Almost all C_4-free graphs have fewer than $(1-\varepsilon)\,\mathrm{ex}(n, C_4)$ edges

József Balogh[*], Wojciech Samotij[†]

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Abstract

A graph is called *H-free* if it contains no copy of H. Let $\mathrm{ex}(n, H)$ denote the Turán number for H, i.e., the maximum number of edges that an n-vertex H-free graph may have. An old result of Kleitman and Winston states that there are $2^{O(\mathrm{ex}(n,C_4))}$ C_4-free graphs on n vertices. Füredi showed that almost all C_4-free graphs of order n have at least $c\,\mathrm{ex}(n, C_4)$ edges for some positive constant c. We prove that there is a positive constant ε such that almost all C_4-free graphs have at most $(1-\varepsilon)\,\mathrm{ex}(n, C_4)$ edges. This resolves a conjecture of Balogh, Bollobás, and Simonovits for the 4-cycle.

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1 Introduction

Let H be a fixed graph. A graph is called *H-free* if it does not contain a copy of H as a (not necessarily induced) subgraph. We will denote the family of all labeled H-free graphs on the vertex set $\{1, \ldots, n\}$ by $\mathcal{F}_n(H)$. Let $\mathrm{ex}(n, H)$ be the Turán number for H, i.e., the maximum number of edges that an H-free graph on n vertices may have. The problem of estimating $|\mathcal{F}_n(H)|$ has been an object of intense study. Erdős conjectured (see [7]) that if H contains a cycle, then $|\mathcal{F}_n(H)| = 2^{(1+o(1))\,\mathrm{ex}(n,H)}$. The conjecture was resolved in the affirmative by Erdős, Frankl, and Rödl [9] under the additional assumption that $\chi(H) \geq 3$. More precise estimates and structural results were obtained by Balogh, Bollobás, and Simonovits [4]. The case of bipartite H is still wide open. For some partial results, see [14] and [15].

The following statement can be proved using the methods from [3] and [4]. Let H be a fixed non-bipartite graph. Then for every positive ε, almost all H-free graphs of order

[*]Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA; and Department of Mathematics, University of Illinois, 1409 W Green Street, Urbana, IL 61801, USA. E-mail address: jobal@math.uiuc.edu. This material is based upon work supported by NSF CAREER Grant DMS-0745185 and DMS-0600303, UIUC Campus Research Board Grants 09072 and 08086, and OTKA Grant K76099.

[†]Department of Mathematics, University of Illinois, 1409 W Green Street, Urbana, IL 61801, USA. Research supported in part by the Trijtzinsky Fellowship and the James D. Hogan Memorial Scholarship Fund. E-mail address: samotij2@illinois.edu.

n have at least $(1/2 - \varepsilon)\,\mathrm{ex}(n, H)$ and at most $(1/2 + \varepsilon)\,\mathrm{ex}(n, H)$ edges. Although it is not unreasonable to claim that a similar concentration occurs also when $\chi(H) = 2$, the case when H is bipartite has proved to be much harder to handle, and not much is known about the distribution of sizes of n-vertex H-free graphs. The main reasons for this difficulty might be that if $\chi(H) = 2$, then all H-free graphs are sparse, i.e., $\mathrm{ex}(n, H) = o(n^2)$, and no Erdős-Simonovits-type stability results are known. On the other hand, there are natural "sparse" settings where a similar concentration around the half does not occur. For example, recently Gerke and McDiarmid [12] proved that the expected number of edges in a random n-vertex planar graph is at least $(13/7 + o(1))n$, which is more than half the number of edges in a maximal planar graph on n vertices, $(3n - 6)/2$. Still, one should expect that the number of edges in a "typical" H-free graph is at least bounded away from the extremal values, $\mathrm{ex}(n, H)$ and 0. Balogh, Bollobás, and Simonovits [2] formalized this intuition in the following conjecture.

Conjecture 1. *For every bipartite graph H that contains a cycle, there is a positive constant c such that almost all H-free graphs on n vertices have at least $c\,\mathrm{ex}(n, H)$ and at most $(1 - c)\,\mathrm{ex}(n, H)$ edges.*

The lower bound is known only for $H = C_4$ (see [11]), C_6 (the argument from [11] can be quite easily extended using methods from [14]), and $K_{s,t}$ for all pairs (s, t) with $2 \leq s \leq t$ such that $s \leq 3$ or $t > (s - 1)!$ (see [5, 6]), whereas the upper bound has not been proved yet for any bipartite H. In this paper we estimate the upper bound in the case $H = C_4$, consequently resolving Conjecture 1 for the 4-cycle.

