct then approximating G-TSP in the plane and G-ST in the plane to within* $O(\log^{\frac{1}{2}} n)$ *is intractable.*

Corollary 21 *If conjecture 19 is correct then approximating WT and WP in 3-D to within* $O(\log^{\frac{1}{2}} n)$ *is intractable.*

This conjecture also implies the following corollary.

Corollary 22 *If conjecture 19 is correct then approximating G-TSP in dimension* $d > 1$ *and G-ST in dimension* $d > 1$ *to within* $O(\log^{\frac{d-1}{d}} n)$ *is hard.*

Proof's sketch: By the simplest generalization of the proof for unconnected sets in 2-D. All the regions are subsets of points of a section of the d-dimensional grid. The section is a single d-dimensional cube, each side of length $\sqrt[d]{n}$. ∎

The following table summarizes our result, assuming correctness of conjecture 19:

G-TSP and G-ST						
Dimension	2-D		3-D		d	
Pairwise Disjoint Sets	Yes	No	Yes	No	Yes	No
connected sets	-	$2-\varepsilon$	$\log^{\frac{1}{2}} n$	$\log^{\frac{1}{2}} n$	$\log^{\frac{d-2}{d-1}} n$	$\log^{\frac{d-2}{d-1}} n$
unconnected sets	$\log^{\frac{1}{2}} n$	$\log^{\frac{1}{2}} n$	$\log^{\frac{2}{3}} n$	$\log^{\frac{2}{3}} n$	$\log^{\frac{d-1}{d}} n$	$\log^{\frac{d-1}{d}} n$

An interesting open problem is whether the square root loss of the approximation hardness factor in the 2-D variant is merely a fault of this reduction or is intrinsic to the plane version of these problems; i.e, is there an approximation with a ratio smaller than $\ln n$ for the plane variants ? Are there approximations to the G-TSP and G-ST that perform better in the plane variants than Slavik's [Sla97] approximations for these problems with triangle inequality only ? Does higher dimension in these problems impel an increase in complexity ?

Other open problems remain for the various parameter settings. A most basic variant of G-TSP and G-ST, namely in 2-D domain, where every region is connected, and regions are pairwise disjoint, remains open, as well as the WT and WP on 2-D (open problem 29 of [Mit00]).

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5 Appendix

5.1 Group Steiner Tree - Pairwise Disjoint Regions

We now show that the factors for G-ST hold even when regions are pairwise disjoint.

Claim 23 *The approximation hardness factors for G-ST (theorems 3) apply even when regions are pairwise disjoint.*

Proof's sketch: We amend the reduction, such that the generated instance X contains only pairwise disjoint sets. For every $e \in E$ and $v \in e$, we redefine the point $p_v \in r_e^{out}$: we replace p_v with a set of new points $p_{v,e}$, shifted to an (arbitrary small) $\varepsilon' > 0$ maximal distance of the original location, such that for any $e_i, e_j \in E$ there are no joint points of the regions r_{e_i}, r_{e_j}. ■

5.2 Group Steiner Tree - Connected Regions in 3-D

Claim 24 *The approximation hardness factors for G-ST (theorems 3) apply in 3-D even when regions are pairwise disjoint, and each region is connected.*

Proof's sketch: The above reduction generates a two dimensional instance, in which regions are not connected. One can show that the approximation hardness result applies for connected regions in dimension three or higher. This is achieved by adding segments to connect the points of every region, in such a way that the optimal solution T doesn't change. For each point of interest p in the plane (i.e. a point which is part of a region) we add a segment from p to a matching point on an enlarged copy of the given plane instance, in which distances are multiplied by a large factor (say, n^4), located parallel to the original instance, at great height (say, n^4) above it. We call that segment the *up-going segment of* p, (see figure 4).

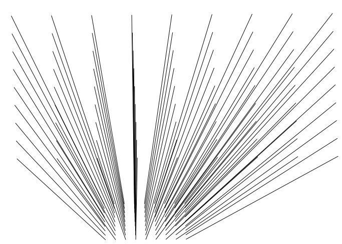

Figure 4: 2-D to 3-D connected

Each region r_e, will be redefined to contain the up-going segments of all $p \in r$. We will now describe new segments which connect the different segments of a region; these new segments will be called the *connecting segments*. Each two up-going segments of the same region are connected using a new, parallel to the plane, segment. The new segments are added at each consecutive multiple of n^2 starting from height n^3. They are added in a way which ensures that none of the points on them are at a distance n^2 or less from other segments. To this end, we may, should the need arise, use three segments rather than one, in order to connect two up-going segments. Each region is once again redefined to contain both its up-going segments and its connecting segments. Note that if original regions were pairwise disjoint then the new regions are as well. Trivially, an optimal solution T never contains points outside the original plane. ■

5.3 G-ST and G-TSP in the plane - Making Each Region Connected

To make each region $r_e \in X_E$ connected (see figure 5), we add segments connecting each of the points on the outer circle, to the closest point on a concentric circle (the *outmost circle*, C), of radius $\rho_{outmost}$ suitably large (say, n); namely, for each $q_v \in P_{outer}$, $q_v = (\rho, \alpha)$ we add the segment

$$l_v = [(\rho, \alpha), (\rho_{outmost}, \alpha)]$$

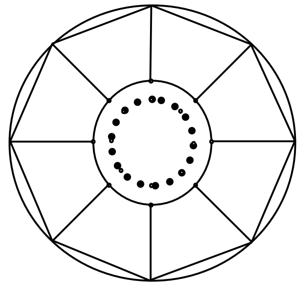

Figure 5: The outmost circle (or polygon).

