Corrádi and Hajnal's theorem for sparse random graphs

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Abstract

In this paper we extend a classical theorem of Corrádi and Hajnal into the setting of sparse random graphs. We show that if $p(n) \gg (\log n/n)^{1/2}$, then asymptotically almost surely every subgraph of $G(n, p)$ with minimum degree at least $(2/3 + o(1))np$ contains a triangle packing that covers all but at most $O(p^{-2})$ vertices. Moreover, the assumption on p is optimal up to the $(\log n)^{1/2}$ factor and the presence of the set of $O(p^{-2})$ uncovered vertices is indispensable. The main ingredient in the proof, which might be of independent interest, is an embedding theorem which says that if one imposes certain natural regularity conditions on all three pairs in a balanced 3-partite graph, then this graph contains a perfect triangle packing.

1 Introduction

1.1 Triangle packings in subgraphs of random graphs

Let H be a fixed graph on h vertices, let G be a graph on n vertices. An arbitrary collection of vertex-disjoint copies of H in G is called an *H-packing* in G. A *perfect H-packing* (an *H-factor*) is an H-packing that covers all vertices of the host graph. In other words, G has an *H-factor* (contains a perfect H-packing) if n is divisible by h and G contains n/h vertex-disjoint copies of H. It has been long known that for every graph H, if the minimum degree of G is sufficiently large, then G contains an H-factor. For example, by the Dirac's Theorem on Hamiltonian cycles [11], if H is a path of length $h - 1$, then $\delta(G) \geq n/2$ guarantees that G has an H-factor. Corrádi and Hajnal [9] proved that $\delta(G) \geq 2n/3$ is sufficient to guarantee a K_3-factor and Hajnal and Szemerédi [15] showed that $\delta(G) \geq (1 - 1/k)n$ suffices to guarantee a K_k-factor for an arbitrary k. Moreover, all these results are easily seen to be best possible.

Finding a similar optimal condition on the minimum degree that guarantees an H-factor for an arbitrary graph H has turned out to be significantly harder. The first result in this direction was obtained by Alon and Yuster [2], who showed that $\delta(G) \geq (1 - 1/\chi(H))n$ implies the existence of $n/h - o(n)$ vertex-disjoint copies of H in G. Later, the same authors [4] showed that $\delta(G) \geq$

$(1 - 1/\chi(H))n + o(n)$ guarantees an H-factor. Finally, Komlós, Sárközy, and Szemerédi [25] showed that merely $\delta(G) \geq (1 - 1/\chi(H))n + c(H)$, where $c(H)$ is a (small) constant depending only on H, suffices. Moreover, it was observed in [4] that there are graphs H for which the above constant $c(H)$ cannot be omitted. Recently, Kühn and Osthus [30] replaced $\chi(H)$ in the above inequality by another parameter $\chi^*(H)$, which depends on the relative sizes of the color classes in the optimal colorings of H and satisfies $\chi(H) - 1 < \chi^*(H) \leq \chi(H)$. Furthermore, they proved that the ratio $(1 - 1/\chi^*(H))$ in the lower bound for $\delta(G)$ is optimal for every H. For further information on H-factors in graphs with large minimum degree, we refer the reader to [29, 30].

An independent direction of research concerned with H-factors has been determining the thresholds for the edge probability p for the property that the Erdős-Rényi random graph $G(n, p)$ contains an H-factor. The case $H = K_2$ was solved by Erdős and Rényi [12], who proved that $\log n/n$ is the threshold for the existence of a perfect matching in $G(n, p)$. The solution for the case when H is a path is a direct consequence of the result of Pósa [32]. Alon and Yuster [3] and, independently, Ruciński [33] determined the threshold for every H whose fractional arboricity[1] is larger than its minimum degree. Later, partial results for the case $H = K_3$ were obtained by Krivelevich [26] and Kim [19] (a related work of Krivelevich, Sudakov, and Szabó [28] studied this case when the host graph is a sparse pseudo-random regular graph). Finally, Johansson, Kahn, and Vu [18] determined the thresholds for all strictly balanced H and determined them up to a sub-polynomial factor for arbitrary H.

Much less is known about common extensions of the results of the above two types. To make it precise, we would like to know whether it is true that for sufficiently large p, a.a.s. every spanning subgraph of $G(n, p)$ with sufficiently large minimum degree has an H-factor. Questions like these can be naturally expressed in the framework of *resilience*, also called *fault tolerance*. Following Sudakov and Vu [34], we state the following definition.

Definition 1.1. Let $\mathcal{P}$ be a monotone increasing graph property. The *local resilience* of a graph G with respect to $\mathcal{P}$ is the minimum number r such that by deleting at most r edges at each vertex of G, one can obtain a graph without $\mathcal{P}$.

Using this terminology, one can restate, e.g., the aforementioned theorem of Corrádi and Hajnal [9] by saying that the local resilience of the complete graph K_n with respect to the property of having a triangle-factor is (at least) $n/3$.

Rephrasing our previous question, we would like to determine the local resilience of the random graph $G(n, p)$ with respect to the property of containing an H-factor for some fixed graph H. Sudakov and Vu [34] showed that it is $(1/2 + o(1))$ when $H = K_2$ and $p \gg \log n/n$ or when H is a path and $p \gg (\log n)^4/n$; Lee and Sudakov [31] showed that the assumption $p \gg \log n/n$ suffices also in the latter case. Recently, Huang, Lee, and Sudakov [17] addressed this problem for an arbitrary H in the case when the edge probability p is a constant.

Theorem 1.2. *Let H be a fixed graph on h vertices, let $p \in (0, 1]$, and let γ be a positive real.*

1. *If H has a vertex that is not contained in any triangle, then a.a.s. every spanning subgraph $G \subset G(n, p)$ with $\delta(G) \geq (1 - 1/\chi(H) + \gamma)np$ has a perfect H-packing, provided that n is divisible by h.*

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[1]The fractional arboricity of a graph H is the quantity $\max\left\{\frac{|E(H')|}{|V(H')| - 1}\right\}$, where the maximum is taken over all subgraphs H' of H with $|V(H')| > 1$.

2. *If every vertex of H is contained in a triangle, then a.a.s. every spanning subgraph $G \subset G(n,p)$ with $\delta(G) \geq (1 - 1/\chi(H) + \gamma)np$ contains an H-packing covering all but at most Dp^{-2} vertices of G, where D is a constant that depends only on $\chi(H)$.*

Moreover, it was shown in [17] that in the case when each vertex of H belongs to some triangle, the Dp^{-2} error term cannot be removed as a.a.s. $G(n,p)$ has a spanning subgraph with large minimum degree such that at least $\Omega(p^{-2})$ of its vertices are not contained in a triangle (and hence they are not contained in a copy of H). For other results on local resilience of random graphs with respect to the property of containing spanning or nearly spanning subgraphs, see [5, 6, 7, 10, 13, 27].

