

COUNTING SUM-FREE SETS IN ABELIAN GROUPS

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ABSTRACT. In this paper we study sum-free sets of order m in finite Abelian groups. We prove a general theorem about independent sets in 3-uniform hypergraphs, which allows us to deduce structural results in the sparse setting from stability results in the dense setting. As a consequence, we determine the typical structure and asymptotic number of sum-free sets of order m in Abelian groups G whose order n is divisible by a prime q with $q \equiv 2 \pmod{3}$, for every $m \geq C(q)\sqrt{n \log n}$, thus extending and refining a theorem of Green and Ruzsa. In particular, we prove that almost all sum-free subsets of size m are contained in a maximum-size sum-free subset of G . We also give a completely self-contained proof of this statement for Abelian groups of even order, which uses spectral methods and a new bound on the number of independent sets of a fixed size in an (n, d, λ) -graph.

1. INTRODUCTION

An important trend in Combinatorics in recent years has been the formulation and proof of various ‘sparse analogues’ of classical extremal results in Graph Theory and Additive Combinatorics. Due to the recent breakthroughs of Conlon and Gowers [12] and Schacht [40], many such theorems, e.g., the theorems of Turán [43] and Erdős and Stone [16] in extremal graph theory, and the theorem of Szemerédi [42] on arithmetic progressions, are now known to extend to sparse random sets. For structural and enumerative results, such as the theorem of Kolaitis, Prömel and Rotschild [30] which states that almost all K_{r+1} -free graphs are r -colourable, perhaps the most natural sparse analogue is a corresponding statement about subsets of a given fixed size m , whenever m is not too small. In this paper, we prove such a result in the context of sum-free subsets of Abelian groups and provide a general framework for solving problems of this type. To be precise, we obtain a sparse analogue of a result of Green and Ruzsa [21], which describes the structure of a typical sum-free subset of an Abelian group.

Sparse versions of classical extremal and Ramsey-type results were first proved for graphs by Frankl and Rödl [17] and Babai, Simonovits and Spencer [6], and for additive structures by Kohayakawa, Łuczak and Rödl [29], and in recent years there has been a tremendous interest in such problems (see, e.g, [18, 35, 36]). Mostly, these results have been in the random setting; for example, Graham, Rödl and Ruciński [20] showed that if $p \gg 1/\sqrt{n}$,

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and $B \subseteq \mathbb{Z}_n$ is a p -random subset¹, then with high probability every 2-colouring of B contains a monochromatic solution of $x + y = z$. The extremal version of this question was open for fifteen years, until it was recently resolved by Conlon and Gowers [12] and Schacht [40]. For more recent proofs, both based on the method introduced in this paper, see [7] and [39].

For problems of the type we are considering, results are known only in a few special cases. Most notably, Osthus, Prömel and Taraz [32], confirming (and strengthening) a conjecture of Prömel and Steger [34], proved that if $m \geq (\frac{\sqrt{3}}{4} + \varepsilon)n^{3/2}\sqrt{\log n}$ then almost all triangle-free graphs with m edges are bipartite; moreover, the constant $\sqrt{3}/4$ is best possible. This result can be viewed as a sparse version of the classical theorem of Erdős, Kleitman and Rothschild [15], which states that almost all triangle-free graphs are bipartite. A similarly sharp result for the existence of monochromatic triangles in two-colourings of $G_{n,p}$ was proved by Friedgut, Rödl, Ruciński and Tetali [18]. It is an interesting open problem to prove such a sharp threshold in the setting of Theorem 1.1, below.

A set $A \subseteq G$, where G is an Abelian group, is said to be *sum-free* if $(A + A) \cap A = \emptyset$, or equivalently, if there is no solution to the equation $x + y = z$ with $x, y, z \in A$. Sum-free subsets of Abelian groups are central objects of interest in Additive Combinatorics, and have been studied intensively in recent years. The main questions are as follows: What are the largest sum-free subsets of G ? How many sum-free sets are there? And what does a typical such set look like? Over forty years ago, Diananda and Yap [13] determined the maximum density $\mu(G)$ of a sum-free set in G whenever $|G|$ has a prime factor $q \not\equiv 1 \pmod{3}$, but it was not until 2005 that Green and Ruzsa [21] completely solved this extremal question for all finite Abelian groups. On the second and third questions, Lev, Łuczak and Schoen [31] and Sapozhenko [38] determined the asymptotic number of sum-free subsets in an Abelian group of even order by showing that almost all such sets² lie in the complement of a subgroup of index 2. Green and Ruzsa [21] extended this result to Abelian groups which have a prime factor $q \equiv 2 \pmod{3}$, and showed also that a general finite Abelian group G has $2^{(1+o(1))\mu(G)|G|}$ sum-free subsets.

We say that G is of Type I if $|G|$ has a prime divisor q with $q \equiv 2 \pmod{3}$, and Type I(q) if q is the smallest such prime. Diananda and Yap [13] proved that if G is of Type I(q), then $\mu(G) = (q + 1)/(3q)$; moreover, they described all sum-free subsets of G with $\mu(G)|G|$ elements. Green and Ruzsa [21] determined the asymptotic number of sum-free subsets of an Abelian group G of Type I, by showing that almost every sum-free set in G is contained in some sum-free set of maximum size. Balogh, Morris and Samotij [8] studied p -random subsets of such groups, and showed that if $p \geq C(q)\sqrt{\log n/n}$, G is an Abelian group of Type I(q) and order n , and G_p is a p -random subset of G , then with high probability every maximum-size sum-free subset of G_p is contained in some sum-free subset of G of maximum size. In the case $G = \mathbb{Z}_{2n}$, they showed that if $p \geq (\frac{1}{\sqrt{3}} + \varepsilon)\sqrt{\log n/n}$ and $A \subseteq G$ is a p -random subset, then with high probability the unique largest sum-free subset of is $A \cap O_{2n}$,

¹A p -random subset of a set X is a random subset of X , where each element is included with probability p , independently of all other elements.

²We say that almost all sets in a family \mathcal{F} of subsets of G satisfy some property \mathcal{P} if the ratio of the number of sets in \mathcal{F} that have \mathcal{P} to the number of all sets in \mathcal{F} tends to 1 as $|G|$ tends to infinity.

where $O_{2n} \subseteq \mathbb{Z}_{2n}$ denotes the set of odd residues modulo $2n$. Moreover, the constant $1/\sqrt{3}$ in this result is best possible.

Let us denote by $\text{SF}(G, m)$ the collection of sum-free subsets of size m in a finite Abelian group G . The following theorem refines the result of Green and Ruzsa [21] to sum-free sets of fixed size m , provided that $m \geq C(q)\sqrt{n \log n}$.

Theorem 1.1. *For every prime $q \equiv 2 \pmod{3}$, there exists a constant $C(q) > 0$ such that the following holds. Let G be an Abelian group of Type I(q) and order n , and let $m \geq C(q)\sqrt{n \log n}$. Then almost every sum-free subset of G of size m is contained in a maximum-size sum-free subset of G , and hence*

$$|\text{SF}(G, m)| = \lambda_q \cdot (\#\{\text{elements of } G \text{ of order } q\} + o(1)) \binom{\mu(G)n}{m}$$

as $n \rightarrow \infty$, where $\lambda_q = 1$ if $q = 2$ and $\lambda_q = 1/2$ otherwise.

Although the factor $\lambda_q \cdot \#\{\text{elements of } G \text{ of order } q\}$ above may appear mysterious, it is a natural consequence of the characterization of maximum-size sum-free sets in groups of Type I, see Theorem 6.2. We remark that the lower bound $m \geq C(q)\sqrt{n \log n}$ is sharp up to a constant factor, since there are at least $(n/2)(\mu(G)n - 3m)^{m-1}/(m-1)!$ sum-free subsets of G which contain exactly one element outside a given maximum-size sum-free subset of G , and this is larger than $m^{1/5} \binom{\mu(G)n}{m}$ if $m \leq \frac{1}{5}\sqrt{n \log n}$. Here, and throughout, \log denotes the natural logarithm.

We shall prove Theorem 1.1 using a new theorem (see Section 2) which describes the structure of a typical independent set³ in a 3-uniform hypergraph \mathcal{H} that satisfies a certain natural ‘stability’ property, see Definition 2.1. The key ingredient in the proof of this theorem is a new method of enumerating independent sets in 3-uniform hypergraphs. In [2], we give another application of this method to the problem of counting sum-free sets of a fixed size in the set $\{1, \dots, n\}$ by proving a refinement of the Cameron-Erdős Conjecture. We shall also use a simplified version of this method to prove a new bound on the number of independent sets in a certain class of expander graphs known as (n, d, λ) -graphs. This bound will be a key ingredient in our completely self-contained proof of Theorem 1.1 in the case $q = 2$ given in Section 7. We remark that a similar method has also recently been applied by Kohayakawa, Lee, Rödl, and Samotij [28] to the problem of counting Sidon sets in $\{1, \dots, n\}$.

First, let us recall the definition of (n, d, λ) -graphs, which are an important class of expanders; for a detailed introduction to expander graphs, we refer the reader to [5] or [23]. Given a graph \mathcal{G} , let $\lambda_1 \geq \dots \geq \lambda_n$ denote the eigenvalues of the adjacency matrix of \mathcal{G} . We call $\max\{|\lambda_2|, |\lambda_n|\}$ the *second eigenvalue* of \mathcal{G} .

Definition 1.2 ((n, d, λ) -graphs). A graph \mathcal{G} is an (n, d, λ) -graph if it has n vertices, is d -regular, and has second eigenvalue at most λ .

Alon and Rödl [4] gave an upper bound on the number of independent sets in an (n, d, λ) -graph, and used their result to give sharp bounds on multicolour Ramsey numbers. When $\lambda = \Omega(d)$ (as $n \rightarrow \infty$), Theorem 1.3 below provides a significantly stronger bound than that

³An independent set in a hypergraph \mathcal{H} is a set of vertices containing no edge of \mathcal{H} .

of Alon and Rödl, for a wider range of m ; it is moreover asymptotically sharp. In fact, we will not assume anything about the second eigenvalue of a graph \mathcal{G} as our bound on the number of independent sets of \mathcal{G} will depend only on the *smallest* eigenvalue of \mathcal{G} . Given a graph \mathcal{G} , let $\lambda(\mathcal{G})$ be the smallest eigenvalue of the adjacency matrix of \mathcal{G} (denoted by λ_n above) and let $I(\mathcal{G}, m)$ be the number of independent sets of size m in \mathcal{G} . Observe that $\lambda(\mathcal{G}) < 0$ for every non-empty \mathcal{G} and that, by definition, every (n, d, λ) -graph satisfies $\lambda(\mathcal{G}) \geq -\lambda$.

Theorem 1.3. *For every $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that the following holds. If \mathcal{G} is an n -vertex d -regular graph with $\lambda(\mathcal{G}) \geq -\lambda$, then*

$$I(\mathcal{G}, m) \leq \binom{\left(\frac{\lambda}{d+\lambda} + \varepsilon\right)n}{m}$$

for every $m \geq Cn/d$.

We remark that the constant $\frac{\lambda}{d+\lambda}$ in Theorem 1.3 is best possible, since there exist n -vertex d -regular graphs with $\lambda(\mathcal{G}) \geq -\lambda$ and $\alpha(\mathcal{G}) = \frac{\lambda}{d+\lambda}n$ for many values of n , d and λ . (Here $\alpha(\mathcal{G})$ denotes the independence number of \mathcal{G} ; it was first proved by Hoffman [22] that $\alpha(\mathcal{G}) \leq \frac{\lambda}{d+\lambda}n$ for every (n, d, λ) -graph \mathcal{G} .) For example, consider a blow-up of the complete graph K_{t+1} , where each vertex is replaced by a set of size $n/(t+1)$ and each edge is replaced by a random d/t -regular bipartite graph with colour classes of size $n/(t+1)$ each. This blown-up graph \mathcal{G} is clearly d -regular, and it is straightforward to show that, with high probability, it satisfies $\lambda(\mathcal{G}) = -d/t$ and $\alpha(\mathcal{G}) = n/(t+1)$.