Edge regions are changed to include the relevant segments and the outmost circle, that is,

$$r_e^{out} = C \cup \bigcup_{v \in e} l_v$$

Vertex regions X_V are left unchanged.

Clearly the shortest tour never exits the outer-circle, therefore all points outside the outer-circle may be ignored in the relevant proofs.

Note also, that the outmost circle can be replaced by a polygon, thus all regions consist of points and segments; in addition, one of the segments of this polygon can be removed, thus having all regions *simply* connected.

5.4 Small regions in the Plane - Group TSP and Group Steiner Tree

Let k-G-ST and k-G-TSP be the variant of G-ST and G-TSP respectively, where each region is comprised of k points. We use a triangular grid (see figure 6) instead of the grid used in the original proof (of theorem 3) to achieve better constants of innapproximability for these variants.

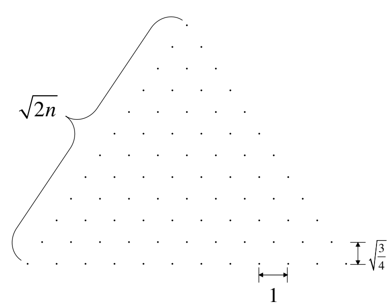

Figure 6: The Triangular Construction

Soundness proofs (of $(1-\varepsilon)n-1$) remains the same. We next give the completeness claims for both problems:

Claim 25 (Completeness - k-G-ST) *If U is a minimal vertex cover of G, then there exists a solution of X (the k-G-ST generated instance) of size at most*

$$|T_N(U)| \leq |U| \cdot \frac{d'}{2} + \sqrt{2n} \cdot \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{n}}{2d'} + \sqrt{2n}$$

for any parameter $0 < d' = \frac{\sqrt{3}}{2}d < \sqrt{n}$.

Proof: The proof is by inspecting the natural tree $T_N(U)$ of a minimal vertex cover U (as demonstrated in figure 7). $T_N(U)$ contains horizontal segments of length $\sqrt{2n}, \sqrt{2n}-d', \sqrt{2n}-2d', ..., d$, another segment of length $\sqrt{2n}$ and at most $|U|$ vertical segments, each of length $\frac{d'}{2}$ at most. ∎

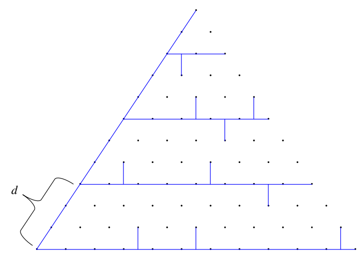

Figure 7: The Natural Tree

Applying the known hardness of Hyper-Graph Vertex-Cover (theorem 2), and choosing $d' = \sqrt[4]{3}\sqrt{k-1-\varepsilon}$ we get, for $|U| = \frac{n}{k-1-\varepsilon}$

$$T_N(U) = \frac{n}{k-1-\varepsilon} \cdot \frac{d'}{2} + \frac{\sqrt{3}n}{2d'} + \sqrt{2n} =$$
$$= \frac{\sqrt[4]{3}n}{\sqrt{k-1-\varepsilon}} + \sqrt{2n}$$

The rest of the proof is as in theorem 3. Therefore,

Corollary 26 *For $k > 4$, k-G-ST is hard to approximate to within* $\frac{\sqrt{k-1}}{\sqrt[4]{3}} - \varepsilon$

Claim 27 (Completeness - k-G-TSP) *If U is a minimal vertex cover of G, then there exists a solution of X (the k-G-TSP generated instance) of size at most*

$$|T_N(U)| \le d'|U| + \sqrt{2n} \cdot \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{n}}{2d'} + 2\sqrt{2n}$$

for any parameter $0 < d' = \frac{\sqrt{3}}{2}d < \sqrt{n}$.

Proof: The proof is by inspecting the natural tour $T_N(U)$ of a minimal vertex cover U (as demonstrated in figure 8). $T_N(U)$ contains horizontal segments of length $\sqrt{2n}, \sqrt{2n}-d', \sqrt{2n}-2d', ..., d$, another segment of length $\sqrt{2n}$, $\frac{\sqrt{2n}}{d}$ segments of length d and at most $2|U|$ segments, each of length $\frac{d'}{2}$ at most. ∎

Applying the known hardness of Hyper-Graph Vertex-Cover (theorem 2), and choosing $d' = \frac{\sqrt[4]{3}}{\sqrt{2}}\sqrt{k-1-\varepsilon}$ we get, for $|U| = \frac{n}{k-1-\varepsilon}$

$$T_N(U) = \frac{d'n}{k-1-\varepsilon} + \frac{\sqrt{3}n}{2d'} + 2\sqrt{2n} =$$
$$= \frac{\sqrt{2}\sqrt[4]{3}n}{\sqrt{k-1-\varepsilon}} + 2\sqrt{2n}$$