In this paper, we extend the result of Huang, Lee, and Sudakov to the sparse random graph setting in the case $H = K_3$. A rather straightforward argument using the conjecture of Kohayakawa, Łuczak, and Rödl [21, Conjecture 23], which is known to be true for triangles, shows that if $p \gg n^{-1/2}$, then a.a.s. every subgraph of $G(n,p)$ whose minimum degree exceeds $(2/3 + o(1))np$ contains a triangle-packing that covers all but at most εn vertices, where ε is an arbitrary positive constant (see Remark 2.8). Our main theorem proves that under the same assumptions, one can make the set of uncovered vertices significantly smaller. More precisely, we prove the following statement.

Theorem 1.3. *For all positive γ, there exist constants C and D such that if $p \geq C(\log n/n)^{1/2}$, then a.a.s. every subgraph $G \subset G(n,p)$ with $\delta(G) \geq (2/3 + \gamma)np$ contains a triangle packing that covers all but at most Dp^{-2} vertices.*

Clearly, the ratio $2/3$ in the statement of Theorem 1.3 is best possible as for every positive γ, a.a.s. $G(n,p)$ has a subgraph G with $\delta(G) \geq (2/3 - \gamma)np$ whose largest triangle packing covers no more than $(1 - \gamma)n$ vertices (e.g., we may let G be the intersection of $G(n,p)$ with the complete 3-partite graph with color classes of sizes $(1 + \gamma)n/3$, $n/3$, and $(1 - \gamma)n/3$). Furthermore, even though it was proved in [18] that $p \gg n^{-2/3}(\log n)^{1/3}$ guarantees that $G(n,p)$ a.a.s. has a triangle-factor, the lower bound on p in Theorem 1.3 cannot be relaxed by more than the $(\log n)^{1/2}$ factor as if $p \ll n^{-1/2}$, then a.a.s. one can remove all triangles from $G(n,p)$ by deleting only $o(np)$ edges incident to every vertex. Finally, the presence of the exceptional set of Dp^{-2} is indispensable, see Proposition 4.6 and [17, Proposition 6.3].

1.2   Embedding theorem for sparse regular triples

One of the main ingredients in the proof of Theorem 1.3 is an embedding theorem for large triangle packings in sparse regular triples. Before we state this result (Theorem 1.4 below), we recall a few basic definitions and briefly summarize what is known about embedding large graphs into regular triples.

Let G be a graph on a vertex set V. Given a pair of disjoint subsets $V_1, V_2 \subset V$, let $e(V_1, V_2)$ denote the number of edges of G with one endpoint in V_1 and the other endpoint in V_2, and let the *density* $d(V_1, V_2)$ of the pair (V_1, V_2) be the quantity $e(V_1, V_2)/(|V_1||V_2|)$. The pair (V_1, V_2) is called (ε, p)-*regular* if for all $V_1' \subset V_1$ and $V_2' \subset V_2$ with $|V_1'| \geq \varepsilon|V_1|$ and $|V_2'| \geq \varepsilon|V_2|$, we have $|d(V_1, V_2) - d(V_1', V_2')| \leq \varepsilon p$. An $(\varepsilon, 1)$-regular pair is simply called ε-*regular*. The concept of regularity, first developed by Szemerédi [35], proved to be of extreme importance in modern combinatorics and played a central rôle in proofs of a range of results in extremal graph theory, Ramsey theory, and others. For example, it is well-known that every triple of sets (V_1, V_2, V_3) such that (V_i, V_j) is ε-regular and has sufficiently large density for all distinct i, j contains a triangle. An ε-regular pair

(V_1, V_2) is called (δ, ε)-*super-regular* if it satisfies the additional condition that every vertex in V_1 has at least $\delta|V_2|$ neighbours in V_2 and, vice versa, every vertex in V_2 has at least $\delta|V_1|$ neighbours in V_1. Komlós, Sárközy, and Szemerédi [24] proved that super-regular triples are even more powerful than mere regular triples. For instance, every triple (V_1, V_2, V_3) such that $|V_1| = |V_2| = |V_3|$ and (V_i, V_j) is (δ, ε)-super-regular and has sufficiently large density for all distinct i, j contains not only a single triangle, but also a family of vertex-disjoint triangles that cover all vertices of the triple.

However, if $p \ll 1$, then the power of (ε, p)-regular pairs turns out to be significantly weaker. For example, Łuczak (see [22]) observed that there are (ε, p)-regular triples which do not contain even a single triangle. Still, Kohayakawa, Łuczak, and Rödl [21] proved that most (ε, p)-regular triples contain a triangle provided that p is sufficiently large and conjectured that an analogous result holds for arbitrary graphs (see the survey [14]).

It is not much of a surprise that even less is known about embedding large graphs into sparse regular pairs. Böttcher, Kohayakawa, and Taraz [7] proved that if the regular pair is a subgraph of a random graph and each part has size n, then (asymptotically almost surely) one can embed into the pair all bipartite graphs with bounded maximum degree whose color classes both have size at most $(1 - \eta)n$, where η is a fixed positive real. Since in an (ε, p)-regular pair (V_1, V_2), each set V_i can have as many as $c_\varepsilon|V_i|$ isolated vertices, one cannot hope to embed spanning graphs into the pair without imposing some further restrictions. Let us now consider sparse regular triples. Observe that imposing merely a minimum degree condition as in the dense case is not sufficient since we can remove all triangles that contain a fixed vertex by deleting all edges in its neighbourhood (this will not effect regularity of the triple since the neighbourhoods of this vertex have size $o(n)$). We suggest one possible strengthening of the notion of super-regularity, which we call *strong-super-regularity*, and show that a sparse strong-super-regular triple in a subgraph of a random graph contains a collection of vertex-disjoint triangles that cover all the vertices of the triple. The definition of a strong-super-regular triple is given in Definition 2.11.

Theorem 1.4. *For all positive δ and ξ there exist $\varepsilon(\delta)$ and $C(\delta, \xi)$ such that if $p(n) \geq C(\log n/n)^{1/2}$, then $G(n, p)$ a.a.s. satisfies the following. Every (δ, ε, p)-strong-super-regular triple (V_1, V_2, V_3) that is a subgraph of $G(n, p)$ with $|V_1| = |V_2| = |V_3| \geq \xi n$ contains a collection of vertex-disjoint triangles that cover all the vertices.*

It is possible that one can derive the same conclusion from weaker assumptions than strong-super-regularity. However, we will later show that the restriction we imposed is not too strong to make our theorem useless, as Theorem 1.4 will form an essential part in the proof of Theorem 1.3.

1.3 Outline of the paper

In Section 2, we recall some known definitions and results and introduce a few notions that will be of great importance in all subsequent sections. Section 3 contains an outline of the proof of Theorem 1.3. In Section 4, we establish some properties of the random graph $G(n, p)$ that we will frequently invoke in subsequent sections. In Sections 5, 6, and 7, we prove a series of technical lemmas that culminate in the proof of Theorem 1.3 and 1.4. For a brief outline of this part of the paper, we refer the reader to Section 3. Finally, Section 8 contains a few concluding remarks.