In Section 7, we shall use Theorem 1.3, together with some basic facts about characters of finite Abelian groups, to give a completely self-contained proof of Theorem 1.1 in the case $q = 2$. For previous results relating the problem of estimating the number of sum-free subsets of groups to that of estimating the number of independent sets in regular graphs, see for example [1, 31, 38]. For other results on counting independent sets in graphs and hypergraphs, see Balogh and Samotij [9, 10], Carroll, Galvin and Tetali [11], Galvin and Kahn [19], Kahn [25, 26], Peled and Samotij [33], Sapozhenko [37] and Zhao [44].

The rest of the paper is organised as follows. In Section 2, we state our structural theorem for 3-uniform hypergraphs, and in Section 3 we prove Theorem 1.3. In Sections 4 and 5, we prove the structural theorem, and in Sections 6 we shall apply it to prove Theorem 1.1. Finally, in Section 7, we shall prove Theorem 1.1 again in the case $q = 2$.

2. A STRUCTURAL THEOREM FOR 3-UNIFORM HYPERGRAPHS

In this section we shall introduce our main tool: a theorem which allows us to deduce structural results for sparse sum-free sets from stability results in the dense setting. Since we wish to apply our result to sum-free sets in various Abelian groups, we shall use the language of (sequences of) general 3-uniform hypergraphs $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$, where throughout the paper $|V(\mathcal{H}_n)| = n$. Throughout the paper, the reader should think of \mathcal{H} as encoding the Schur triples (that is, triples (x, y, z) with $x + y = z$) in an additive structure.

We now define the stability property with which we shall be able to work. Let $\alpha \in (0, 1)$ and let $\mathcal{B} = (\mathcal{B}_n)_{n \in \mathbb{N}}$, where \mathcal{B}_n is a family of subsets of $V(\mathcal{H}_n)$. We shall write $|\mathcal{B}_n|$ for the

number of sets in \mathcal{B}_n , and set $\|\mathcal{B}_n\| = \max\{|B| : B \in \mathcal{B}_n\}$. Besides, our notation is standard; thus $e(\mathcal{H})$ always denotes the number of edges in a hypergraph \mathcal{H} , and $\alpha(\mathcal{H})$ denotes the size of the largest independent set.

Definition 2.1. A sequence of hypergraphs $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ is said to be (α, \mathcal{B}) -stable if for every $\gamma > 0$ there exists $\beta > 0$ such that the following holds. If $A \subseteq V(\mathcal{H}_n)$ with $|A| \geq (\alpha - \beta)n$, then either $e(\mathcal{H}_n[A]) \geq \beta e(\mathcal{H}_n)$, or $|A \setminus B| \leq \gamma n$ for some $B \in \mathcal{B}_n$.

We remark that, in our applications, α will typically be chosen to be the extremal density of the sequence \mathcal{H} , and \mathcal{B}_n will be the collection of extremal sets in \mathcal{H}_n . That is, both the largest independent set in \mathcal{H}_n , and each set $B \in \mathcal{B}_n$, will have size $(\alpha + o(1))n$. Thus, roughly speaking, the reader may think of a sequence of hypergraphs \mathcal{H} as being (α, \mathcal{B}) -stable if for every $A \subseteq V(\mathcal{H}_n)$ such that $|A|$ is almost as large as $\alpha(\mathcal{H}_n)$, the set A is either very close to an extremal set $B \in \mathcal{B}_n$, or it contains many (i.e., a positive fraction of all) edges of \mathcal{H}_n . Observe that classical stability results, such as that of Erdős and Simonovits [14, 41], are typically of this form.

We shall need two further technical conditions on \mathcal{H} . Let

$$\Delta_2(\mathcal{H}_n) = \max_{T \subseteq V(\mathcal{H}_n), |T|=2} |\{e \in \mathcal{H}_n : T \subseteq e\}|,$$

and note that if \mathcal{H} encodes Schur triples then $\Delta_2(\mathcal{H}_n) \leq 3$. Also define

$$\delta(\mathcal{H}_n, \mathcal{B}_n) := \min_{B \in \mathcal{B}_n} \min_{v \in V(\mathcal{H}_n) \setminus B} |\{e \in \mathcal{H}_n : |e \cap B| = 2 \text{ and } v \in e\}|.$$

The following theorem is the key ingredient in the proof of Theorem 1.1.

Theorem 2.2. *Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of 3-uniform hypergraphs which is (α, \mathcal{B}) -stable, with $e(\mathcal{H}_n) = \Theta(n^2)$ and $\Delta_2(\mathcal{H}_n) = O(1)$. Suppose that $|\mathcal{B}_n| = n^{O(1)}$, that $\alpha(\mathcal{H}_n) \geq \|\mathcal{B}_n\| \geq \alpha n$, and that $\delta(\mathcal{H}_n, \mathcal{B}_n) = \Omega(n)$. Then there exists a constant $C = C(\mathcal{H}, \mathcal{B}) > 0$ such that if*

$$m \geq C\sqrt{n \log n},$$

then almost every independent set in \mathcal{H}_n of size m is a subset of some $B \in \mathcal{B}_n$.

We shall prove Theorem 2.2 in Sections 4 and 5. In Section 6, we shall use it to prove Theorem 1.1.

3. INDEPENDENT SETS IN REGULAR GRAPHS WITH NO SMALL EIGENVALUES

As a warm-up for the proof of Theorem 2.2, we shall prove a bound on the number of independent sets in regular graphs with no small eigenvalues, Theorem 1.3, which improves a theorem of Alon and Rödl [4]. This result will be a key tool in our self-contained proof of Theorem 1.1 in the case $q = 2$, see Section 7. Moreover, many of the ideas from the proof of Theorem 1.3 will be used again in the proof of Theorem 2.2. We remark that the technique of enumerating independent sets in graphs used in this section was pioneered by Kleitman and Winston [27] and our proof of Theorem 1.3, below, requires little more than their original method.

Given a graph \mathcal{G} on n vertices, and an integer $m \in [n]$, let $I(\mathcal{G})$ denote the number of independent sets in \mathcal{G} , and recall that $I(\mathcal{G}, m)$ denotes the number of independent sets of size m in \mathcal{G} . Alon [1] proved that if \mathcal{G} is a d -regular graph on n vertices, then $I(\mathcal{G}) \leq 2^{n/2+o(n)}$ (as $d \rightarrow \infty$), resolving a conjecture of Granville (see [1]), and suggested that the unique \mathcal{G} that maximizes $I(\mathcal{G})$ among all such (i.e., n -vertex d -regular) graphs might be a disjoint union of copies of $K_{d,d}$. This conjecture was proven (using an entropy method) by Kahn [25] for bipartite graphs, and recently in full generality by Zhao [44] (by reducing to the bipartite case).

Theorem 3.1 (Kahn [25], Zhao [44]). *Let \mathcal{G} be a d -regular graph on n vertices. Then*

$$I(\mathcal{G}) \leq (2^{d+1} - 1)^{n/2d},$$

where equality holds if and only if \mathcal{G} is a disjoint union of copies of $K_{d,d}$.

Since $-d$ is an eigenvalue of $K_{d,d}$, any d -regular graph \mathcal{G} containing a copy of $K_{d,d}$ satisfies $\lambda(\mathcal{G}) = -d$. One might hope that a stronger bound on $I(\mathcal{G})$ holds for d -regular graphs \mathcal{G} with $\lambda(\mathcal{G}) > -d$. Alon and Rödl [4] proved such a bound on $I(\mathcal{G}, m)$ for the slightly narrower class of (n, d, λ) -graphs and used their result to give sharp bounds on Ramsey numbers.

Theorem 3.2 (Alon and Rödl [4]). *Let \mathcal{G} be an (n, d, λ) -graph. Then*

$$I(\mathcal{G}, m) \leq \left(\frac{emd^2}{4\lambda n \log n} \right)^{2(n/d) \log n} \binom{2\lambda n/d}{m}$$

for every $m \geq 2(n/d) \log n$.

If $\lambda = \Omega(d)$ as $n \rightarrow \infty$, then Theorem 1.3 improves the above result in three ways: it provides a stronger bound for a wider range of values of m in a wider class of graphs. Theorem 1.3 is an immediate consequence of the following theorem, combined with the Alon-Chung Expander Mixing Lemma (Lemma 3.4, below).

Theorem 3.3. *For every $\varepsilon, \delta > 0$, there exists a constant $C = C(\varepsilon, \delta)$ such that the following holds. Let \mathcal{G} be a graph on n vertices, let $d \in \mathbb{N}$, and suppose that $2e(\mathcal{G}[A]) \geq \varepsilon|A|d$ for every $A \subseteq V(\mathcal{G})$ with $|A| \geq (\alpha + \delta)n$. Then*

$$I(\mathcal{G}, m) \leq \binom{(\alpha + 2\delta)n}{m}$$

for every $m \geq Cn/d$.

The assumption in Theorem 3.3 that $2e(\mathcal{G}[A]) \geq \varepsilon|A|d$ for every $A \subseteq V(\mathcal{G})$ with $|A| \geq (\alpha + \delta)n$ might seem somewhat strong. However, it follows from Lemma 3.4 that it is satisfied (with $\delta = \varepsilon$) by every (n, d, λ) -graph with $\lambda/(d + \lambda) \leq \alpha$. For two sets $S, T \subseteq V(\mathcal{G})$, let $e(S, T)$ denote the number of pairs $(x, y) \in S \times T$ such that $\{x, y\} \in E(\mathcal{G})$; in particular, we have $e(S, S) = 2e(\mathcal{G}[S])$ for every $S \subseteq V(\mathcal{G})$. The following result is proved⁴ in [3].

⁴Although the result is stated in [3] in a slightly different form, its proof there in fact implies Lemma 3.4.

Lemma 3.4 (Alon-Chung [3]). *Let \mathcal{G} be an n -vertex d -regular graph. Then for all $A \subseteq V(\mathcal{G})$,*

$$2e(\mathcal{G}[A]) \geq \frac{d}{n}|A|^2 + \frac{\lambda(\mathcal{G})}{n}|A|(n - |A|).$$

We first deduce Theorem 1.3 from Theorem 3.3 and Lemma 3.4.

Proof of Theorem 1.3. We claim that if \mathcal{G} is an n -vertex d -regular graph with $\lambda(\mathcal{G}) \geq -\lambda$ and $\alpha = \lambda/(d + \lambda)$, then $2e(\mathcal{G}[A]) \geq \varepsilon|A|d$ for every $A \subseteq V(\mathcal{G})$ with $|A| \geq (\alpha + \varepsilon)n$. This implies that \mathcal{G} satisfies the assumption of Theorem 3.3 (with $\delta = \varepsilon$), and so the theorem follows.