The rest of the proof is as in theorem 3. Therefore,

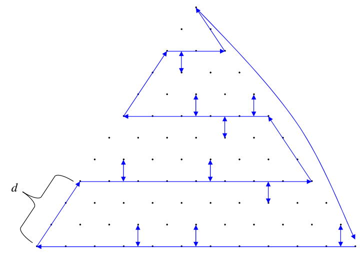

Figure 8: The Natural Tour

Corollary 28 *For $k > 4$, k-G-TSP is hard to approximate to within* $\frac{\sqrt{k-1}}{\sqrt{2}\sqrt[4]{3}} - \varepsilon$

For low k values, for example, $k = 4$ one can achieve a somewhat better factors, using the following theorem and a more subtle estimation of $T_N(U)$.

Theorem 29 *[DGKR02, Hol02], For $k = 4$,Gap-E4-Vertex-Cover-$[(\frac{1}{2}+\varepsilon),(1-\varepsilon)]$ $\in$NP-Hard*

Note that the horizontal segments of the natural tour (or tree) can be shifted up or down, thus, if d is an integer, we may cover $\frac{1}{d}$ fraction of U using these segments. Hence we have

Claim 30 (Completeness) *If U is a minimal vertex cover of G, then for any parameter $d' = \frac{\sqrt{3}}{2}d, d \in [\sqrt{n}]$. there exists a solution of X (the generated instance) of size*

$$|T_N(U)| \leq \frac{d-1}{d}|U| \cdot \frac{d'}{2} + \sqrt{2n} \cdot \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{n}}{2d'} + \sqrt{2n} =$$
$$= \frac{\sqrt{3}(d-1)}{4}|U| + \frac{n}{d} + \sqrt{2n}$$

for k-G-ST and

$$|T_N(U)| \leq \frac{d-1}{d} d'|U| + \sqrt{2n} \cdot \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{n}}{2d'} + 2\sqrt{2n} =$$
$$= \frac{\sqrt{3}(d-1)}{2}|U| + \frac{n}{d} + 2\sqrt{2n}$$

for k-G-TSP

We therefore have:

Corollary 31 *If Gap-Ek-Vertex-Cover-$[\frac{n}{t}$,$(1-\varepsilon)n]$ is intractable, then so is approximating k-G-ST in the plane to within a factor of*

$$\max_{d \in [\sqrt{n}]} \left\{ \frac{1}{\frac{\sqrt{3}(d-1)}{4t} + \frac{1}{d}} - \varepsilon \right\}$$

and k-G-TSP in the plane to within a factor of

$$\max_{d \in [\sqrt{n}]} \left\{ \frac{1}{\frac{\sqrt{3}(d-1)}{2t} + \frac{1}{d}} - \varepsilon \right\}$$

By applying corollary 31 to theorem 29, and assigning $d = 2$ we have:

Corollary 32 *4-G-ST in the plane is NP-hard to approximate to within* $\frac{8}{\sqrt{3}+4} - \varepsilon > 1.39$ *and 4-G-TSP in the plane is NP-hard to approximate to within* $\frac{4}{\sqrt{3}+2} - \varepsilon > 1.07$.

5.5 Minimum Watchman Tour and Path

We show a gap preserving reduction from G-TSP to WT.

Proof's sketch: (WT) Each set of points is represented by an octopus shaped figure (see below), comprised of hollow square shaped pipes. The pipes converge at the top and are linked on their other side to the set of points in the plane level. Hidden at the converging point of the pipes is a point, which can only be seen by looking through any of the pipe openings in the plane level.

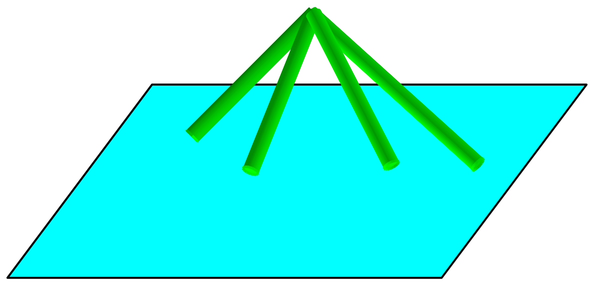

Figure 9: The WT Construction

Therefore by corollary 7 we get the same hardness factors for WT in 3-D.

■

Claim 33 *If there exists an efficient α-approximation for WP then an efficient 2α-approximation for WT also exists.*

Proof: The size of a minimal path of WP, T^*_{WP}, is smaller than the size of a minimal tour of WT, T^*_{WT}, of the same given instance (as one can take a tour and transform it to a path by removing one of its edges). On the other hand, T^*_{WP} is at most $2T^*_{WP}$ (as one can take two copies of the same path to have a tour). Thus

$$T^*_{WP} < T^*_{WT} \leq 2T^*_{WP}$$

■

Thus the hardness factors of WT apply for WP too.