For a fixed real number $\varepsilon \in (0, 1)$, let $\mathcal{F}_n^\varepsilon(H)$ denote the subfamily of $\mathcal{F}_n(H)$ consisting only of graphs that have at least $(1 - \varepsilon)\,\mathrm{ex}(n, H)$ edges. Our main result is the following.

Theorem 2. *There exist a positive constant ε such that*

$$\lim_{n\to\infty} \frac{|\mathcal{F}_n^\varepsilon(C_4)|}{|\mathcal{F}_n(C_4)|} = 0. \tag{1}$$

We would like to remark that our result seems to be closely related to another classical Turán-type problem proposed by Erdős and Spencer (see [8]). Given two graphs G and H, let $\mathrm{ex}(G, H)$ be the maximum number of edges in an H-free subgraph of G. Trivially, asymptotically almost surely we have

$$(1/2 + o(1))\,\mathrm{ex}(n, C_4) \leq \mathrm{ex}(G(n, 1/2), C_4) \leq \mathrm{ex}(n, C_4), \tag{2}$$

but no better estimates are known. Since actually the proof of Theorem 2 shows that for some positive ε,

$$\left|\mathcal{F}_n^\varepsilon(C_4)\right| = 2^{-\Omega(n^{3/2})} \cdot \left|\mathcal{F}_n(C_4)\right|, \tag{3}$$

it follows that if Erdős' conjecture was true for C_4, i.e., $|\mathcal{F}_n(C_4)| = 2^{(1+o(1))\,\mathrm{ex}(n,C_4)}$, then (3) would imply that for some positive constant ε', $|\mathcal{F}_n^{\varepsilon'}(C_4)| \leq 2^{(1-2\varepsilon')\,\mathrm{ex}(n,C_4)}$, provided that n is sufficiently large. Hence, by the union bound, the probability of the appearance of a subgraph from $\mathcal{F}_n^{\varepsilon'}(C_4)$ in $G(n, 1/2)$ would be at most

$$2^{(1-2\varepsilon')\,\mathrm{ex}(n,C_4)} \cdot (1/2)^{(1-\varepsilon')\,\mathrm{ex}(n,C_4)},$$

which is clearly $o(1)$. This would improve the upper bound in (2) to $(1 - \varepsilon') \cdot \mathrm{ex}(n, C_4)$.

Remark. A straightforward modification of the proof of Theorem 2, combined with the argument from [6], resolves Conjecture 1 for all $K_{2,t}$. More specifically, the tools developed in [6] allow one to re-prove a version of Lemma 5 with $K_{2,2}$ replaced by $K_{2,t}$. Then, a virtually identical computation to the one in the proof of Theorem 2 implies that for some small positive ε, $\left|\mathcal{F}_n^\varepsilon(K_{2,t})\right| = 2^{-\Omega(n^{3/2})} \cdot \left|\mathcal{F}_n(K_{2,t})\right|$. We omit the details.

The remainder of this paper is organized as follows. In Section 2, we introduce some notation and prove a few technical lemmas. The proof of the main result, Theorem 2, is given in Section 3.

2 Notation and technical lemmas

Given an arbitrary set A, we will denote its power set, i.e., the set of all subsets of A, by $\mathcal{P}(A)$. For a positive integer k, the set of all functions from A to the set $\{0, \dots, k-1\}$ will be abbreviated by k^A. Given a graph G and a subset of its vertices $A \subseteq V(G)$, the subgraph of G induced on $V(G) - A$ will be denoted by $G - A$. The number of edges in G is $e(G)$. Given a vertex $v \in V(G)$, its degree in G is denoted by $\deg_G(v)$ or simply $\deg(v)$ whenever G is clear from the context. The minimum degree of G is $\delta(G) = \min_{v \in V(G)} \deg_G(v)$. Finally, for any $A \subseteq V(G)$ and $v \in V(G)$, we let $\deg(v, A)$ denote the number of neighbors of v in A.