1.4 Notation

Let G be a graph with vertex set V and edge set E. For a vertex $v \in V$, we denote its neighbourhood in G by $N(v)$ and let $\deg(v)$ be its degree. The minimum degree of the graph is denoted by $\delta(G)$. For a set $X \subset V$, we let $e(X)$ be the number of edges of G with both endpoints in the set X, and $\deg(v, X) = |N(v) \cap X|$. We say that two edges are independent if they do not share a vertex. For two subsets $X, Y \subset V$, we let $e(X, Y)$ be the number of ordered pairs (x, y) such that $x \in X$, $y \in Y$ and xy is an edge of G; note that $e(X, X) = 2e(X)$. If X and Y are disjoint, we refer to the quantity $e(X, Y)/(|X||Y|)$, denoted by $d(X, Y)$, as the density of the pair (X, Y). With a slight abuse of notation, we will sometimes write (X, Y) to denote the set of all edges xy with $x \in X$ and $y \in Y$. Let $X, Y, Z \subset V$ be three pairwise disjoint sets. We say that the triple (X, Y, Z) is balanced if $|X| = |Y| = |Z|$. The minimum density of the triple is the minimum of the numbers $d(X, Y)$, $d(X, Z)$, and $d(Y, Z)$. A triangle across (X, Y, Z) is any triangle with one vertex in each of X, Y, and Z. When the implicit graph we are considering is not clear from the context, we will use subscripts to prevent ambiguity. For example, $\deg_G(v)$ is the degree of v in the graph G.

We write $y = 1 \pm x$ to abbreviate $y \in [1 - x, 1 + x]$. We omit floor and ceiling signs whenever they are not crucial. Throughout the paper, log will always denote the natural logarithm. Finally, we often use subscripts such as in $c_{3.6}$ to explicitly indicate that the constant $c_{3.6}$ is defined in Claim/Lemma/Proposition/Theorem 3.6.

2 Preliminaries

2.1 Sparse regularity lemma

Let G be a graph on a vertex set V. Recall that a pair (V_1, V_2) of disjoint subsets of V is (ε, p)-regular if for all $V_1' \subset V_1$ and $V_2' \subset V_2$ with $|V_1'| \geq \varepsilon|V_1|$ and $|V_2'| \geq \varepsilon|V_2|$, $|d(V_1, V_2) - d(V_1', V_2')| \leq \varepsilon p$. We call a triple (V_1, V_2, V_3) of disjoint subsets of V (ε, p)-regular if (V_i, V_j) forms an (ε, p)-regular pair for every $\{i, j\} \subset \{1, 2, 3\}$. Let $\mathcal{G}(K_3, (n_1, n_2, n_3), (d_{12}, d_{23}, d_{31}), (\varepsilon, p))$ be the collection of all (ε, p)-regular triples (V_1, V_2, V_3) such that $|V_i| = n_i$ for all i and $d(V_i, V_j) = d_{ij}p$ for every $\{i, j\} \subset \{1, 2, 3\}$.

Below we establish two simple hereditary properties of regular pairs.

Proposition 2.1. *Let positive reals ε_1, ε_2, and p satisfying $\varepsilon_1 < \varepsilon_2 \leq 1/2$ be given. Let (V_1, V_2) be an (ε_1, p)-regular pair and for $i \in \{1, 2\}$, let $V_i' \subset V_i$ be an arbitrary subset with $|V_i'| \geq \varepsilon_2|V_i|$. Then (V_1', V_2') is an $(\varepsilon_1/\varepsilon_2, p)$-regular pair of density $d(V_1, V_2) \pm \varepsilon_1 p$.*

Proof. By regularity of the pair (V_1, V_2), for every pair of subsets $V_i'' \subset V_i'$ such that $|V_i''| \geq (\varepsilon_1/\varepsilon_2)|V_i'| \geq \varepsilon_1|V_i|$ for $i \in \{1, 2\}$, we have

$$|d(V_1'', V_2'') - d(V_1', V_2')| \leq |d(V_1'', V_2'') - d(V_1, V_2)| + |d(V_1', V_2') - d(V_1, V_2)| \leq 2\varepsilon_1 p.$$

Since $\max\{\varepsilon_1/\varepsilon_2, 2\varepsilon_1\} = \varepsilon_1/\varepsilon_2$, the pair (V_1', V_2') is $(\varepsilon_1/\varepsilon_2, p)$-regular. The density condition immediately follows from the definition of regularity. $\square$

Proposition 2.2. *Let (V_1, V_2) be an (ε, p)-regular pair in a graph G and let G' be a subgraph of G obtained by removing at most $\varepsilon^3 p|V_1||V_2|$ edges from (V_1, V_2). Then (V_1, V_2) is $(3\varepsilon, p)$-regular in G'.*

Proof. For $i \in \{1, 2\}$, let U_i be a subset of V_i of size at least $\varepsilon|V_i|$. Note that

$$|d_G(U_1, U_2) - d_{G'}(U_1, U_2)| \leq \frac{e_G(U_1, U_2) - e_{G'}(U_1, U_2)}{|U_1||U_2|} \leq \frac{\varepsilon^3 p |V_1||V_2|}{|U_1||U_2|} \leq \varepsilon p.$$

The conclusion easily follows from the triangle inequality. $\qquad\qquad\qquad\qquad\square$

An (ε, p)-*regular partition* of an n-vertex graph G is a partition $(V_i)_{i=0}^k$ of its vertex set such that (i) the exceptional class V_0 has size at most εn, (ii) $V_1, \ldots, V_k$ have equal sizes, and (iii) all but at most εk^2 of the pairs (V_i, V_j) are (ε, p)-regular. Given a collection of subsets $(W_i)_{i=0}^k$ of the vertex set $V(G)$, the (δ, ε, p)-*reduced graph* R of the collection is the graph on the vertex set $[k]$ such that $i, j \in [k]$ are adjacent if and only if W_i and W_j form an (ε, p)-regular pair of density at least δp. Note that when considering reduced graphs, the partition $(W_i)_{i=0}^k$ is not necessarily a regular partition and we ignore the set W_0. For a graph R' on the vertex set $[k]$, we say that G is (δ, ε, p)-regular over R' if for every edge $\{i, j\}$ of R', the pair (V_i, V_j) is ε-regular with density at least δ. Let η and b be reals such that $\eta \in (0, 1]$, and $b \geq 1$. We say that G is (η, b, p)-*upper-uniform* if $d(V_1, V_2) \leq bp$ for all disjoint sets V_1, V_2 with $|V_1|, |V_2| \geq \eta|V|$. With the above definitions at hand, we may now state a version of Szemerédi's regularity lemma for upper-uniform graphs (see, e.g., [20, 23]).