To prove the claim, suppose that $A \subseteq V(\mathcal{G})$ satisfies $|A| \geq (\alpha + \varepsilon)n$. By Lemma 3.4,

$$2e(\mathcal{G}[A]) \geq \frac{d}{n} \cdot |A|^2 - \frac{\lambda}{n} \cdot |A|(n - |A|) = \frac{|A|}{n} \left[(d + \lambda)|A| - \lambda n \right].$$

Our lower bound on $|A|$ and the choice of $\alpha = \lambda/(d + \lambda)$ now give

$$2e(\mathcal{G}[A]) \geq |A| \left[(d + \lambda)(\alpha + \varepsilon) - \lambda \right] = \varepsilon|A|(d + \lambda) \geq \varepsilon|A|d,$$

as required. \square

In the proof of Theorem 3.3, we shall use an algorithm which uniquely encodes every independent set I of size m in \mathcal{G} as a pair $(S, I \setminus S)$, where $|S| \leq 2n/\varepsilon d \leq 2m/C\varepsilon$, and $I \setminus S$ is contained in some set A , with $|A| \leq (\alpha + \delta)n$, which depends only on S . At all times, it maintains a partition of $V(\mathcal{G})$ into sets S , X , and A (short for Selected, eXcluded, and Available), such that $S \subseteq I \subseteq A \cup S$.

At each stage of the algorithm, we will need to order the vertices of A with respect to their degrees. For the sake of brevity and clarity of the presentation, let us make the following definition.

Definition 3.5 (Max-degree order). Given a graph \mathcal{G} and a set $A \subseteq V(\mathcal{G})$, the *max-degree order* on A is the following linear order $(v_1, \dots, v_{|A|})$ on the elements of A : For every $i \in \{1, \dots, |A|\}$, v_i is the maximum-degree vertex in the graph $\mathcal{G}[A \setminus \{v_1, \dots, v_{i-1}\}]$; we break ties by giving preference to vertices that come earlier in some predefined ordering of $V(\mathcal{G})$.

We are now ready to describe the Basic Algorithm.

The Basic Algorithm. Set $A = V(\mathcal{G})$ and $S = X = \emptyset$. Now, while $|A| > (\alpha + \delta)n$, we repeat the following:

- (a) Let i be the minimal index (in the max-degree order on A) such that $v_i \in I$.
- (b) Move v_i from A to S .
- (c) Move v_1, \dots, v_{i-1} from A to X (since they are not in I by the choice of i).
- (d) Move $N(v_i)$ from A to X (since I is independent and $v_i \in I$).

Finally, when $|A| \leq (\alpha + \delta)n$, we output S (which is a subset of $V(\mathcal{G})$) and $I \setminus S$ (which is a subset of A).

We remark that, as well as in [27], algorithms similar to the one above have been considered before to bound the number of independent sets in graphs [4, 37] and hypergraphs [9, 10].

Proof of Theorem 3.3. The theorem is an easy consequence of the following two statements:

$$I(\mathcal{G}, m) \leq \sum_{t=1}^{t_0} \binom{n}{t} \binom{(\alpha + \delta)n}{m-t}, \quad (1)$$

where $t_0 = \frac{n}{\varepsilon d} + 1$,⁵ and

$$\binom{n}{t} \binom{(\alpha + \delta)n}{m-t} \leq \frac{1}{m} \binom{(\alpha + 2\delta)n}{m} \quad (2)$$

if $t \leq 2n/\varepsilon d$ and $m \geq Cn/d$. We shall prove (1) using the Basic Algorithm; (2) will follow from a straightforward calculation.

To prove (1), let $t \in \mathbb{N}$ be the number of elements of S at the end of the algorithm, and note that we have at most $\binom{n}{t}$ choices for S . Now, crucially, we observe that A is uniquely determined by S (given the original ordering of $V(\mathcal{G})$); indeed, step (a) of the Basic Algorithm requires no knowledge of I , only of S and \mathcal{G} . Since $|A| \leq (\alpha + \delta)n$, it follows that we have at most $\binom{(\alpha + \delta)n}{m-t}$ choices for $I \setminus S$.

It will thus suffice to show that, given our assumptions on \mathcal{G} , the algorithm terminates in at most $t_0 = \frac{n}{\varepsilon d} + 1 \leq \frac{2n}{\varepsilon d}$ steps. We shall show that A loses at least εd elements at each step of the algorithm (except perhaps the last), from which this bound follows immediately. Indeed, consider a step of the algorithm (not the last), in which a vertex v_i is moved to S , and set $A' = A \setminus \{v_1, \dots, v_{i-1}\}$. Since this is not the last step, we have $|A'| \geq (\alpha + \delta)n$, and so $2e(\mathcal{G}[A']) \geq \varepsilon|A'|d$, by our assumption on \mathcal{G} . Thus $|N(v_i) \cap A'| \geq \varepsilon d$, since v_i is the vertex of maximum degree in $\mathcal{G}[A']$, and hence A loses at least εd elements in this step, as claimed.

To prove (2), we use the fact that, if t is replaced by $t + 1$, then the left-hand side is multiplied by

$$\frac{n-t}{t+1} \cdot \frac{m-t}{(\alpha + \delta)n - m + t + 1}. \quad (3)$$

We claim that (3) is at most $\frac{2}{\delta} \cdot \left(\frac{m}{t}\right)^2$. To prove this, we consider two cases: if $m \leq (\alpha + \delta/2)n$, then (3) is at most

$$\frac{n}{t} \cdot \frac{2m}{\delta n} = \frac{2m}{\delta t}$$

while if $m > (\alpha + \delta/2)n$ then it is at most

$$\frac{n}{t} \cdot \frac{m}{t} \leq \frac{2}{\delta} \cdot \left(\frac{m}{t}\right)^2,$$

since $n \leq 2m/\delta$, and our assumptions imply that $\alpha(\mathcal{G}) < (\alpha + \delta)n$, so we may assume that $m < (\alpha + \delta)n$. Thus, for each t with $t \leq t_0 \leq \frac{2n}{\varepsilon d}$,

$$\binom{n}{t} \binom{(\alpha + \delta)n}{m-t} \leq \binom{(\alpha + \delta)n}{m} \prod_{r=1}^t \frac{2}{\delta} \cdot \left(\frac{m}{r}\right)^2 \leq \left(\frac{1}{t_0!}\right)^2 \left(\frac{2m}{\delta}\right)^{2t_0} \binom{(\alpha + \delta)n}{m}.$$

⁵Here, and throughout the paper, we shall (for simplicity) omit irrelevant ceiling and floor signs.

Since $\binom{b}{c} \leq \left(\frac{b}{a}\right)^c \binom{a}{c}$ for every $a > b > c \geq 0$, $k! > (k/e)^k$ for every $k \in \mathbb{N}$, the function $t_0 \mapsto \left(\frac{em}{\delta t_0}\right)^{t_0}$ is increasing on the interval $(0, m)$, and $t_0 \leq \frac{2n}{\varepsilon d} \leq \frac{2m}{C\varepsilon}$, this is at most

$$\left(\frac{2em}{\delta t_0}\right)^{2t_0} \binom{(\alpha + \delta)n}{m} \leq \left(\frac{C\varepsilon e}{\delta}\right)^{\frac{4}{C\varepsilon}m} \left(\frac{\alpha + \delta}{\alpha + 2\delta}\right)^m \binom{(\alpha + 2\delta)n}{m} \leq \frac{1}{m} \cdot \binom{(\alpha + 2\delta)n}{m}$$

as required. In the final inequality we used the fact that C is sufficiently large as a function of δ and ε , and that m is sufficiently large (as a function of δ), since $m \geq Cn/d$. \square

4. ALGORITHM ARGUMENT

In this section, we shall introduce a more powerful algorithm than that used in Section 3. We shall use this algorithm in the proof of Theorem 2.2 to bound the number of independent sets of size m which contain at least δm elements of $V(\mathcal{H}_n) \setminus B$ for every $B \in \mathcal{B}_n$. We shall show that when $m \gg \sqrt{n}$, then the number of such independent sets is exponentially small. The model example that the reader should keep in mind when reading this section is as follows: \mathcal{H}_n is the hypergraph of Schur triples in an n -element Abelian group G of Type I(q), where q is some prime satisfying $q \equiv 2 \pmod{3}$, and \mathcal{B}_n denotes the collection of maximum-size sum-free subsets of G . Note that independent sets in \mathcal{H}_n are exactly sum-free sets in G .

Given a hypergraph \mathcal{H}_n , a family of sets \mathcal{B}_n , and $\delta > 0$, we define

$$\text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m) := \left\{ I \in \text{SF}(\mathcal{H}_n, m) : |I \setminus B| \geq \delta m \text{ for every } B \in \mathcal{B}_n \right\},$$

where $\text{SF}(\mathcal{H}_n, m)$ denotes the collection of independent sets in \mathcal{H}_n of size m . The following theorem shows that there are very few sets in $\text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ which are far from every set $B \in \mathcal{B}_n$.

Theorem 4.1. *Let $\alpha > 0$ and let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of 3-uniform hypergraphs which is (α, \mathcal{B}) -stable, has $e(\mathcal{H}_n) = \Theta(n^2)$ and $\Delta_2(\mathcal{H}_n) = O(1)$. If $\|\mathcal{B}_n\| \geq \alpha n$, then for every $\delta > 0$, there exists a $C > 0$ such that the following holds. If $m \geq C\sqrt{n}$ and n is sufficiently large, then*

$$|\text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)| \leq (1 - \varepsilon)^m \binom{\|\mathcal{B}_n\|}{m}$$

for some $\varepsilon = \varepsilon(\mathcal{H}, \delta) > 0$.

The rest of this section is devoted to the proof of Theorem 4.1; in Section 5 we shall use it to prove Theorem 2.2.

We shall describe an algorithm which encodes every independent set I in \mathcal{H}_n and produces short output for every $I \in \text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$. As before, our algorithm will maintain a partition of $V(\mathcal{H}_n)$ into sets S , X , and A (short for Selected, eXcluded, and Available), such that $S \subseteq I \subseteq A \cup S$. We shall also maintain a set $T \subseteq S$ (for Temporary), and the corresponding *link graph* \mathcal{G}_T , i.e., the graph with vertex set $V(\mathcal{H}_n)$, and edge set

$$E(\mathcal{G}_T) = \left\{ \{u, v\} \subseteq V(\mathcal{H}_n) : \{u, v, w\} \in \mathcal{H}_n \text{ for some } w \in T \right\}. \quad (4)$$

We note the following easy property of the link graph for future reference.

Observation 4.2. $\Delta(\mathcal{G}_T) \leq |T| \cdot \Delta_2(\mathcal{H}_n)$.

In the algorithm described below, we shall frequently consider the max-degree order (defined in Section 3) on the vertices of the graph $\mathcal{G}_T[A]$.

4.1. The Algorithm. The idea of the algorithm is quite simple: we apply the Basic Algorithm of Section 3 to the graph $\mathcal{G}_T[A]$ as long as it is reasonably dense. If $\mathcal{G}_T[A]$ becomes too sparse, then there are four possibilities: either we have arrived at a set A which has at most $(\alpha - \beta)n$ elements, or a set A which is almost contained in some $B \in \mathcal{B}_n$; or if not, then we can use the (α, \mathcal{B}) -stability of \mathcal{H} to find either a new set T for which \mathcal{G}_T is dense (see Case 2 below), or a set of size $\Omega(n)$ that contains very few elements of I . Therefore, after moving relatively few vertices of \mathcal{H}_n to S , our choice for $I \setminus S$ is limited to a set $A \subseteq V(\mathcal{G})$ that is either small or almost contained in some $B \in \mathcal{B}_n$. It follows that if I was far from every $B \in \mathcal{B}_n$, then (in both cases) the number of ways to choose $I \setminus S$ from A is very small.