First, let us recall that $\mathrm{ex}(n, C_4) = (1/2 + o(1)) \cdot n^{3/2}$ (see, e.g., [10]). Suppose that G is a C_4-free graph on n vertices which has at least $(1-\varepsilon)\,\mathrm{ex}(n, C_4)$ edges. If the minimum degree of G is smaller than $3n^{1/2}/4$, then by removing a vertex of smallest degree from G, we will increase the *relative edge density* (the ratio $e(G)/v(G)^{3/2}$) of the resulting graph. Since removing vertices cannot create a copy of C_4 in our graph, the relative edge density after any number of such removals will not exceed $1/2 + o(1)$, and hence we cannot continue removing low degree vertices indefinitely. It follows that after deleting a relatively small set of low degree vertices from our graph G, we will obtain a graph whose minimum degree is at least about $3n^{1/2}/4$. We formalize the above discussion in Lemma 3.

Lemma 3. *For every positive constant α, there is a positive ε such that the following holds. Let $G \in \mathcal{F}_n^\varepsilon(C_4)$, where n is large enough. Then there is a set $X \subseteq V(G)$ with $|X| \le \alpha n$ such that $\delta(G - X) \ge (3/4 - \alpha)n^{1/2}$.*

Proof. Let $\varepsilon = \alpha^2/3$. Define an ordering of the vertices of G as follows. Let v_1 be a vertex of minimum degree in G. Provided that $v_1, \dots, v_i$ have already been chosen, we let v_{i+1} be a vertex of minimum degree in $G - \{v_1, \dots, v_i\}$. Since every subgraph of G is C_4-free, and $\mathrm{ex}(n, C_4) \le (2n^{3/2} + n)/4$ (see, e.g., [10]), the function f defined by

$$f(k) = \frac{2(n-k)^{3/2} + (n-k)}{4} - e(G - \{v_1, \dots, v_k\})$$

is non-negative for all k.

Let k_0 be the smallest number such that $\delta(G - \{v_1, \dots, v_k\}) \ge (3/4 - \alpha)n^{1/2}$, or $k_0 = n$ in case such a number does not exist. If $k_0 \le \alpha n$, then $X = \{v_1, \dots, v_{k_0}\}$ is good for our purposes. Otherwise for all k with $k \le \alpha n$, we have

$$\begin{aligned} f(k) - f(k-1) &= \frac{(n-k)^{3/2} - (n-k+1)^{3/2}}{2} - \frac{1}{4} + \deg(v_k, V(G) - \{v_1, \ldots, v_{k-1}\}) \\ &\le -\frac{3}{4}(n-k)^{1/2} + \left(\frac{3}{4} - \alpha\right) n^{1/2} \\ &\le -\frac{3}{4}\left(1-\alpha\right) n^{1/2} + \left(\frac{3}{4} - \alpha\right) n^{1/2} \le -\frac{\alpha}{4} n^{1/2}, \end{aligned}$$

where the first inequality follows from the fact that $(a+1)^{3/2} - a^{3/2} \ge 3a^{1/2}/2$ for all non-negative a, which can be proved using elementary calculus. Since f is non-negative, it follows that

$$-f(0) \le f(\alpha n) - f(0) = \sum_{k=1}^{\alpha n} \big(f(k) - f(k-1)\big) \le \alpha n \cdot \left(-\frac{\alpha}{4} n^{1/2}\right) = -\frac{\alpha^2}{4} n^{3/2},$$

and hence

$$e(G) = \frac{2n^{3/2} + n}{4} - f(0) \le \frac{2-\alpha^2}{4} n^{3/2} + \frac{n}{4} < (1 - \frac{3}{2}\varepsilon + o(1)) \cdot \mathrm{ex}(n, C_4),$$

which is a contradiction. □

The next lemma formalizes the following intuition. A random partition of the vertex set of a graph into two sets of sizes a and b splits the neighborhood of each vertex roughly in proportion $a : b$.

Lemma 4. *For all positive reals $\beta, \rho \in (0,1)$, there exists an n_0 such that the following holds. Let $n \ge n_0$ and let G be an n-vertex graph with $\delta(G) \ge \log^2 n$. Then there exists an $A \subseteq V(G)$ with $|A| \in \big((1-\rho)\beta n, (1+\rho)\beta n\big)$ such that for all $v \in V(G)$,*

$$(1-\rho)\beta \cdot \deg(v) \le \deg(v, A) \le (1+\rho)\beta \cdot \deg(v).$$

Proof. Let us pick, randomly and independently, each vertex of G with probability β. Let A be the set of selected vertices. By the Chernoff bound (see, e.g., Corollary A.1.14 in [1]),

$$\Pr\Big[\big||A| - \beta n\big| \ge \rho\beta n\Big] \le 2e^{-cn},$$

where c is some positive constant depending only on β and ρ. Similarly, for every vertex v we have

$$\Pr\Big[\big|\deg(v, A) - \beta\deg(v)\big| \ge \rho\beta\deg(v)\Big] \le 2e^{-c\deg(v)} \le 2e^{-c\delta(G)} \le 2e^{-c\log^2 n}.$$

By the union bound, the set A has all the required properties with probability tending to 1 as n goes to infinity. Hence, provided that n is large enough, there exists a set A satisfying all the required conditions. □

The key ingredient in the proof of Theorem 2 is Lemma 5, which builds on a slick counting argument of Kleitman and Winston from [15], which was later used in many papers, including [13, 14, 16].