Theorem 2.3. *For every positive ε, b, and k_0 with $b, k_0 \geq 1$, there exist constants $\eta(\varepsilon, b, k_0)$ and $K(\varepsilon, b, k_0)$ with $K \geq k_0$ such that for every positive p, every (η, b, p)-upper-uniform graph with at least k_0 vertices admits an (ε, p)-regular partition $(V_i)_{i=0}^k$ such that $k_0 \leq k \leq K$, and each part forms a regular pair with at least $(1 - \varepsilon)k$ other parts.*

The version of the regularity lemma stated above is slightly different from those given in [20, 23], which say that the total number of irregular pairs is at most εk^2. However, by using some standard techniques, one can derive the 'minimum degree' version from the results in [20, 23].

2.2 Typical vertices and super-regularity

We start this section by introducing the notions of typical vertices and triples.

Definition 2.4. Let (V_1, V_2, V_3) be a triple of sets (not necessarily regular) with densities $d_{ij}p$ between V_i and V_j.

(A) Fix a vertex $v \in V_1$ and for $i \in \{2, 3\}$, let $N_i = N(v) \cap V_i$. We say that v is ε-*typical* if for $i \in \{2, 3\}$,

    (i) $|N_i| = (1 \pm \varepsilon)d_{1i}p|V_i|$ and

    (ii) there exists $N_i' \subset N_i$ satisfying $|N_i'| \geq (1 - \varepsilon)|N_i|$ such that (N_2', N_3') is an (ε, p)-regular pair with density $(1 \pm \varepsilon)d_{23}p$.

(B) The triple (V_1, V_2, V_3) is ε-*typical* if it is (ε, p)-regular and for each i, all but at most $\varepsilon|V_i|$ vertices in V_i are ε-typical.

Remark 2.5. Since the property of being ε-typical depends not only on ε but also on p, we should rather speak of (ε, p)-typical vertices and triples. Nevertheless, since the parameter p will be always clear from the context, we will suppress it from the notation for the sake of brevity.

It turns out that an overwhelming majority of all regular triples are also typical. The following lemma, which is a straightforward generalization of [14, Lemma 5.1], makes the above statement precise. We omit its proof as it can be easily read out from the proof of [14, Lemma 5.1].

Lemma 2.6. *For all positive β, δ, ε', and ξ, there exist constants $\varepsilon_0(\beta, \delta, \varepsilon')$ and $C(\delta, \varepsilon', \xi)$ such that if $\varepsilon \leq \varepsilon_0$, $d_{12}, d_{13}, d_{23} \geq \delta$, $\xi n \leq n_1, n_2, n_3 \leq n$, and $p \geq Cn^{-1/2}$, then all but at most*

$$\beta^{\delta \xi^2 n^2 p} \binom{n_1 n_2}{d_{12} p n_1 n_2} \binom{n_1 n_3}{d_{13} p n_1 n_3} \binom{n_2 n_3}{d_{23} p n_2 n_3}$$

graphs in $\mathcal{G}(K_3, (n_1, n_2, n_3), (d_{12}, d_{13}, d_{23}), (\varepsilon, p))$ are ε'-typical provided that n is sufficiently large.

The following proposition justifies why the notion of ε-typical triples can be useful for our purposes.

Proposition 2.7. *For every positive α, δ, and p, there exists an $\varepsilon(\alpha, \delta)$ such that every ε-typical (ε, p)-regular triple (V_1, V_2, V_3) of minimum density at least δp contains $(1 - \alpha) \min_i |V_i|$ vertex-disjoint triangles.*

Proof. Note that without loss of generality, we may assume that $\alpha \leq 1/2$. Furthermore, let $\varepsilon = \min\{\delta/4, \alpha/20\}$ and let $\varepsilon' = 2\varepsilon/\alpha$. Let us greedily remove triangles from (V_1, V_2, V_3) until we cannot do it anymore and denote the remaining triple by (V_1', V_2', V_3'). If $|V_i'| \leq \alpha|V_i|$ for some i, then there is nothing left to prove, so we may assume that $|V_i'| > \alpha|V_i|$ for all i. Let W_i be the set of all those vertices in V_i' that were ε-typical in the original triple and note that $|W_i| \geq |V_i'| - \varepsilon|V_i| \geq (\alpha/2)|V_i|$. By Proposition 2.1, the triple (W_1, W_2, W_3) is (ε', p)-regular and the density of each pair (W_i, W_j) is at least $(d_{ij} - \varepsilon)p$. Since $\varepsilon' < 1/2$, there is a vertex $v \in W_1$ with $\deg(v, W_i) \geq (d_{ij} - \varepsilon - \varepsilon')p|W_i|$ for $i \in \{2, 3\}$. For $i \in \{2, 3\}$, let N_i and N_i' be the sets from the definition of an ε-typical vertex for v and let $M_i = N_i \cap W_i$ and $M_i' = N_i' \cap W_i$. Since

$$|M_i'| \geq |M_i| - |N_i \setminus N_i'| \geq (d_{1i} - \varepsilon - \varepsilon')p|W_i| - \varepsilon(1 + \varepsilon)d_{1i}p|V_i|$$
$$\geq \left[(1 - \varepsilon/\delta - \varepsilon'/\delta)(\alpha/2) - \varepsilon(1 + \varepsilon)\right]d_{1i}p|V_i| \geq \varepsilon(1 + \varepsilon)d_{1i}p|V_i| \geq \varepsilon|N_i'|$$

and (N_2', N_3') was (ε, p)-regular with density at least $(1 - \varepsilon)\delta p$ and $(1 - \varepsilon)\delta p > \varepsilon p$, the pair (M_2', M_3') has positive density. It follows that (W_1, W_2, W_3) contains a triangle, but this is impossible. $\square$

Remark 2.8. It is quite easy to see that the combination of Theorems 2.3 and 2.16, Lemma 2.6 (see Proposition 4.8), and Proposition 2.7 implies the following statement. For all positive constants γ and ε, there exists a C such that if $p(n) \geq Cn^{-1/2}$, then a.a.s. every subgraph $G \subset G(n, p)$ with $\delta(G) \geq (2/3 + \gamma)np$ contains a triangle packing that covers all but at most εn vertices of G.

The following concept will serve us as a generalization of super-regularity to the sparse setting.

Definition 2.9. A triple (V_1, V_2, V_3) is (δ, ε, p)-*super-regular* if each pair (V_i, V_j) is (ε, p)-regular with density at least δp and for every i, all vertices in V_i are ε-typical.

We close this section with the following proposition, which tells us how to trim a typical regular triple in order to get a super-regular one.