We begin by choosing some constants. Given \mathcal{H} , α , \mathcal{B} and δ as in the statement of the theorem, let $\gamma = \gamma(\alpha, \delta) > 0$ be sufficiently small, and note that since \mathcal{H} is (α, \mathcal{B}) -stable, there exists $\beta > 0$ so that if $|A| \geq (\alpha - \beta)n$ and $|A \setminus B| > \gamma n$ for every $B \in \mathcal{B}_n$, then $e(\mathcal{H}_n[A]) \geq \beta e(\mathcal{H}_n)$. Let us choose $\beta > 0$ sufficiently small so that $e(\mathcal{H}_n) \geq \beta n^2$ and $\Delta_2(\mathcal{H}_n) \leq 1/\beta$ for all sufficiently large n . Let $C = C(\beta) > 0$ be sufficiently large (to be specified below), and set

$$d = \frac{Cn}{m} \leq \frac{m}{C} \leq \frac{n}{C}.$$

We are ready to describe the Main Algorithm; this is the key step in our proof of Theorem 2.2.

The Main Algorithm. We initiate the algorithm with $T = S \subseteq I$, a deterministically chosen subset of I of size d (the first d elements of I in our ordering of $V(\mathcal{H}_n)$, say), and with $A = V(\mathcal{H}_n) \setminus S$ and $X = \emptyset$. Now, while $|A| > (\alpha - \beta)n$ and $|A \setminus B| > \gamma n$ for every $B \in \mathcal{B}_n$, we repeat the following steps:

Case 1: If the average degree in $\mathcal{G}_T[A]$ is at least $\beta^4 d$, then:

- (a) Let i be the minimal index in the max-degree order on $V(\mathcal{G}_T[A])$ such that $v_i \in I$.
- (b) Move v_i from A to S .
- (c) Move v_1, \dots, v_{i-1} from A to X (since they are not in I by the choice of i).
- (d) Move $N(v_i)$ from A to X (since I is independent and $v_i \in I$).

Case 2: If the average degree of $\mathcal{G}_T[A]$ is less than $\beta^4 d$, then we find a new set T as follows. Since \mathcal{H} is (α, \mathcal{B}) -stable, $|A| > (\alpha - \beta)n$, and $|A \setminus B| > \gamma n$ for every $B \in \mathcal{B}_n$, then A contains at least $\beta e(\mathcal{H}_n)$ edges of \mathcal{H}_n . Set

$$Z = \left\{ z \in A : e(\mathcal{G}_z[A]) \geq \beta^2 n \right\},$$

where $\mathcal{G}_z[A] = G_{\{z\}}[A]$. We call the elements of Z *useful*. Since $e(\mathcal{H}_n) \geq \beta n^2$, we have $e(\mathcal{H}_n[A]) \geq \beta^2 n^2$, and so

$$\sum_{z \in A} e(\mathcal{G}_z[A]) = 3e(\mathcal{H}_n[A]) \geq 3\beta^2 n^2.$$

Moreover, we have $e(\mathcal{G}_z[A]) \leq \Delta(\mathcal{H}_n) \leq \Delta_2(\mathcal{H}_n)n \leq n/\beta$ for every $z \in V(\mathcal{G})$. Thus, by the pigeonhole principle, it follows that $|Z| \geq 2\beta^3n$.

Now we have two subcases:

- (a) If I contains fewer than d useful elements, then move these elements from A to S and move the other useful elements from A to X .
- (b) If I contains more than d useful elements, choose d of them u_1, \dots, u_d (the first d in our ordering, say) and move them from A to S . Moreover, set $T = \{u_1, \dots, u_d\}$.

Finally, when $|A| \leq (\alpha - \beta)n$, or $|A \setminus B| \leq \gamma n$ for some $B \in \mathcal{B}_n$, then we output S (which is a subset of $V(\mathcal{G})$) and $I \setminus S$ (which is a subset of A) and stop.

We shall show that the Main Algorithm encodes at most β^2m elements of I in S , and that S determines A . Theorem 4.1 then follows from some simple counting.

4.2. Proof of Theorem 4.1. We begin by proving three straightforward claims about the Main Algorithm; these, together with some simple counting, will be enough to prove the theorem. The following statements all hold under the assumptions of Theorem 4.1.

Claim 1. *The Main Algorithm passes through Case 1 at most $2n/(\beta^4d)$ times, and through Case 2 at most $1/\beta^5$ times.*

Proof. We prove the second statement first. To do so, simply observe that each time we pass through Case 2(a), we move at least $2\beta^3n - d \geq \beta^3n$ vertices from A to X , and each time we pass through Case 2(b), we obtain a graph $\mathcal{G}_T[A]$ with at least $\beta^2nd/\Delta_2(\mathcal{H}_n) - O(d^2) \geq 2\beta^4nd$ edges. In the latter case, we must remove at least β^4nd edges from $\mathcal{G}_T[A]$ before we can return to Case 2. Since $\Delta_2(\mathcal{H}_n) \leq 1/\beta$, it follows that $\Delta(\mathcal{G}_T) \leq |T|/\beta = d/\beta$, by Observation 4.2. Thus we must remove at least β^5n vertices from A before returning to Case 2, and hence the algorithm can pass through Case 2 at most $1/\beta^5$ times before the set A shrinks to size αn , as claimed.

To prove the first statement, note that each time we pass through Case 1 on two successive steps of the algorithm, we remove at least β^4d vertices of A in the first of these. Indeed, since $\mathcal{G}_T[A]$ (for the second step) has average degree at least β^4d , then by the definition of the max-degree order, the vertex we removed in the first step must have had forward degree at least β^4d . By the argument above, there are at most $1/\beta^5$ steps at which this fails to hold, and therefore the algorithm passes through Case 1 at most

$$\frac{n}{\beta^4d} + \frac{1}{\beta^5} \leq \frac{2n}{\beta^4d}$$

times, as claimed. □

The next claim is a simple consequence of Claim 1 and our choice of d .

Claim 2. *If $C \geq 3/\beta^7$, then $|S| \leq \beta^2m$ at the end of the Main Algorithm.*

Proof. Each time the Main Algorithm passes through Case 1, $|S|$ increases by one; each time it passes through Case 2, $|S|$ increases by at most d . Thus, by Claim 1 and our choice of d ,

$$|S| \leq d + \frac{2n}{\beta^4 d} + \frac{d}{\beta^5} \leq \frac{m}{C} + \frac{2m}{\beta^4 C} + \frac{2m}{C\beta^5} \leq \beta^2 m$$

if $C \geq 3/\beta^7$, as claimed. \square

We next make the key observation that the set S contains all the information we need to recover the final set A produced by the algorithm.

Claim 3. *The set A is uniquely determined by the set S of selected elements.*

Proof. This follows because all steps of the Main Algorithm are deterministic, and every element of I which we need to observe is placed in S . Indeed, in Case 1 we observe only that $v_i \in I$, and that the elements $v_1, \dots, v_{i-1} \notin I$. Since $v_i \in S$ and $v_1, \dots, v_{i-1} \notin S$, this can be deduced from S . In Case 2, the set Z does not depend on I . If at most $d-1$ elements of Z are in S , then we are in Case 2(a) and the remaining elements of Z are in X ; otherwise, we are in Case 2(b) and the first d elements of $S \cap Z$ (in the order on $V(\mathcal{H}_n)$) form the set T . Thus, inductively, we see that at each stage of the algorithm, the set A is determined by the set S . \square

After all this preparation, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be an (α, \mathcal{B}) -stable sequence of 3-uniform hypergraphs as in the statement of the theorem, let $\delta > 0$ be arbitrary, and choose $\gamma = \gamma(\alpha, \delta) > 0$ to be sufficiently small. Since \mathcal{H} is (α, \mathcal{B}) -stable, there exists $\beta > 0$ such that if $|I| \geq (\alpha - 2\beta)n$, then either $|I \setminus B| < \gamma n$ for some $B \in \mathcal{B}_n$, or I is not an independent set in \mathcal{H}_n . In particular, note that if $\gamma < \delta(\alpha - 2\beta)$, then either $m \leq (\alpha - 2\beta)n$ or $\text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ is empty. We choose such a β sufficiently small, set $C = 3/\beta^7$, and choose $\varepsilon = \varepsilon(\beta) > 0$ to be sufficiently small. Finally, let n be sufficiently large.

Applying the Main Algorithm to each independent set $I \in \text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$, that is, to every independent set in \mathcal{H}_n such that $|I| = m$ and $|I \setminus B| \geq \delta m$ for every $B \in \mathcal{B}_n$, we obtain (for each such I) a pair (A, S) such that $S \subseteq I \subseteq A \cup S$. There are two cases to deal with, corresponding to the two possibilities that can occur at the end of the Main Algorithm. We first show that the number of sets I as above that yield a pair (A, S) with $|A \setminus B| \leq \gamma n$ for some $B \in \mathcal{B}_n$ is much smaller than the claimed upper bound on $|\text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)|$.

Claim 4. *There are at most $\delta^m \binom{\|\mathcal{B}_n\|}{m}$ sets $I \in \text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ such that the Main Algorithm ends because $|A \setminus B| \leq \gamma n$ for some $B \in \mathcal{B}_n$.*

Proof. We claim first that the number of such sets I is at most

$$\sum_{t=0}^{\beta^2 m} \sum_{r \geq \delta m} \binom{n}{t} \binom{\gamma n}{r-t} \binom{\|\mathcal{B}_n\|}{m-r}. \quad (5)$$

Indeed, let S and A be the selected and available sets at the end of the algorithm, set $t = |S|$, and recall that $t \leq \beta^2 m$ by Claim 2. We have at most $\binom{n}{t}$ choices for S and, by Claim 3,

the set S determines the set A . Let $B \in \mathcal{B}_n$ be such that $|A \setminus B| \leq \gamma n$ and recall that $|I \setminus B| \geq \delta m$ by our assumption on I . Thus we must choose at least $\delta m - t$ elements of $A \setminus B$, and the remaining elements from B .

Note that $t \leq \delta m/2$ by our choice of β and so either $m \leq (2\gamma/\delta)n$ or the number of choices for I is zero. Since $\|\mathcal{B}_n\| \geq \alpha n$ and γ is small, it follows that the summand in (5) is maximized (for a given t) when $r = \delta m$. Now, using the inequalities $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and

$$\binom{a}{b-c} \leq \left(\frac{b}{a-b}\right)^c \binom{a}{b}, \quad (6)$$

which holds for every $a > b > c \geq 0$, and since $t \leq \delta m/2$, $m \leq (2\gamma/\delta)n \leq (\alpha/2)n$ and $\|\mathcal{B}_n\| \geq \alpha n$, we can bound each summand in (5) from above by

$$\left(\frac{en}{t}\right)^t \left(\frac{2e\gamma n}{\delta m}\right)^{\delta m-t} \left(\frac{m}{\alpha n - m}\right)^{\delta m} \binom{\|\mathcal{B}_n\|}{m}.$$

Since $t \leq \delta m/2$ and $t \mapsto (c/t)^t$ is increasing on $(0, c/e)$, this is at most

$$\left(\frac{\delta m}{2\gamma t}\right)^t \left(\frac{2e\gamma n}{\delta m} \cdot \frac{2m}{\alpha n}\right)^{\delta m} \binom{\|\mathcal{B}_n\|}{m} \leq \left(\frac{1}{\gamma}\right)^{\delta m/2} \left(\frac{4e\gamma}{\alpha\delta}\right)^{\delta m} \binom{\|\mathcal{B}_n\|}{m} \leq \delta^{2m} \binom{\|\mathcal{B}_n\|}{m}$$

if $\gamma \leq (\alpha\delta/4e)^2 \cdot \delta^{4/\delta}$. Since $m^2 \cdot \delta^{2m} \leq \delta^m$, the claim follows. \square

Finally, we deal with the case in which $|A| \leq (\alpha - \beta)n$ for some $B \in \mathcal{B}_n$.