Lemma 5. *Let G be an n-vertex C_4-free graph with $\delta(G) \geq \delta \geq n^{1/2}/2$. Suppose we want to add to G a new vertex v of degree d, where $d \geq \frac{1}{2}n^{1/2}$, so that the resulting graph remains C_4-free and, moreover, we have already chosen pd neighbors of v, where $p \in [0,1]$. Then the number of ways we can select the remaining $(1-p)d$ neighbors of v is at most*

$$2^{o(\log^3 n)} \cdot \binom{n/\delta - pd}{(1-p)d}.$$

Proof. We slightly modify the argument from [15]. The key idea there was the following. We proceed in steps, adding edges between v and G one at a time and bounding the number of choices we can make. At the beginning, all n vertices of G are *eligible* to become neighbors of v. At each step we list all eligible vertices in the following way. After i vertices have already been listed, the $(i+1)$st vertex is a vertex that is connected by a path of length 2 to the greatest number of eligible vertices not yet on the list. We then locate the vertex with the smallest index (in the ordered list we have just constructed) that we want to connect to v and add the appropriate edge to our graph. All vertices preceding the chosen vertex in our list, as well as all vertices connected to it by a path of length 2, become ineligible. Since none of the discarded vertices can become a neighbor of v (otherwise the graph we are constructing would contain a C_4), using the above procedure we can create all possible neighborhoods of v that give rise to a C_4-free supergraph of G.

The key observation in [15] is that after merely $z = n\delta^{-2}\log n \leq 4\log n$ steps the set of eligible vertices, which contains all the remaining $d-z$ neighbors of v, has size at most n/δ. If z' out of the first z neighbors of v were the already chosen ones, we have to select the remaining $(1-p)d - z + z'$ vertices to complete the neighborhood of v from a set of size at most $n/\delta - pd + z'$. Hence the overall number of choices for the $(1-p)d$ new neighbors of v is bounded by

$$\sum_{z' \leq z} \binom{n}{z-z'} \binom{n/\delta - pd + z'}{(1-p)d - z + z'}. \tag{4}$$

Clearly, if $z' \leq z \leq n/2$, then $\binom{n}{z-z'} \leq \binom{n}{z} \leq n^z$ and

$$\binom{n/\delta - pd + z'}{(1-p)d - z + z'} \leq \binom{n/\delta - pd + z}{(1-p)d - z + z'} \leq n^z \cdot \binom{n/\delta - pd + z}{(1-p)d} \leq n^{2z} \cdot \binom{n/\delta - pd}{(1-p)d},$$

where the last two inequalities follow from the fact that $n/\delta - pd + z \leq n$, combined with the following easily verifiable inequalities: $\binom{a}{b-c} \leq a^c \cdot \binom{a}{b}$ for all $a \geq b \geq c \geq 0$, and $\binom{a+c}{b} \leq (a+c)^c \cdot \binom{a}{b}$ for all $a \geq b \geq 0$ and $c \geq 0$. Consequently, we can bound (4) by

$$(z+1) \cdot n^{3z} \cdot \binom{n/\delta - pd}{(1-p)d} \leq 2^{o(\log^3 n)} \cdot \binom{n/\delta - pd}{(1-p)d}.$$

□

Finally, we need the following well-known estimate (see, e.g., [17, Lemma 9.2]). Let

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

be the binary entropy function.