Proposition 2.10. *For all positive ε' and δ, there exists an $\varepsilon(\varepsilon', \delta)$ such that the following holds. Let (V_1, V_2, V_3) be an (ε, p)-regular triple, where for each i and j, the density of (V_i, V_j) is $d_{ij}p$, where $d_{ij} \geq \delta$. For each i, let X_i be an arbitrary subset of V_i with $|X_i| \leq \varepsilon|V_i|$. Then every vertex $v \in V_1 \setminus X_1$ that is ε-typical and satisfies $\deg(v, X_j) \leq \varepsilon p|V_j|$ for every $j \in \{2, 3\}$ becomes an ε'-typical vertex in $(V_1 \setminus X_1, V_2 \setminus X_2, V_3 \setminus X_3)$.*

Proof. Let $v \in V_1$ be any such vertex and for $j \in \{2, 3\}$, let $N_j = N(v) \cap V_j$. Since v is ε-typical, $(1 - \varepsilon)d_{1j}p|V_j| \leq |N_j| \leq (1 + \varepsilon)d_{1j}p|V_j|$. Moreover, there exist subsets $N'_j \subset N_j$ satisfying $|N'_j| \geq (1-\varepsilon)|N_j|$ such that (N'_2, N'_3) is an (ε, p)-regular pair with density $d_v p$, where $d_v \in [(1-\varepsilon)d_{23}, (1+\varepsilon)d_{23}]$. Moreover, let $M_j = N_j \setminus X_j$ and similarly let $M'_j = N'_j \setminus X_j$.

For each i, let $W_i = V_i \setminus X_i$ and recall that $(1 - \varepsilon)|V_i| \leq |W_i| \leq |V_i|$. For every i and j, let $d'_{ij}p$ be the density of the pair (W_i, W_j). Since (V_i, V_j) is (ε, p)-regular, $d'_{ij} \in [d_{ij} - \varepsilon, d_{ij} + \varepsilon]$. It follows that

$$|M_j| \leq |N_j| \leq (1 + \varepsilon)d_{1j}p|V_j| \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}(d'_{1j} + \varepsilon)p|W_j|.$$

And by the given condition $\deg(v, X_j) \leq \varepsilon p|V_j|$, we have

$$|M_j| \geq |N_j| - \varepsilon p|V_j| \geq (1 - \varepsilon - \varepsilon/\delta)d_{1j}p|V_j| \geq (1 - \varepsilon - \varepsilon/\delta)(d'_{1j} - \varepsilon)p|W_j|.$$

Moreover, since $|N_j| \geq (1 - \varepsilon)\delta p|V_j|$, we have

$$|M'_j| \geq |N'_j| - \varepsilon p|V_j| \geq (1 - \varepsilon)|N_j| - \varepsilon p|V_j| \geq (1 - \varepsilon - \varepsilon/((1 - \varepsilon)\delta))|N_j|$$
$$\geq (1 - \varepsilon - \varepsilon/((1 - \varepsilon)\delta))|M_j| \geq |M_j|/2.$$

By Proposition 2.1, the pair (M'_2, M'_3) is $(2\varepsilon, p)$-regular with density $d'_v p$ satisfying

$$(1 - \varepsilon)(d'_{23} - \varepsilon) - \varepsilon \leq d'_v \leq (1 + \varepsilon)(d'_{23} + \varepsilon) + \varepsilon.$$

Therefore, if ε is sufficiently small, then $\deg(v, W_j) = |M_j| \in [(1 - \varepsilon')pd'_{1j}|W_j|, (1 + \varepsilon')pd'_{1j}|W_j|]$, $|M'_j| \geq (1-\varepsilon')|M_j|$, and (M'_2, M'_3) is (ε', p)-regular with density $d'_v p$, where $d'_v \in [(1-\varepsilon')d_{23}, (1+\varepsilon')d_{23}]$. It follows that v is ε'-typical in (W_1, W_2, W_3). $\qquad\square$

2.3 Good edges and good vertices

As we established in Section 2.2 (see Remark 2.8), imposing certain regularity conditions on the vertices of a regular triple suffices to guarantee the existence of an almost perfect triangle packing. In order to assure that a triangle-factor can be found, we will need to impose some conditions also on the edges of the triple. With hindsight (see the discussion in Section 3.2), we now introduce the notions of *good edges* and *good vertices*.

Definition 2.11. Let (V_1, V_2, V_3) be a triple of sets (not necessarily regular) with densities $d_{ij}p$ between V_i and V_j.

(A) We say that an edge between V_2 and V_3 is ε-*good* if its endpoints have at least $(1 - \varepsilon)d_{12}d_{13}p^2|V_1|$ common neighbourhoods in V_1.

(B) We say that an ε-typical vertex $v \in V_1$ is ε-*good* if $(N(v) \cap V_2, N(v) \cap V_3)$ contains at most $\varepsilon d_{12}d_{13}d_{23}p^3|V_2||V_3|$ edges that are not ε-good.

(C) We say that the triple (V_1, V_2, V_3) is (δ, ε, p)-*strong-super-regular* if it is (δ, ε, p)-super-regular and for every i, all vertices in V_i are ε-good.

Next, we show that super-regular triples are not very far from being strong-super-regular. More precisely, we prove that requiring a triple to be merely typical (recall Definition 2.4) and all pairs in this triple to have non-zero densities forces most of its edges and vertices to be good.

Proposition 2.12. *Let ε, ε', and δ be positive constants satisfying $3\varepsilon + \varepsilon/\delta \leq \varepsilon'$ and let (V_1, V_2, V_3) be an ε-typical (ε, p)-regular triple, where the density $d_{ij}p$ of each pair (V_i, V_j) is at least δp. Then there are at most $4\varepsilon d_{23}p|V_2||V_3|$ edges between V_2 and V_3 which are not ε'-good.*

Proof. For an ε-typical vertex $v \in V_2$, let $N_1 = N(v) \cap V_1$ and $N_3 = N(v) \cap V_3$. Recall from Definition 2.4 that $|N_i| \geq (1 - \varepsilon)d_{2i}p|V_i|$ and that there exist $N_i' \subset N_i$ with $|N_i'| \geq (1 - \varepsilon)|N_i|$ for $i \in \{1, 3\}$ such that (N_1', N_3') is (ε, p)-regular and has density at least $(1 - \varepsilon)d_{13}p$. It follows that at least $(1 - \varepsilon)|N_3'|$ vertices $w \in N_3'$ have at least $((1 - \varepsilon)d_{13} - \varepsilon)p|N_1'|$ common neighbours with v in N_1'. Since

$$((1 - \varepsilon)d_{13} - \varepsilon)p|N_1'| \geq (1 - \varepsilon - \varepsilon/\delta)(1 - \varepsilon)d_{13}p|N_1| \geq (1 - 3\varepsilon - \varepsilon/\delta)d_{12}d_{13}p^2|V_1|,$$

and $3\varepsilon + \varepsilon/\delta \leq \varepsilon'$, each such edge $\{v, w\}$ is ε'-good. Since there are at least $(1 - \varepsilon)|V_2|$ typical vertices in V_2 and

$$(1 - \varepsilon)|N_3'| \geq (1 - \varepsilon)^3 d_{23}p|V_3| \geq (1 - 3\varepsilon)d_{23}p|V_3|,$$

the total number of ε'-good edges between V_2 and V_3 is at least $(1 - 4\varepsilon)d_{23}p|V_2||V_3|$. Finally, since the number of edges between V_2 and V_3 is exactly $d_{23}p|V_2||V_3|$, the total number of non-ε'-good edges is at most $4\varepsilon d_{23}p|V_2||V_3|$. $\square$