Claim 5. *There are at most $(1 - \varepsilon)^m \binom{\|\mathcal{B}_n\|}{m}$ sets $I \in \text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ such that the Main Algorithm ends because $|A| \leq (\alpha - \beta)n$.*

Proof. As in the previous claim, we have $t = |S| \leq \beta^2 m$, by Claim 2, and the set S determines the set A , by Claim 3. Thus, the number of choices for I is at most

$$\sum_{t=0}^{\beta^2 m} \binom{n}{t} \binom{(\alpha - \beta)n}{m-t}. \quad (7)$$

Now, recall that either $m \leq (\alpha - 2\beta)n$ or $\text{SF}_{\geq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ is empty. Thus, estimating each summand in (7) as in the proof of Claim 4, we obtain

$$\binom{n}{t} \binom{(\alpha - \beta)n}{m-t} \leq \left(\frac{en}{t}\right)^t \left(\frac{m}{(\alpha - \beta)n - m}\right)^t \binom{(\alpha - \beta)n}{m} \leq \left(\frac{em}{\beta t}\right)^t \binom{(\alpha - \beta)n}{m}.$$

Now, using the inequality $\binom{b}{c} \leq \left(\frac{b}{a}\right)^c \binom{a}{c}$, which is valid for all $a > b > c \geq 0$, and recalling that $t \leq \beta^2 m$ and that $t \mapsto (c/t)^t$ is increasing on $(0, c/e)$, we get

$$\left(\frac{em}{\beta t}\right)^t \binom{(\alpha - \beta)n}{m} \leq \left(\frac{e}{\beta^3}\right)^{\beta^2 m} \left(\frac{\alpha - \beta}{\alpha}\right)^m \binom{\alpha n}{m} \leq \left(\frac{e}{\beta^3}\right)^{\beta^2 m} e^{-\beta m/\alpha} \binom{\alpha n}{m}.$$

Since $\|\mathcal{B}_n\| \geq \alpha n$, the right-hand side is at most $\frac{1}{m}(1 - \varepsilon)^m \binom{\|\mathcal{B}_n\|}{m}$ if $\beta > 0$ and $\varepsilon = \varepsilon(\beta) > 0$ are sufficiently small, as required. \square

Combining Claims 4 and 5, we obtain Theorem 4.1. \square

5. JANSON ARGUMENT

In this section, we shall complete the proof of Theorem 2.2 by showing that, under certain conditions, almost all independent (i.e., sum-free) sets I of size m in \mathcal{H}_n either satisfy $I \subseteq B$ for some $B \in \mathcal{B}_n$, or $|I \setminus B| \geq \delta m$ for every $B \in \mathcal{B}_n$. The key properties of \mathcal{H} which we will use are that $\Delta_2(\mathcal{H}_n) = O(1)$, and that $\delta(\mathcal{H}_n, \mathcal{B}_n) = \Omega(n)$; our key tool will be Janson's inequality. An argument similar to that presented in this section was used in [8] to study sum-free sets in random subsets of Abelian groups.

Given a hypergraph \mathcal{H}_n , a family of sets \mathcal{B}_n and $\delta > 0$, we define

$$\text{SF}_{\leq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m) := \left\{ I \in \text{SF}(\mathcal{H}_n, m) : |I \setminus B| \leq \delta m \text{ for some } B \in \mathcal{B}_n \right\}.$$

The following proposition shows that, if $\delta > 0$ is sufficiently small, then almost all independent sets in $\text{SF}_{\leq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ are contained in some $B \in \mathcal{B}_n$. We write 2^B to denote the power set of B , i.e., the family of all subsets of B .

Proposition 5.1. *Let $\alpha > 0$, let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of 3-uniform hypergraphs with $\Delta_2(\mathcal{H}_n) = O(1)$, and let $\mathcal{B} = (\mathcal{B}_n)_{n \in \mathbb{N}}$ be a family of sets with $\|\mathcal{B}_n\| \geq \alpha n$. For every $\beta > 0$, there exist constants $\delta > 0$ and C_0 such that the following holds. If $\delta(\mathcal{H}_n, \mathcal{B}_n) \geq \beta n$ and $C \geq C_0$, then*

$$\left| \text{SF}_{\leq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m) \setminus \bigcup_{B \in \mathcal{B}_n} 2^B \right| \leq n^{-C} |\mathcal{B}_n| \binom{\|\mathcal{B}_n\|}{m}$$

for every $m \geq C\sqrt{n \log n}$.

Note that δ and C_0 in the statement of Proposition 5.1 may depend on \mathcal{H} , \mathcal{B} , α and β . We begin by recalling the Janson inequalities, and some basic facts about the hypergeometric distribution.

5.1. The hypergeometric distribution. The following well-known inequality (see [24, page 17], for example) allows us to deduce bounds in the hypergeometric distribution from results on product measure. For completeness we give a proof.

Lemma 5.2 (Pittel's inequality). *Let $m, n \in \mathbb{N}$, and set $p = m/n$. For any property \mathcal{Q} on $[n]$ we have*

$$\mathbb{P}(\mathcal{Q} \text{ holds for a random } m\text{-set}) \leq 3\sqrt{m} \cdot \mathbb{P}(\mathcal{Q} \text{ holds for a random } p\text{-subset of } [n]).$$

Moreover, if \mathcal{Q} is monotone decreasing and $m \leq n - 1$, then

$$\mathbb{P}(\mathcal{Q} \text{ holds for a random } m\text{-set}) \leq C \cdot \mathbb{P}(\mathcal{Q} \text{ holds for a random } p\text{-subset of } [n])$$

for some absolute constant $C > 0$.

Proof. For the first part, simply note that a random p -subset of $[n]$ has size $m = pn$ with probability at least $1/(3\sqrt{m})$. If \mathcal{Q} is monotone decreasing, say, then we apply the 'Local LYM inequality' to \mathcal{Q}_m , the set of m -sets in \mathcal{Q} , and deduce that

$$\mathbb{P}(\mathcal{Q} \text{ holds for a random } k\text{-set}) \geq \mathbb{P}(\mathcal{Q} \text{ holds for a random } m\text{-set})$$

for every $k \leq m$. It is well-known that the median of the binomial distribution lies between $\lfloor pn \rfloor$ and $\lceil pn \rceil$, and if $m \leq n - 1$ then it is easy to see that $\text{Bin}(n, p) = \lceil pn \rceil$ has probability at most $(1 - 1/n)^{n-1} \rightarrow 1/e$ as $n \rightarrow \infty$. Thus, if $m \leq n - 1$ and n is sufficiently large, then a random p -subset of $[n]$ has size at most $m = pn$ with probability at least $1/2 - 1/e + o(1)$ as $n \rightarrow \infty$, and the result follows. \square

The following result is an easy corollary of Janson's inequality (see [5, 24]), combined with Pittel's inequality.

Lemma 5.3 (Hypergeometric Janson Inequality). *Suppose that $\{U_i\}_{i \in I}$ is a family of subsets of an n -element set X and let $m \in \{0, \dots, n\}$. Let*

$$\mu = \sum_{i \in I} (m/n)^{|U_i|} \quad \text{and} \quad \Delta = \sum_{i \sim j} (m/n)^{|U_i \cup U_j|},$$

where the second sum is over ordered pairs (i, j) such that $i \neq j$ and $U_i \cap U_j \neq \emptyset$. Let R be a uniformly chosen random m -subset of X . Then

$$\mathbb{P}(U_i \not\subseteq R \text{ for all } i \in I) \leq C \cdot \max \left\{ e^{-\mu/2}, e^{-\mu^2/(2\Delta)} \right\},$$

where $C > 0$ is the constant in Pittel's inequality.

We now return to the proof of Proposition 5.1.

5.2. Proof of Proposition 5.1. We begin by partitioning $\text{SF}_{\leq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m)$ according to the set $B \in \mathcal{B}_n$ such that $|I \setminus B| \leq \delta m$, and also according to the set $S = I \setminus B$. (Technically there could be more than one such set B , so in fact this might be a cover, rather than a partition.) Set

$$I(B, S) := \left| \left\{ I \in \text{SF}(\mathcal{H}_n, m) : I \setminus B = S \right\} \right|.$$

We shall prove the following lemma.

Lemma 5.4. *Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of 3-uniform hypergraphs with $\Delta_2(\mathcal{H}_n) = O(1)$. For every sufficiently small $\beta > 0$, there exists a constant C_0 such that the following holds. Let $C \geq C_0$, let $B \subseteq [n]$ with $\delta(\mathcal{H}_n, B) \geq \beta n$, and let $S \subseteq [n] \setminus B$ satisfy $k := |S| \leq m/2$. Then*

$$I(B, S) \leq \left(|B|^{-5Ck} + e^{-\beta^3 m} \right) \binom{|B|}{m-k}$$

for every $m \geq C\sqrt{n \log n}$.

Recall from (4) the definition of the link graph \mathcal{G}_S of a set S in a 3-uniform hypergraph \mathcal{H}_n . In order to prove Lemma 5.4, we shall apply the following lemma to the subgraph of the link graph of S induced by B . The lemma is a straightforward consequence of the Hypergeometric Janson's inequality.

Lemma 5.5. *For every $\beta > 0$, there exists a constant C_0 such that the following holds. Let \mathcal{G} be a graph on n vertices with maximum degree at most d . If*

$$e(\mathcal{G}) \geq 4\beta dn,$$

then for every $C \geq C_0$,

$$I(\mathcal{G}, m) \leq \left(n^{-Cd} + e^{-\beta m} \right) \binom{n}{m}$$

for every $m \geq C\sqrt{n \log n}$.

Proof. Let $\{U_i\}_{i \in I}$ be the collection of pairs of vertices which span an edge of \mathcal{G} , so $U_i \not\subseteq R$ for all $i \in I$ if and only if R is an independent set in \mathcal{G} . It is easy to see that, letting μ and Δ to be the quantities defined in the statement of Lemma 5.3,

$$\mu = e(\mathcal{G}) \frac{m^2}{n^2} \quad \text{and} \quad \Delta \leq \binom{d}{2} \left(\frac{2e(\mathcal{G})}{d} \right) \left(\frac{m}{n} \right)^3 \leq e(\mathcal{G}) \frac{dm^3}{n^3}.$$

Thus, by our bounds on $e(\mathcal{G})$ and m , and assuming $C \geq 4/\beta$,

$$\mu \geq 4Cd \log n \quad \text{and} \quad \frac{\mu^2}{\Delta} \geq e(\mathcal{G}) \frac{m}{dn} \geq 4\beta m.$$

By the Hypergeometric Janson Inequality,

$$\begin{aligned} I(\mathcal{G}, m) / \binom{n}{m} &\leq C \cdot \max \left\{ e^{-\mu/2}, e^{-\mu^2/(2\Delta)} \right\} \\ &\leq C \cdot \max \left\{ n^{-2Cd}, e^{-2\beta m} \right\} \leq \max \left\{ n^{-Cd}, e^{-\beta m} \right\}, \end{aligned}$$

as required. \square

We can now easily deduce Lemma 5.4 from Lemma 5.5.