Lemma 6. *Let ℓ and λ satisfy $0 < \ell = \lambda n < n$. Then*

$$\frac{1}{n+1} 2^{nH(\lambda)} \leq \binom{n}{\ell} \leq 2^{nH(\lambda)}.$$

3 Proof of Theorem 2

Very vaguely, the idea of the proof can be summarized as follows. If G is a C_4-free graph with large minimum degree, then the number of ways in which we can remove a certain proportion of its edges is much larger than the number of ways we can add the same number of edges back, so that the resulting graph remains C_4-free. In other words, every $G \in \mathcal{F}_n^\varepsilon(C_4)$ has many more different subgraphs $F \in \mathcal{F}_n(C_4)$ than the number of supergraphs in $\mathcal{F}_n^\varepsilon(C_4)$ that any such F can possibly have. This implies that $|\mathcal{F}_n^\varepsilon(C_4)| = o(|\mathcal{F}_n(C_4)|)$. In what follows we will formalize the above discussion. We would like to remark that a similar technique was also used in [4].

Consider an arbitrary mapping

$$\varphi\colon \mathcal{F}_n^\varepsilon(C_4) \to \mathcal{P}\big(\mathcal{F}_n(C_4) \times 2^{[n]} \times n^{[n]}\big).$$

For a triple $T \in \mathcal{F}_n(C_4) \times 2^{[n]} \times n^{[n]}$, let

$$\psi(T) = \big\{G \in \mathcal{F}_n^\varepsilon(C_4) \colon T \in \varphi(G)\big\}.$$

Counting appearances of all triples T in the images $\varphi(G)$, where G ranges over $\mathcal{F}_n^\varepsilon(C_4)$ and T ranges over $\mathcal{F}_n(C_4) \times 2^{[n]} \times n^{[n]}$, yields

$$\sum_G |\varphi(G)| = \sum_T |\psi(T)|. \tag{5}$$

Equality (5) implies an obvious bound on the size of $\mathcal{F}_n^\varepsilon(C_4)$, namely,

$$\big|\mathcal{F}_n^\varepsilon(C_4)\big| \le (2n)^n \cdot \frac{\sup_T |\psi(T)|}{\inf_G |\varphi(G)|} \cdot \big|\mathcal{F}_n(C_4)\big|. \tag{6}$$

Now, inequality (6) combined with any $o((2n)^{-n})$ bound on $\sup_T |\psi(T)| / \inf_G |\varphi(G)|$ (for a carefully chosen φ) will imply (1).

Since the remainder of the proof gets somewhat technical, we will start by giving its short and informal outline. In Lemma 3, we have already noted that every C_4-free graph G with many edges, i.e., one with e(G) close to the extremal number $\mathrm{ex}(n, C_4)$, contains an almost spanning subgraph G_0 with minimum degree at least about $3n^{1/2}/4$. Next, using Lemma 4, we find a subset $A \subseteq V(G_0)$ of size about βn such that the minimum degree of $G' = G_0 - A$ is still at least almost $3n^{1/2}/4$ and all the vertices in A have (approximately) at least $3n^{1/2}/4$ neighbors in $V(G')$. Given such a set A, we define $\varphi(G)$ to be the set of all graphs obtained from G by deleting 10% of the cross-edges incident to each vertex in A, together with all the necessary information to identify the set A and reconstruct all relevant degrees after such a deletion. Finally, given a triple T consisting of a graph F, a set $A \subseteq V(F)$ and the list of degrees that the vertices in A had in the original C_4-free graph $G \supseteq F$, we prove, using Lemma 5, an upper bound on the number of supergraphs $G \supseteq F$ with $T \in \varphi(G)$. Combining this upper bound with a lower bound on $|\varphi(G)|$ and (6), we conclude that $\big|\mathcal{F}_n^\varepsilon(C_4)\big| \le 2^{-\Omega(n^{3/2})} \cdot \big|\mathcal{F}_n(C_4)\big|$.

Let us start by rigorously defining the mapping φ. Fix some very small constants α, β, and ρ (we will specify them later), and let ε be as in the statement of Lemma 3. Furthermore, let $p = 0.9$. Suppose that $n \ge n_0/(1-\alpha)$, where n_0 is as in the statement of Lemma 4.

Finally, fix some $G \in \mathcal{F}_n^\varepsilon(C_4)$. By Lemma 3, there is a subset $X \subseteq V(G)$ of size at most αn such that $\delta(G - X) \geq (3/4 - \alpha)n^{1/2}$. Now, by Lemma 4, we can find an $A \subseteq V(G) - X$ with

$$(1-\rho)(1-\alpha)\beta n \leq |A| \leq (1+\rho)\beta n, \tag{7}$$

such that if we let $G' = G - X - A$ and $\gamma = (1 - (1+\rho)\beta)(3/4 - \alpha)$, then

$$\delta(G') \geq (1 - (1+\rho)\beta) \cdot \delta(G - X) \geq \gamma n^{1/2}, \tag{8}$$

and for every vertex $v \in A$,

$$\deg(v, V(G')) \geq (1 - (1+\rho)\beta) \cdot \deg_{G-X}(v) \geq \gamma n^{1/2}. \tag{9}$$