Proposition 2.13. *For every ε' and δ, there exists a positive $\varepsilon(\varepsilon', \delta)$ such that the following holds. Let (V_1, V_2, V_3) be an ε-typical (ε, p)-regular triple with minimum density at least δp. Moreover, assume that the endpoints of no edge in (V_2, V_3) have more than $4p^2|V_1|$ common neighbours in V_1. Then V_1 contains at most $\varepsilon'|V_1|$ vertices that are not ε'-good.*

Proof. Let $\varepsilon = \min\{\varepsilon'/4, \varepsilon'\delta/4, (\varepsilon'\delta)^2/32\}$. By Proposition 2.12, at most $4\varepsilon d_{23}p|V_2||V_3|$ edges in (V_2, V_3) are not ε'-good. Let $\alpha|V_1|$ be the number of ε-typical vertices in V_1 that are not ε'-good. By definition, the neighbourhood of every such vertex contains at least $\varepsilon'd_{12}d_{13}d_{23}p^3|V_2||V_3|$ edges that are not ε'-good. Therefore, our assumption on the maximum number of common neighbours of the endpoints of edges in (V_2, V_3) implies that

$$\alpha|V_1| \cdot \varepsilon'd_{12}d_{13}d_{23}p^3|V_2||V_3| \leq 4p^2|V_1| \cdot 4\varepsilon d_{23}p|V_2||V_3|$$

and hence $\alpha \leq 16\varepsilon/(\varepsilon'd_{12}d_{13}) \leq \varepsilon'/2$. Finally, since at most $\varepsilon|V_1|$ vertices in V_1 are not ε-typical and $\varepsilon \leq \varepsilon'/2$, the number of vertices in V_1 that are not ε'-good is at most $\varepsilon'|V_1|$. $\square$

We end this section by showing that the neighbourhood of every typical (good) vertex contains a subgraph with bounded maximum degree and many (good) edges.

Proposition 2.14. *Let ε, δ, and p be positive constants with $\varepsilon < 1/2$. Let (V_1, V_2, V_3) be a triple of sets such that for all i and j, the density of (V_i, V_j) is $d_{ij}p$, where $d_{ij} \geq \delta$. Then for every ε-typical vertex $v \in V_1$, there exist sets $N_j'' \subset N(v) \cap V_j$ for $j \in \{2, 3\}$ such that*

(i) *there are at least $(1 - 6\varepsilon - 2\varepsilon/\delta)d_{12}d_{13}d_{23}p^3|V_2||V_3|$ edges in (N_2'', N_3'') and if v is ε-good, then there are at least that many ε-good edges in (N_2'', N_3''), and*

(ii) *for all j and k with $\{j,k\} = \{2,3\}$, no vertex in N_j'' has more than $(1+2\varepsilon/\delta)(1+\varepsilon^2)d_{1k}d_{23}p^2|V_k|$ neighbours in N_k''.*

Proof. Fix an ε-typical vertex $v \in V_1$. For each $j \in \{2,3\}$, let $N_j = N(v) \cap V_j$. Since v is ε-typical, there are $N_j' \subset N_j$ with $|N_j'| \geq (1 - \varepsilon)|N_j| \geq (1 - \varepsilon)^2 pd_{1j}|V_j|$ such that (N_2', N_3') is (ε, p)-regular with density $d_{23}'p$, where $(1 - \varepsilon)d_{23} \leq d_{23}' \leq (1 + \varepsilon)d_{23}$. Let N_j'' be the set of vertices in N_j' that have at most $(d_{23}' + \varepsilon)p|N_k'|$ neighbours in N_k' and note that $|N_j''| \geq (1 - \varepsilon)|N_j'|$ by (ε, p)-regularity of (N_2', N_3'). Since $1/d_{23}' \leq 1/((1 - \varepsilon)d_{23}) \leq 2/\delta$, we have

$$(d_{23}' + \varepsilon)p|N_k'| \leq (1 + 2\varepsilon/\delta)d_{23}'p|N_k| \leq (1 + 2\varepsilon/\delta)(1 + \varepsilon)^2 d_{1k}d_{23}p^2|V_k|,$$

and then (ii) follows. Note that $|N_j''| \geq (1 - \varepsilon)|N_j'| \geq (1 - \varepsilon)^2 d_{1j}p|V_j|$ and

$$e(N_2'', N_3'') \geq (d_{23}' - \varepsilon)p|N_2''||N_3''| \geq (1 - 2\varepsilon/\delta)d_{23}'p|N_2''||N_3''| \geq (1 - 2\varepsilon/\delta)(1 - \varepsilon)^5 d_{12}d_{13}d_{23}p^3|V_2||V_3|.$$

Moreover, if v is ε-good, then at most $\varepsilon d_{12}d_{13}d_{23}p^3|V_2||V_3|$ edges in (N_2'', N_3'') are not ε-good. Now (i) follows. $\square$

2.4 Graph theory

The following proposition, which we will be using several times in the proof of our main result, is a simple corollary from Hall's marriage theorem [16] and gives a sufficient condition for a bipartite graph to have a perfect matching.

Proposition 2.15. *Let H be a bipartite graph on the vertex set $A \cup B$ with $|A| = |B|$. Suppose that there is an integer L such that*

(i) $|N(S)| \geq |S|$ *for each $S \subset A$ with $|A \setminus S| \geq L$ and*

(ii) $|N(T)| \geq |T|$ *for each $T \subset B$ with $|T| \leq L$.*

Then H has a perfect matching.

Recall that the following theorem was proved by Corrádi and Hajnal [9].

Theorem 2.16. *Every graph on n vertices with minimum degree at least $2n/3$ contains a perfect K_3-packing provided that n is divisible by 3.*

2.5 Bounding large deviations

Throughout the proof, we will extensively use the following standard estimate on the tail probabilities of binomial random variables, see [1, Appendix A]. We denote by $\mathrm{Bi}(n, p)$ the binomial random variable with parameters n and p, i.e., the number of successes in a sequence of n independent Bernoulli trials with success probability p.

Theorem 2.17 (Chernoff's inequality)**.** *Let $p \in (0, 1)$ and let n be a positive integer. Then for every positive a with $a \leq 2np/3$,*

$$P\big(|\mathrm{Bi}(n, p) - np| > a\big) \leq \exp(-a^2/(6np)).$$

3   Outline of the proof of Theorem 1.3

Let G be a subgraph of $G(n, p)$ with minimum degree at least $(2/3+o(1))np$. Throughout this section, we will tacitly condition on a few events that hold in $G(n, p)$ asymptotically almost surely. The proof of Theorem 1.3 breaks down into the following four simple steps.

1. Apply the sparse regularity lemma (Theorem 2.3) and Theorem 2.16 to partition the vertex set of G into regular triples with positive density and a small exceptional set of vertices.

2. Remove from G a collection of vertex-disjoint triangles so that all but at most $O(p^{-2})$ remaining vertices lie in balanced super-regular triples.