Proof of Lemma 5.4. If I is an independent set in \mathcal{H}_n containing S , then $I \setminus S$ is an independent set in \mathcal{G}_S , so

$$I(B, S) \leq I(\mathcal{G}_S[B], m - k),$$

where $k = |S|$. Choose $\beta > 0$ sufficiently small so that $\Delta_2(\mathcal{H}_n) \leq 1/(4\beta)$. Let $d := |S|/(4\beta)$, note that $d \geq 5|S|$ if $\beta < 1/20$, and that by Observation 4.2, $\Delta(\mathcal{G}_S) \leq |S| \cdot \Delta_2(\mathcal{H}_n) \leq d$. Recall that $\delta(\mathcal{H}_n, B) \geq \beta n$, and observe that therefore

$$e(\mathcal{G}_S[B]) \geq \frac{\beta n \cdot |S|}{\Delta_2(\mathcal{H}_n)} \geq 2\beta^2 |S| \cdot n \geq 8\beta^3 d \cdot |B|.$$

Thus, by Lemma 5.5, we have

$$I(\mathcal{G}_S[B], m - k) \leq \left(|B|^{-5Ck} + e^{-\beta^3 m} \right) \binom{|B|}{m - k}$$

for every $m \geq C\sqrt{n \log n}$, as required. \square

Finally, let us deduce Proposition 5.1 from Lemma 5.4.

Proof of Proposition 5.1. Summing over all sets $B \in \mathcal{B}_n$ and subsets $S \subseteq [n] \setminus B$ with $|S| \leq \delta m$, and applying Lemma 5.4, we have

$$\begin{aligned} \left| \text{SF}_{\leq}^{(\delta)}(\mathcal{H}_n, \mathcal{B}_n, m) \setminus \bigcup_{B \in \mathcal{B}_n} 2^B \right| &\leq \sum_{B \in \mathcal{B}_n} \sum_{k=1}^{\delta m} \sum_{S \subseteq [n] \setminus B: |S|=k} I(B, S) \\ &\leq |\mathcal{B}_n| \sum_{k=1}^{\delta m} \binom{n}{k} \left(n^{-4(C+k)} + e^{-\beta^3 m} \right) \binom{\|\mathcal{B}_n\|}{m-k}, \end{aligned}$$

for every $m \geq C\sqrt{n \log n}$, since $\delta(\mathcal{H}_n, \mathcal{B}_n) \geq \beta n$ and $\Delta_2(\mathcal{H}_n) = O(1)$ together imply that $|B| = \Theta(n)$ for every $B \in \mathcal{B}_n$. We consider three cases.

Case 1: If $n^{-4(C+k)} \geq e^{-\beta^3 m}$, then

$$\binom{n}{k} \left(n^{-4(C+k)} + e^{-\beta^3 m} \right) \binom{\|\mathcal{B}_n\|}{m-k} \leq n^{-2C} \binom{\|\mathcal{B}_n\|}{m},$$

since $\binom{\|\mathcal{B}_n\|}{m-k} \leq \binom{n}{k} \binom{\|\mathcal{B}_n\|}{m}$.

Case 2: If $n^{-4(C+k)} \leq e^{-\beta^3 m}$ and $m \leq \alpha n/2$, then by (6) we have

$$\binom{\|\mathcal{B}_n\|}{m-k} \leq \left(\frac{m}{\alpha n - m} \right)^k \binom{\|\mathcal{B}_n\|}{m} \leq \left(\frac{2m}{\alpha n} \right)^k \binom{\|\mathcal{B}_n\|}{m},$$

since $\|\mathcal{B}_n\| \geq \alpha n$. Thus, using the bound $\binom{n}{k} \leq \left(\frac{en}{k} \right)^k$, we have

$$\binom{n}{k} \left(n^{-4(C+k)} + e^{-\beta^3 m} \right) \binom{\|\mathcal{B}_n\|}{m-k} \leq 2 \cdot e^{-\beta^3 m} \left(\frac{2em}{\alpha k} \right)^k \binom{\|\mathcal{B}_n\|}{m} \leq e^{-\beta^3 m/2} \binom{\|\mathcal{B}_n\|}{m},$$

if $\delta = \delta(\alpha, \beta) > 0$ is sufficiently small, since $k \leq \delta m$.

Case 3: If $\|\mathcal{B}_n\|^{-4(C+k)} \leq e^{-\beta^3 m}$ and $m \geq \alpha n/2$, then we again use the (trivial) bound $\binom{\|\mathcal{B}_n\|}{m-k} \leq \binom{n}{k} \binom{\|\mathcal{B}_n\|}{m}$, to obtain

$$\binom{n}{k} \left(n^{-4(C+k)} + e^{-\beta^3 m} \right) \binom{\|\mathcal{B}_n\|}{m-k} \leq 2e^{-\beta^3 m} \binom{n}{k}^2 \binom{\|\mathcal{B}_n\|}{m} \leq e^{-\beta^3 m/2} \binom{\|\mathcal{B}_n\|}{m},$$

if $\delta = \delta(\alpha, \beta) > 0$ is sufficiently small, since $\binom{n}{k} \leq \left(\frac{2m}{k} \right)^k \leq \left(\frac{2e}{\alpha \delta} \right)^{\delta m} \leq e^{-\beta^3 m/6}$ for $k \leq \delta m$.

Since $e^{-\beta^3 m/2} \ll n^{-2C}$ for $m \geq C\sqrt{n \log n}$, the claimed bound follows. \square

We finish this section by observing that Theorem 4.1 and Proposition 5.1 together imply Theorem 2.2.

Proof of Theorem 2.2. Let $\mathcal{H} = (\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of 3-uniform hypergraphs which is (α, \mathcal{B}) -stable, where $\mathcal{B} = (\mathcal{B}_n)_{n \in \mathbb{N}}$ is a family of sets, and $\alpha > 0$. Suppose that $\alpha(\mathcal{H}_n) \geq \|\mathcal{B}_n\| \geq \alpha n$, and that there exists $\beta > 0$ such that $e(\mathcal{H}_n) \geq \beta n^2$, $\Delta_2(\mathcal{H}_n) \leq 1/\beta$, $|\mathcal{B}_n| \leq n^{1/\beta}$

and $\delta(\mathcal{H}_n, \mathcal{B}_n) \geq \beta n$ for every $n \in \mathbb{N}$. Let $\delta = \delta(\beta) > 0$ be sufficiently small, and let $C = C(\beta, \delta) > 0$ be sufficiently large. We claim that if

$$m \geq C\sqrt{n \log n},$$

then almost every independent set in \mathcal{H}_n of size m is a subset of some $B \in \mathcal{B}_n$.

Indeed, by Theorem 4.1, the number of independent sets I in \mathcal{H}_n of size m for which $|I \setminus B| \geq \delta m$ for every $B \in \mathcal{B}_n$ is at most

$$(1 - \varepsilon)^m \binom{\|\mathcal{B}_n\|}{m}$$

for some $\varepsilon > 0$ and by Proposition 5.1, the number of such sets for which $1 \leq |I \setminus B| \leq \delta m$ for some $B \in \mathcal{B}_n$ is at most

$$n^{-C} |\mathcal{B}_n| \binom{\|\mathcal{B}_n\|}{m}.$$

Since $|\mathcal{B}_n| \leq n^{1/\beta}$, $C > 1/\beta$, and $\alpha(\mathcal{H}_n) \geq \|\mathcal{B}_n\|$, the result follows. \square

6. ABELIAN GROUPS OF TYPE I

In this section, we shall use Theorem 2.2 to prove Theorem 1.1 for all $q > 2$. We remark that the proof below can also be adapted to cover the case $q = 2$; however, since we shall give a different proof of the case $q = 2$ in Section 7, we leave the details to the reader. (If \mathcal{H}_n denotes the hypergraph that encodes Schur triples in a group G of even order n and \mathcal{B}_n denotes the collection of maximum-size sum-free subsets of G , then it is not always true that $\delta(\mathcal{H}_n, \mathcal{B}_n) = \Omega(n)$. This problem can be easily overcome by considering triples of the form $(x, x, 2x)$, cf. the proof of the 1-statement in [8, Theorem 1.2].)

In order to prove that our hypergraph is (α, \mathcal{B}) -stable, we shall use the following result (see [8, Corollary 2.8]), which follows immediately by combining results of Green and Ruzsa [21] and Lev, Łuczak, and Schoen [31]. Let $\text{SF}_0(G)$ denote the collection of maximal-size sum-free subsets of G and recall that each $B \in \text{SF}_0(G)$ has size $\mu(G)|G|$.

Proposition 6.1. *Let G be a finite Abelian group of Type I(q), where $q \equiv 2 \pmod{3}$ and let $0 < \gamma < \gamma(q)$ and $0 < \beta < \beta_0(\gamma, q)$ be sufficiently small. Let $A \subseteq G$, and suppose that*

$$|A| \geq (\mu(G) - \beta)|G|.$$

Then one of the following holds:

- (a) $|A \setminus B| \leq \gamma|G|$ for some $B \in \text{SF}_0(G)$.
- (b) A contains at least $\beta|G|^2$ Schur triples.

We shall also use the following classification of extremal sum-free sets for Type I groups.

Theorem 6.2 (Diananda and Yap [13]). *Let G be a finite Abelian group of Type I(q), where $q \equiv 2 \pmod{3}$. Then every $B \in \text{SF}_0(G)$ is a union of cosets of some subgroup H of G of index q , B/H is an arithmetic progression in G/H , and $B \cup (B + B) = G$.*

In other words, for every $B \in \text{SF}_0(G)$, there exists a homomorphism $\varphi: G \rightarrow \mathbb{Z}_q$ such that $B = \varphi^{-1}(\{k + 1, \dots, 2k + 1\})$, where $q = 3k + 2$.

Combining Theorem 6.2 with Kronecker's Decomposition Theorem, we easily obtain the following well-known corollary.

Corollary 6.3. *Let G be an arbitrary group of Type I. Then $|\text{SF}_0(G)| \leq |G|$.*

It is now straightforward to deduce Theorem 1.1 from Theorem 2.2, Proposition 6.1, and Corollary 6.3.

Proof of Theorem 1.1 for $q \neq 2$. Let $q \equiv 2 \pmod{3}$ be an odd prime, let $C = C(q)$ be sufficiently large, and let G_n be an Abelian group of Type I(q), with $|G_n| = n$. We shall show that if $m \geq C(q)\sqrt{n \log n}$, then almost every sum-free set of size m in G_n is contained in a member of $\text{SF}_0(G)$.

Let $X \subseteq \mathbb{N}$ be an infinite set such that, for every $n \in X$, q is the smallest prime divisor of n with $q \equiv 2 \pmod{3}$. For each $n \in X$, let G_n be an Abelian group of Type I(q), with $|G_n| = n$, and define $\mathcal{H} = (\mathcal{H}_n)_{n \in X}$ to be the sequence of hypergraphs on vertex set $V(\mathcal{H}_n) = G_n$ which encodes Schur triples. To be precise, let $V(\mathcal{H}_n) = G_n$, let $\{x, y, z\} \in \binom{G_n}{3}$ be an edge of \mathcal{H}_n whenever $x + y = z$, and observe that every sum-free subset of G_n is an independent set in \mathcal{H}_n .⁶ Let $\mathcal{B}_n = \text{SF}_0(G_n)$, the collection of maximum size sum-free subsets of G_n .

We claim that \mathcal{H} and \mathcal{B} satisfy the conditions of Theorem 2.2. Indeed, \mathcal{H}_n is 3-uniform, has $\Theta(n^2)$ edges, and satisfies $\Delta_2(\mathcal{H}_n) = 3$. Setting $\alpha = \mu(G)$, we have $\alpha(\mathcal{H}_n) = \|\mathcal{B}_n\| = \alpha n$ and $|\mathcal{B}_n| \leq n$, by Corollary 6.3. Moreover, the statement that \mathcal{H} is (α, \mathcal{B}) -stable is exactly Proposition 6.1. Thus it will suffice to show that $\delta(\mathcal{H}_n, \mathcal{B}_n) = \Omega(n)$.