We define $\varphi(G)$ to be the set of all triples $(F, X \cup A, f)$, where

$$f(v) = \begin{cases} \deg(v, V(G')) & \text{if } v \in X \cup A, \\ 0 & \text{otherwise,} \end{cases}$$

and F is any subgraph of G obtained by deleting, for each vertex $v \in X \cup A$, a set of $(1-p) \cdot \deg(v, V(G'))$ edges connecting v to $V(G')$. Also, note that since G is C_4-free, no two paths of length 2 starting at some $v \in X \cup A$ can reach the same vertex, and hence $f(v) \cdot \delta(G') \leq |V(G')| \leq n$, which together with (8) implies that $f(v) \leq n^{1/2}/\gamma$.

Claim 7. *For every $G \in \mathcal{F}_n^\varepsilon(C_4)$,*

$$\big|\varphi(G)\big| \geq 2^{H(p)(1-\rho)(1-\alpha)\gamma\beta n^{3/2} - O(n \log n)}.$$

Proof. It suffices to count the number of subgraphs F appearing in the definition of $\varphi(G)$. By our bounds on the size of A and the degrees of vertices in A, this is at least

$$\begin{aligned} \prod_{v \in A} \binom{\deg(v, V(G'))}{p \deg(v, V(G'))} &\geq (n+1)^{-|A|} \cdot 2^{H(p) \sum_{v \in A} \deg(v, V(G'))} \\ &\geq (n+1)^{-(1+\rho)\beta n} \cdot 2^{H(p)(1-\rho)(1-\alpha)\gamma\beta n^{3/2}}, \end{aligned}$$

where the first inequality follows from Lemma 6, and the second inequality follows from (7) and (9). □

Let $T = (F, S, f)$ be a triple from the image of φ. The way we defined φ guarantees that the set S has size at most $(1+\rho)\beta n + \alpha n$, and $F - S$ has minimum degree at least $\gamma n^{1/2}$. By Lemmas 5 and 6, we get the following bound on the size of $\psi(T)$:

$$\begin{aligned} \big|\psi(T)\big| &\leq 2^{o(n \log^3 n)} \cdot \prod_{v \in S} \binom{(n - |S|)/(\gamma n^{1/2}) - pf(v)}{(1-p)f(v)} \\ &\leq 2^{o(n \log^3 n)} \cdot \prod_{v \in S} \binom{n^{1/2}/\gamma - pf(v)}{(1-p)f(v)} \\ &\leq 2^{o(n \log^3 n)} \cdot \prod_{v \in S} 2^{(n^{1/2}/\gamma - pf(v)) H\left(\frac{(1-p)f(v)}{n^{1/2}/\gamma - pf(v)}\right)} \\ &\leq 2^{o(n \log^3 n)} \cdot 2^{((1+\rho)\beta + \alpha)sn^{3/2}}, \end{aligned}$$

where (letting $d = f(v)n^{-1/2}$ and noting that $\gamma \le d \le \gamma^{-1}$)

$$s = \sup_d \left[(1/\gamma - pd) \cdot H\left(\frac{(1-p)d}{1/\gamma - pd} \right) \right].$$

If we set $\alpha = \rho = 10^{-10}$, $\beta = 10^{-5}$, and $p = 0.9$, then numerical computations performed in *Mathematica* show that $s \le 0.3467$, and hence

$$\left|\psi(T)\right| \le 2^{3468 \cdot 10^{-9} n^{3/2} + o(n^{3/2})}.$$

On the other hand, Claim 7 gives

$$\left|\varphi(G)\right| \ge 2^{3517 \cdot 10^{-9} n^{3/2} - o(n^{3/2})}.$$

It follows that

$$\frac{\sup_T |\psi(T)|}{\inf_G |\varphi(G)|} \le 2^{-49 \cdot 10^{-9} \cdot n^{3/2} + o(n^{3/2})},$$

and therefore, by (6),

$$\left|\mathcal{F}_n^{\varepsilon}(C_4)\right| \le 2^{-4 \cdot 10^{-8} \cdot n^{3/2}} \cdot \left|\mathcal{F}_n(C_4)\right|,$$

provided that n is large enough. □

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