3. Decompose each of those super-regular triples into a triangle packing, a balanced strong-super-regular triple, and a set of $O(p^{-1})$ leftover vertices.

4. Find a triangle-factor in each strong-super-regular triple.

Since step 1 is a straightforward application of the regularity lemma (Theorem 2.3) and Theorem 2.16, we will only describe the basic ideas of steps 2, 3, and 4 in this section. The details of these steps will be given in Sections 5, 6, and 7, respectively.

3.1   Step 2

In order to construct super-regular triples from the regular triples we obtained in step 1, we first move all non-typical vertices to the exceptional set V_0. Since we have no control over V_0 and $|V_0|$ can be linear in n, we need to cover most of it with vertex-disjoint triangles. At the same time, we do not want to use too many vertices from any of the regular triples in order not to destroy their structure, i.e., to keep them close to being super-regular. This will be achieved by an application of Lemma 4.5, which allows us to find such triangles. After we absorb the exceptional vertices into a triangle packing, some triples in the remaining graph might become imbalanced. Since in order for any triple to have a triangle-factor (or at least an almost perfect triangle packing), the sizes of all three of its parts must be equal, we have to balance the sizes of the remaining triples. We will do that by adding to our triangle packing some triangles whose vertices lie in two different triples, see Lemma 5.1. Finally, since at the beginning we removed all non-typical vertices from each triple and later we did not alter it too much, we can make every triple super-regular by deleting at most $O(p^{-1})$ of its vertices (see Proposition 2.10).

3.2   Steps 3 and 4

Our general strategy for finding a triangle-factor in a super-regular triple (V_1, V_2, V_3) can be summarized as follows.

(i) For each $\{i, j\} \subset \{1, 2, 3\}$, randomly select a small set M_{ij} of independent edges in (V_i, V_j).

(ii) Find an almost perfect triangle packing that does not hit any endpoints of the edges in any M_{ij}.

(iii) Match the remaining vertices with the edges in the sets M_{ij} in order to extend the triangle packing to a triangle-factor.

Assume that the first two steps have been performed. Then, in order to verify Hall's condition (see Proposition 2.15) to prove that an appropriate matching can be found in (iii), we need to know, in particular, that the endpoints of each edge in M_{23} have many common neighbours in the remaining part of V_1 and that each vertex in V_1 is incident to both endpoints of many edges in M_{23} (and that similar conditions hold for other choices of indices). Therefore, it would be convenient if M_{23} consisted only of good edges and V_1 contained only good vertices (see Section 2.3). Unfortunately, super-regular triples can generally contain vertices that are not good. This is the reason why in step 3, we need to break down each super-regular triple into a triangle packing and a strong-super-regular triple.

Therefore, we will perform the above described process twice. First, in step 3, we will absorb all the non-good vertices into a small triangle packing by performing (i) and (iii), see Theorem 6.6. In step 4, once we are left with a balanced strong-super-regular triple (after deleting at most $O(p^{-1})$ further vertices), we can finally perform (i)–(iii), now using only good edges to construct M_{ij}s, to find a triangle-factor inside this triple, see Theorem 1.4.

4 Properties of Random Graphs

In this section we establish several properties of the random graph that will be useful in later sections.

Proposition 4.1. *For every positive real ρ, there exists a constant $C(\rho)$ such that if $p \geq C(\log n/n)^{1/2}$, then $G(n,p)$ a.a.s. satisfies the following properties.*

(i) *Every vertex has degree $(1 \pm \rho)np$.*

(ii) *Every pair of distinct vertices has $(1 \pm \rho)np^2$ common neighbours.*

(iii) *For all $X, Y \subset V$ with $|X|, |Y| \geq \rho np$, we have $e(X, Y) = (1 \pm \rho)|X||Y|p$. In particular, $e(X) = e(X, X)/2 = (1 \pm \rho)|X|^2p/2$ for all X of size at least ρnp.*

Proposition 4.2. *For every $\rho \in (0, 1/2)$, $G(n, p)$ satisfies the following.*

(i) *Let D be a positive real. For a fixed set $W \subset V$, with probability $1 - e^{(1-\rho^2 D/12)n}$, all but at most $nDp^{-1}/|W|$ vertices in $V \setminus W$ satisfy*
$$\deg(v, W) = (1 \pm \rho)|W|p.$$

(ii) *For every positive real ξ, there exists a constant $D(\rho, \xi)$ such that a.a.s. the following holds. For all $W \subset V$ with $|W| \geq \xi n$, all but at most Dp^{-1} vertices in $V \setminus W$ satisfy*
$$\deg(v, W) = (1 \pm \rho)|W|p.$$

Proof. To prove (i), as a first step, we fix a set $W \subset V$. We may assume that $Dp^{-1}n/|W| \leq n$ as otherwise, the claim is vacuously true. Suppose that there are $Dp^{-1}n/|W|$ vertices $v \in V \setminus W$ such that $\deg(v, W) \neq (1 \pm \rho)|W|p$. Then there exists a set $B \subset V \setminus W$ of size $Dp^{-1}n/(2|W|)$ such that either $\deg(v, W) > (1 + \rho)|W|p$ for all $v \in B$ or $\deg(v, W) < (1 - \rho)|W|p$ for all $v \in B$. This clearly implies that $e(B, W) \neq (1 \pm \rho)|B||W|p$ for some B as above. Since $e(B, W)$ is a sum of independent binomial random variables and $\mathbb{E}[e(B, W)] = |B||W|p = Dn/2$, by Chernoff's inequality,
$$P\big(e(B, W) \neq (1 \pm \rho)|B||W|p\big) \leq e^{-\rho^2 Dn/12}.$$

By the union bound, the probability that such a set B exists is at most $2^n e^{-\rho^2 Dn/12}$.

Now that (i) is proved, we easily get (ii) by applying the union bound. $\qquad\square$

Proposition 4.3. *For all $\xi \in (0,1)$, there exists a $C(\xi)$ such that if $p \geq C(\log n/n)^{1/2}$, then a.a.s. for every $x \in [1, \xi n/2]$, $G(n,p)$ does not contain a set W of x vertices and a set E of x independent edges outside W (i.e., no edge in E has an endpoint in W) such that either the endpoints of each edge in E have at least ξnp^2 common neighbours in W or each vertex in W is adjacent to both endpoints of at least ξnp^2 edges in E.*