Claim. *For each $B \in \text{SF}_0(G)$ and every $x \in G \setminus B$,*

$$\left| \left\{ \{y, z\} \in \binom{B}{2} : x = y + z \right\} \right| \geq \frac{n}{2q} - \frac{1}{2}.$$

Proof of claim. Let $B \in \text{SF}_0(G)$ and let $x \in G \setminus B$. By Theorem 6.2, there exists a subgroup H of G of index q such that B is a union of cosets of H and $B \cup (B + B) = G$. It follows that $x = y + z$ for some $y, z \in B$, and that $y + h, z - h \in B$ for every $h \in H$. Thus,

$$\{y + h, z - h\} \in C(x) := \left\{ \{y, z\} \in \binom{B}{2} : x = y + z \right\}$$

whenever $h \in H$ and $y + h \neq z - h$. Moreover, since $|G|$ is odd, there is at most one $h \in H$ such that $2h = z - y$, so $|C(x)| \geq (|H| - 1)/2 = n/2q - 1/2$, as required. \square

Thus the pair $(\mathcal{H}, \mathcal{B})$ satisfies the conditions of Theorem 2.2 and hence if $C(q)$ is sufficiently large and $m \geq C(q)\sqrt{n \log n}$, then almost every sum-free set of size m in G_n is contained in some $B \in \mathcal{B}_n$, as required.

Finally, let us deduce that if G is an Abelian group of Type I(q), and $m \geq C(q)\sqrt{n \log n}$, then

$$|\text{SF}(G, m)| = \frac{1}{2} \cdot (\#\{\text{elements of } G \text{ of order } q\} + o(1)) \binom{\mu(G)n}{m}.$$

⁶But not vice-versa, since \mathcal{H}_n does not contain the Schur triples in G_n of the form $(x, x, 2x)$. Thus, by bounding $I(\mathcal{H}_n, m)$ we are in fact proving a statement which is slightly stronger than Theorem 1.1.

Indeed, it suffices to observe that $|\text{SF}_0(G)| = \#\{\text{elements of } G \text{ of order } q\}/2$, by Theorem 6.2, and that each pair $B, B' \in \text{SF}_0(G)$ intersect in at most $(1 - 1/q)\mu(G)|G|$ elements. The result now follows from some easy counting. \square

7. ABELIAN GROUPS OF EVEN ORDER

In this section, we shall prove the following theorem, which implies Theorem 1.1 in the case $q = 2$. We shall use Theorem 1.3 and some ideas from Section 5, but otherwise this section is self-contained. In particular, we shall not use Proposition 6.1 and thus we give a new proof of the main theorem of [31] and [38].

Theorem 7.1. *If G is an Abelian group of order n , then*

$$|\text{SF}(G, m)| = (\#\{\text{elements of } G \text{ of order } 2\} + o(1)) \binom{n/2}{m}$$

for every $m \geq 4\sqrt{n \log n}$.

We remark that we shall prove the theorem for *all* finite Abelian groups, not just those of even order. We begin by partitioning the collection of sum-free sets into two pieces. Given an Abelian group G , let

$$\text{SF}_{\leq}^{(\delta)}(G, m) := \left\{ I \in \text{SF}(G, m) : |I \cap H| \leq \delta m \text{ for some } H \leq G \text{ with } [G : H] = 2 \right\},$$

and

$$\text{SF}_{\geq}^{(\delta)}(G, m) := \left\{ I \in \text{SF}(G, m) : |I \cap H| \geq \delta m \text{ for every } H \leq G \text{ with } [G : H] = 2 \right\}.$$

Note that if $|G|$ is odd then $\text{SF}_{\leq}^{(\delta)}(G, m)$ is empty. We shall prove the following proposition using the method of Section 5.

Proposition 7.2. *Let G be an Abelian group of order n , and let $\delta > 0$ be sufficiently small. Then*

$$|\text{SF}_{\leq}^{(\delta)}(G, m)| \leq (\#\{\text{elements of } G \text{ of order } 2\} + o(1)) \cdot \binom{n/2}{m}$$

for every $m \geq 4\sqrt{n \log n}$.

For sets in $\text{SF}_{\geq}^{(\delta)}(G, m)$, i.e., far from any $H \leq G$ of index 2, we shall prove the following stronger bound using Theorem 1.3.

Proposition 7.3. *Let G be an Abelian group of order n , and let $\delta > 0$. If $\varepsilon = \varepsilon(\delta) > 0$ is sufficiently small and $C = C(\varepsilon, \delta)$ is sufficiently large, then*

$$|\text{SF}_{\geq}^{(\delta)}(G, m)| \leq 2^{-\varepsilon m} \binom{n/2}{m}$$

for every $m \geq C\sqrt{n}$ and every sufficiently large $n \in \mathbb{N}$.

We begin by proving Proposition 7.2. Given $S \subseteq G$, define \mathcal{G}_S^* to be the graph with vertex set $G \setminus S$ and edge set $\{xy : x - y \in S\}$, and note that if I is a sum-free set in G with $S \subseteq I$, then $I \setminus S$ is an independent set in \mathcal{G}_S^* .

Proof of Proposition 7.2. Let G be an Abelian group of even order n , let H be a subgroup of G of index 2, and let $S \subseteq H$ satisfy $|S| = k \leq \delta m$. Set $\gamma = 1/65$. We claim that for every $m \geq 4\sqrt{n \log n}$, there are at most

$$\left(n^{-4k} + e^{-\gamma m}\right) \binom{n/2}{m-k} \quad (8)$$

sum-free subsets I of G of order m with $I \cap H = S$.

Observe first that the graph $\mathcal{G}_S^*[G \setminus H]$ is d -regular, where $d = |S \cup (-S)| \in [k, 2k]$. Indeed, for each $x \in G \setminus H$, let

$$N(x) = \{y \in G \setminus H : x - y \in S \text{ or } y - x \in S\}.$$

Since $S \subseteq H$, it follows that $x - S$ and $x + S$ are in $G \setminus H$, and hence $|N(x)| = |S \cup (-S)|$, as claimed. Since $|S| = k$, we have $k \leq d \leq 2k$.

Now, by the Hypergeometric Janson Inequality, Lemma 5.3, there are at most

$$C \cdot \max \left\{ e^{-km^2/4n}, e^{-m/64} \right\} \binom{n/2}{m-k} \leq \left(n^{-4k} + e^{-\gamma m}\right) \binom{n/2}{m-k}$$

independent sets of size $m - k$ in $\mathcal{G}_S^*[G \setminus H]$. This follows because $k \leq \delta m$, so

$$\mu \geq \left(\frac{kn}{4}\right) \left(\frac{(m-k)^2}{(n/2)^2}\right) \geq \frac{km^2}{2n} \quad \text{and} \quad \Delta \leq \binom{2k}{2} \frac{n}{2} \left(\frac{(m-k)^3}{(n/2)^3}\right) \leq \frac{8k^2 m^3}{n^2},$$

and $m \geq 4\sqrt{n \log n}$. Since every sum-free subset $I \subseteq G$ with $I \cap H = S$ induces an independent set in $\mathcal{G}_S^*[G \setminus H]$, then (8) follows.

Finally, summing (8) over subgroups H and sets S , we obtain

$$\begin{aligned} |\text{SF}_{\leq}^{(\delta)}(G, m)| &\leq \#\{H \leq G : [G : H] = 2\} \sum_{k=0}^{\delta m} \binom{n}{k} \left(n^{-4k} + e^{-\gamma m}\right) \binom{n/2}{m-k} \\ &\leq \left(\#\{\text{elements of } G \text{ of order } 2\} + O\left(\frac{1}{n^2}\right)\right) \binom{n/2}{m} \end{aligned} \quad (9)$$

for every $m \geq 4\sqrt{n \log n}$. To see the last inequality, observe that the number of subgroups H of index 2 in G is exactly the number of elements of G of order 2 and consider three cases as in the proof of Proposition 5.1. Indeed, if $n^{-4k} \geq e^{-\gamma m}$ or $m \geq n/4$, then each summand in (9) is at most $(n^{-2k} + e^{-\gamma m/2}) \binom{n/2}{m}$ by the trivial bound $\binom{n/2}{m-k} \leq \binom{n}{k} \binom{n/2}{m}$. But if $n^{-4k} \leq e^{-\gamma m}$ and $m \leq n/4$, then by (6),

$$\binom{n}{k} \binom{n/2}{m-k} \leq \left(\frac{en}{k}\right)^k \left(\frac{2m}{n-2m}\right)^k \binom{n/2}{m} \leq \left(\frac{4em}{k}\right)^k \binom{n/2}{m} \leq e^{O(\sqrt{\delta m})} \binom{n/2}{m}$$

since $k \leq \delta m$. Thus, if $\delta > 0$ is chosen small enough, then each summand in (9) is at most $e^{-\gamma m/2} \binom{n/2}{m}$, as required. \square

We next turn to the proof of Proposition 7.3. We shall divide into two cases: either the smallest eigenvalue $\lambda(I)$ of I (see below) is at most $(\delta - 1)|I|$, in which case we shall use some basic facts about characters of finite Abelian groups to show that there are few such

sets; or $\lambda(I)$ is larger, in which case we shall find a small subset $S \subseteq I$ such that \mathcal{G}_S^* is a d -regular graph with smallest eigenvalue satisfying $\lambda > (\delta/4 - 1)d$, and apply Theorem 1.3. We begin with the following key definition.

Definition 7.4 (The smallest eigenvalue of S). Given a finite Abelian group G , and a subset $0 \notin S \subseteq G$, let

$$\lambda(S) := \min \{ \operatorname{Re}(\lambda) : A(S)v = \lambda v \text{ for some } v \neq \mathbf{0} \},$$

where $\operatorname{Re}(\lambda)$ is the real part of the complex number λ , $A(S)$ is the adjacency matrix of the directed Cayley graph on G , i.e., the $(0, 1)$ -matrix with $A(x, y) = 1$ iff $y - x \in S$, and $\mathbf{0}$ is the zero vector.

Next, we recall some well-known simple properties of characters of finite Abelian groups.

7.1. Characters of finite Abelian groups. Recall that a *character* of a group G is a homomorphism from G into the multiplicative group of non-zero complex numbers, i.e., a function $\chi: G \rightarrow \mathbb{C}^*$ such that $\chi(a + b) = \chi(a)\chi(b)$ for all $a, b \in G$. A character χ is called trivial if $\chi(x) = 1$ for all $x \in G$; we will denote the trivial character by χ_T . The set of all characters of G is denoted by \hat{G} . The following well-known statement establishes a relation between the smallest eigenvalue of the matrix $A(S)$ and the characters of G . For completeness, we shall sketch a proof.

Lemma 7.5. *For every $0 \notin S \subseteq G$,*

$$\lambda(S) = \min \left\{ \operatorname{Re} \left(\sum_{s \in S} \chi(s) \right) : \chi \in \hat{G} \right\} \geq -|S|.$$

Proof of Lemma 7.5. Let us start by breaking up the adjacency matrix $A(S)$ into $|S|$ pieces as follows:

$$A(S) = \sum_{s \in S} A_s, \quad \text{where} \quad A_s(x, y) = \begin{cases} 1 & \text{if } y - x = s \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for every $s, x \in G$ and $\chi \in \hat{G}$,

$$(A_s \chi)(x) = \chi(x + s) = \chi(x)\chi(s) = (\chi(s)\chi)(x),$$

and so every $\chi \in \hat{G}$ is an eigenvector of each A_s with $\chi(s)$ being the corresponding eigenvalue. Hence, every $\chi \in \hat{G}$ is an eigenvector of $A(S)$, with eigenvalue $\sum_{s \in S} \chi(s)$. Since the characters of G form an orthogonal basis of \mathbb{C}^G , it follows that the set of eigenvalues of $A(S)$ is exactly

$$\left\{ \sum_{s \in S} \chi(s) : \chi \in \hat{G} \right\}.$$

The inequality $\lambda(S) \geq -|S|$ follows since $|\chi(s)| = 1$ for every $s \in S$ and $\chi \in \hat{G}$. □

Let us note for future reference the following fact from the proof above.

Fact 1. *For every $0 \notin S \subseteq G$, the characters of G form a basis of eigenvectors of the matrix $A(S)$.*

7.2. Sum-free sets with small smallest eigenvalue. In this section, we shall prove the following lemma.

Lemma 7.6. *Let G be an Abelian group of order n , and let $\delta > 0$ be sufficiently small. If $\varepsilon = \varepsilon(\delta) > 0$ is sufficiently small and $C = C(\varepsilon, \delta)$ is sufficiently large, then*

$$\left| \left\{ I \in \text{SF}_{\geq}^{(\delta)}(G, m) : \lambda(I) \leq (\delta - 1)|I| \right\} \right| \leq 2^{-\varepsilon m} \binom{n/2}{m}.$$

for every $m \geq C\sqrt{n}$ and every sufficiently large $n \in \mathbb{N}$.

Proof. For each character χ of G and each $A \subseteq G$, define $\lambda(A, \chi) = \text{Re} \left(\sum_{x \in A} \chi(x) \right)$. We shall bound the number of $I \in \text{SF}_{\geq}^{(\delta)}(G, m)$ such that $\lambda(I, \chi) \leq (\delta - 1)|I|$ by

$$\frac{2^{-\varepsilon m}}{n} \binom{n/2}{m}.$$

The desired bound will follow since $\lambda(I) = \min_{\chi \in \hat{G}} \lambda(I, \chi)$ and there are at most $|G|$ characters of G . We split into two cases, depending on the number of different values taken by χ .

Case 1. $|\text{range}(\chi)| = 2$.

Since χ is a group homomorphism, it corresponds to a subgroup H of G of index 2, namely, $H = \chi^{-1}(1)$. Since $|I \cap H| \geq \delta n$ for every such H , we have

$$\lambda(I, \chi) = \text{Re} \left(\sum_{x \in I} \chi(x) \right) = |I \cap H| - |I \setminus H| \geq (2\delta - 1)|I|$$

and hence in this case, there are no $I \in \text{SF}_{\geq}^{(\delta)}(G, m)$ such that $\lambda(I, \chi) \leq (\delta - 1)|I|$.

Case 2. $|\text{range}(\chi)| \geq 3$.

Let $k = |\text{range}(\chi)|$ and recall that $\text{range}(\chi) = U_k$, where U_k is the multiplicative group of k th roots of unity. Observe that $|\chi^{-1}(\xi)| = n/k$ for every $\xi \in U_k$, and consider the open arc \mathcal{C} of length $\pi/3$ centred at -1 on the complex unit circle. Set $K := \chi^{-1}(\mathcal{C})$, and note that $|\mathcal{C} \cap U_k| \leq \lceil k/6 \rceil \leq k/3$ and hence $|K| \leq n/3$.

Let $c > 0$ and observe that the number of sets I such that $|K \cap I| \geq (1 - c)|I|$ is at most

$$\sum_{\ell=0}^{cm} \binom{n - |K|}{\ell} \binom{|K|}{m - \ell} \leq n \binom{n/3}{(1 - c)m} \binom{2n/3}{cm} \leq \left(\frac{2}{3} + c' \right)^m \binom{n/2}{m},$$

where $c'(c) \rightarrow 0$ as $c \rightarrow 0$.

Now suppose that $|K \cap I| \leq (1 - c)|I|$. We claim that, if $\delta > 0$ is sufficiently small, then

$$\lambda(I, \chi) = \sum_{x \in I} \text{Re}(\chi(x)) \geq \left(c - 1 - c \cdot \cos(5\pi/6) \right) \cdot |I| > (\delta - 1)|I|. \quad (10)$$

To see this, note that, by our assumption, $\chi(x)$ can lie within the open arc of length $\pi/3$ centred at -1 for at most $(1-c)|I|$ elements $x \in I$. Since each of the others contribute at least $\cos(5\pi/6)$ in the direction of v , (10) follows. The proof is now complete. \square

7.3. Sum-free sets with large smallest eigenvalue. We shall prove the following statement using Theorem 1.3. Together with Lemma 7.6, this gives Proposition 7.3 and hence Theorem 7.1.

Lemma 7.7. *Let G be an Abelian group of order n , and let $\delta > 0$. If $\varepsilon = \varepsilon(\delta) > 0$ is sufficiently small and $C = C(\varepsilon, \delta)$ is sufficiently large, then*

$$\left| \left\{ I \in \text{SF}_{\geq}^{(\delta)}(G, m) : \lambda(I) \geq (\delta - 1)|I| \right\} \right| \leq 2^{-\varepsilon m} \binom{n/2}{m}$$

for every $m \geq C\sqrt{n}$ and every sufficiently large $n \in \mathbb{N}$.

The idea of the proof is as follows: we choose a set $S \subseteq I$ of size εm and observe that, since I is sum-free, $I \setminus S$ is an independent set in \mathcal{G}_S^* . The key point is that, for some such S , our bound on $\lambda(I)$ implies the existence of a non-trivial bound on $\lambda(\mathcal{G}_S^*)$. Combined with Theorem 1.3, this implies that there are only very few choices for $I \setminus S$, and hence for I itself.

The first step is the following lemma, which shows that our bound on $\lambda(I)$ allows us to find a small set S such that $\lambda(S)/|S|$ is also bounded away from minus one.

Lemma 7.8. *Let $\varepsilon, \delta > 0$ and $m, n \in \mathbb{N}$, and suppose that $m/\log n$ is sufficiently large. If G is an Abelian group of order n , and $I \in \text{SF}_{\geq}^{(\delta)}(G, m)$ satisfies $\lambda(I) \geq (\delta - 1)|I|$, then there exists a set $S \subseteq I$ of size εm such that*

$$\lambda(S) \geq \left(\frac{\delta}{2} - 1 \right) |S|. \quad (11)$$

Proof. Recall the definition of $\lambda(A, \chi)$ from the proof of Lemma 7.6. Since $\lambda(I) \geq (\delta - 1)|I|$, it follows from Lemma 7.5 that $\lambda(I, \chi) \geq (\delta - 1)|I|$ for every $\chi \in \hat{G}$. Choose a subset $S \subseteq I$ of size εm uniformly at random; we claim that $\lambda(S, \chi)$ is tightly concentrated around the mean, i.e., around $\varepsilon\lambda(I, \chi)$. Indeed, by Chernoff's inequality, we have

$$\mathbb{P}\left(\lambda(S, \chi) \leq (\delta/2 - 1)\varepsilon|I|\right) \leq e^{-\Omega(m)} \leq \frac{1}{2n},$$

where the implicit constant depends on ε and δ , and the final inequality holds because $m/\log n$ is sufficiently large. There are exactly n characters in \hat{G} , and so, by the union bound, the probability that S does not satisfy (11) is at most $1/2$. Thus there exists a set S as claimed. \square

Next, we show that this bound on $\lambda(S)$ implies a similar bound on $\lambda(\mathcal{G}_S^*)$. Recall that the adjacency matrix of \mathcal{G}_S^* is $A(S \cup (-S))$, and hence

$$\lambda(\mathcal{G}_S^*) = \lambda(S \cup (-S)).$$

We shall use the following lemma, which bounds $\lambda(\mathcal{G}_S^*)$ in terms of $\lambda(S)$.

Lemma 7.9. *Let $0 \notin S \subseteq G$ and $\delta > 0$. If $\lambda(S) \geq (\delta - 1)|S|$, then*

$$\lambda(\mathcal{G}_S^*) \geq \left(\frac{\delta}{2} - 1\right) |S \cup (-S)|.$$

Proof of Lemma 7.9. By Fact 1, the characters of G are a basis of eigenvectors of both $A(S)$ and $A((-S) \setminus S)$. Thus, by Lemma 7.5,

$$\begin{aligned} \lambda(\mathcal{G}_S^*) &= \lambda(S \cup (-S)) \geq \lambda(S) + \lambda((-S) \setminus S) \\ &\geq (\delta - 1)|S| - |(-S) \setminus S| \geq \left(\frac{\delta}{2} - 1\right) |S \cup (-S)|, \end{aligned}$$

as required. The last inequality follows from the fact that $|(-S) \setminus S| \leq |(-S)| = |S|$. \square

We can now complete the proof of Lemma 7.7.

Proof of Lemma 7.7. Let $I \in \text{SF}_{\geq}^{(\delta)}(G, m)$ and suppose that $\lambda(I) \geq (\delta - 1)|I|$. By Lemmas 7.8 and 7.9, there exists a set $S \subseteq I$ with $|S| = \varepsilon m$, such that

$$\lambda(\mathcal{G}_S^*) \geq \left(\frac{\delta}{4} - 1\right) |S \cup (-S)|.$$

Since I is sum-free, $I \setminus S$ is an independent set in \mathcal{G}_S^* . We claim that \mathcal{G}_S^* satisfies the conditions of Theorem 1.3. Indeed, \mathcal{G}_S^* is a d_S -regular graph on n vertices, where $d_S = |S \cup (-S)|$, and

$$|I \setminus S| = (1 - \varepsilon)m \geq \frac{C(\varepsilon, \delta)n}{\varepsilon m} \geq \frac{C(\varepsilon, \delta)n}{d_S},$$

since $m \geq C\sqrt{n}$. Note also that

$$\frac{|\lambda(\mathcal{G}_S^*)|}{d_S + |\lambda(\mathcal{G}_S^*)|} \leq \frac{1 - (\delta/4)}{2 - (\delta/4)} \leq \frac{1}{2} - \frac{\delta}{16}.$$

Hence, by Theorem 1.3,

$$I(\mathcal{G}_S^*, (1 - \varepsilon)m) \leq \binom{(1/2 - \delta/20)n}{(1 - \varepsilon)m},$$

and so

$$\left| \left\{ I \in \text{SF}_{\geq}^{(\delta)}(G, m) : \lambda(I) \leq (1 - \delta)|I| \right\} \right| \leq \binom{n}{\varepsilon m} \binom{(1/2 - \delta/20)n}{(1 - \varepsilon)m} \leq 2^{-\varepsilon m} \binom{n/2}{m}$$

if $\varepsilon = \varepsilon(\delta) > 0$ is sufficiently small. This proves the lemma. \square

Finally, note that Lemmas 7.6 and 7.7 imply Proposition 7.3.

7.4. Proof of Theorem 7.1. First, observe that if $|G|$ is even, then the claimed lower bound on $|\text{SF}(G, m)|$ is a straightforward consequence of the fact that $|\text{SF}_0(G)|$ is equal to the number of elements of G of order 2, and that each pair of distinct $B, B' \in \text{SF}_0(G)$ intersects in $|G|/4$ elements (see Theorem 6.2). If $|G|$ is odd, then Theorem 7.1 only gives an upper bound on $|\text{SF}(G, m)|$.

For the upper bound, observe that by Propositions 7.2 and 7.3, we have

$$\begin{aligned} |\text{SF}(G, m)| &\leq |\text{SF}_{\leq}^{(\delta)}(G, m)| + |\text{SF}_{\geq}^{(\delta)}(G, m)| \\ &\leq (\#\{\text{elements of } G \text{ of order } 2\} + o(1)) \binom{n/2}{m} + 2^{-\varepsilon m} \binom{n/2}{m}, \end{aligned}$$

for every $m \geq 4\sqrt{n \log n}$, as required.

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