Proof. Fix x, W, and E as in the statement of this proposition. For a vertex $w \in W$ and an edge $\{u,v\} \in E$, let $B(u,v,w)$ denote the event that w is adjacent to both u and v. Let X be the random variable denoting the number of events $B(u,v,w)$ that occur in $G(n,p)$. Note that each of the "bad" events described in the statement of this lemma implies that $X \geq \xi np^2 x \geq 2x^2 p^2$. Moreover, observe that $X \leq x^2$, so we can restrict our attention to the case $x \geq \xi np^2$. Since all $B(u,v,w)$ are mutually independent, X has binomial distribution with parameters x^2 and p^2, and hence by Chernoff's inequality,

$$P(X \geq \xi np^2 x) \leq e^{-c\xi np^2 x}$$

for some absolute positive constant c. Since for each x, there are at most $\binom{n}{x} n^{2x}$ pairs (W,E) with $|W| = |E| = x$, the probability that some "bad" event occurs is at most

$$\sum_{x=\xi np^2}^{\xi n/2} \binom{n}{x} n^{2x} e^{-c\xi np^2 x}.$$

Finally, note that

$$\binom{n}{x} n^{2x} e^{-c\xi np^2 x} \leq e^{3x \log n - c\xi np^2 x} \leq n^{-2},$$

provided that $np^2 \geq (4/c\xi) \log n$. $\qquad\square$

Proposition 4.4. *Let $p \gg n^{-1/2}$. For every positive reals ε and ρ, there exists a positive real $D(\varepsilon, \rho)$ such that $G(n,p)$ a.a.s. satisfies the following property. For every set X with $|X| \geq Dp^{-2}$, there are at most $\varepsilon n^2 p$ edges $\{v,w\}$ in $G[V \setminus X]$ such that v and w do not have $(1 \pm \rho)|X|p^2$ common neighbours in X.*

Proof. The constant $D(\varepsilon, \rho)$ will be chosen later. Let X be a fixed set of size at least Dp^{-2}. Without loss of generality we may assume that $\rho \leq 1/2$.

First expose the edges between X and $V \setminus X$ and call a pair of vertices $\{v,w\} \in V \setminus X$ *bad* if v and w do not have $(1 \pm \rho)|X|p^2$ common neighbours in X. By Proposition 4.2 (i), with probability $1 - e^{(1-\rho^2 D^{1/2}/50)n}$, there are at most $D^{1/2}p^{-1}n/|X|$ vertices that do not satisfy $\deg(v,X) = (1 \pm \rho/2)|X|p$. Even if each of these vertices forms bad pairs with all n vertices, there are at most $D^{1/2}p^{-1}n^2/|X|$ such bad pairs. For each vertex that satisfies $\deg(v,X) = (1 \pm \rho/2)|X|p$, again by Proposition 4.2 (i), with probability $1 - e^{(1-\rho^2 D^{1/2}/50)n}$, there are at most $2D^{1/2}p^{-1}n/(|X|p)$ other vertices w which do not have $(1 \pm \rho)|X|p^2$ common neighbours with w in X. Therefore, if D is sufficiently large, then with probability at least $1 - e^{-2n}$, the total number of bad pairs is at most

$$\frac{D^{1/2}p^{-1}n^2}{|X|} + \frac{2D^{1/2}p^{-1}n^2}{|X|p} \leq 3D^{-1/2}n^2 \leq \varepsilon n^2/2.$$

Finally, expose the edges within $V \setminus X$. By Chernoff's inequality, with probability $1 - e^{-\varepsilon n^2 p/20}$, at most $\varepsilon n^2 p$ bad pairs will form an edge.

Since $n^2 p \gg n$, if we fix the set X, both of the above events happen with probability at least $1 - e^{-2n}$. Since there are at most 2^n choices for X, we can take the union bound over all choices of X to derive the conclusion. $\qquad \square$

Using the above propositions, we now prove the following generalization of [17, Lemma 6.4].

Lemma 4.5. *There exist C, D, and ε such that if $p \geq C(\log n/n)^{1/2}$, then $G(n,p)$ a.a.s. has the following property. For every spanning subgraph $G' \subset G(n,p)$ with $\delta(G') \geq (2/3)np$ and every set $T \subset V(G')$ with $|T| \leq \varepsilon n$, all but at most Dp^{-2} vertices of $V \backslash T$ are contained in a triangle of G which does not intersect T.*

Proof. For the sake of brevity, denote $G(n,p)$ by G and let $V = V(G)$. Let ε be a small positive constant (we will fix it later), let $C = C_{4.1}(\varepsilon)$, and let $D_0(\varepsilon)$ be a constant satisfying $D_0 \geq \max\{D_{4.2(ii)}(\varepsilon, \varepsilon), D_{4.4}(\varepsilon, \varepsilon), 1\}$.

Without loss of generality we may assume that $|T| = \varepsilon n$. Let $X_0 \subset V \setminus T$ be an arbitrary set of size $2D_0 p^{-2}$. By assuming that the events from Propositions 4.1, 4.2 (ii), and 4.4 hold, we will show that there exists a triangle in G' which intersects X_0 but not T. This will prove that there are at most $2D_0 p^{-2}$ vertices that are not contained in triangles that do not hit T. Let T' be the collection of all the vertices v that satisfy $\deg(v, X_0 \cup T) \geq (1 + \varepsilon)|X_0 \cup T|p$ and note that $|T'| \leq D_0 p^{-1}$ by Proposition 4.2 (ii). Let $T'' = T \cup T'$ and $X = X_0 \setminus T'$. Note that $|T''| \leq 2\varepsilon n$ and $|X| \geq |X_0| - |T'| \geq D_0 p^{-2}$. Let D be the constant defined by $|X| = Dp^{-2}$ and note that $D \geq D_0$. It suffices to show that there exists a triangle in G' which contains a vertex from X but not from T''.

Let $Y = V \setminus (X \cup T'')$ and fix a vertex $x \in X$. Note that

$$\deg_{G'}(x, Y) = \deg_{G'}(x) - \deg_{G'}(x, X \cup T'') \geq \deg_{G'}(x) - \deg_{G'}(x, X_0 \cup T) - |T'|$$
$$\geq (2/3)np - (1 + \varepsilon)|T \cup X_0|p - D_0 p^{-1} \geq (2/3 - 3\varepsilon)np,$$

where the last two inequalities follow from the fact that $x \notin T'$ and our assumption on p. Finally, let $N_x = N_G(x) \cap Y$ and fix an arbitrary subset $N'_x \subset N_{G'}(x) \cap Y$ of size $(2/3 - 3\varepsilon)np$.

It suffices to show that the number of triangles $xy_1 y_2$ in G' such that $x \in X$ and $y_1, y_2 \in N'_x$ is nonzero. Let this number be M. To bound M from below, first bound the number of triangles $xy_1 y_2$ in G such that $x \in X, y_1, y_2 \in N_x$, and $y_1 y_2$ is an edge of the graph G' (we will later subtract the number triangles whose y_1 or y_2 is not in N'_x). Let this number be M_0. Since $|X \cup T''|, |Y| \geq \varepsilon n$, by Proposition 4.1 (iii), we have

$$e_{G'}(Y) \geq e_{G'}(V) - e_{G'}(Y, X \cup T'') - e_{G'}(X \cup T'') \geq \frac{2}{3} \cdot \frac{n^2 p}{2} - 4\varepsilon n^2 p = \left(\frac{2}{3} - 8\varepsilon\right)\frac{n^2 p}{2}.$$

By Proposition 4.4, the number of edges $\{v, w\}$ in $G'[Y]$ that form a triangle in G with fewer than $(1 - \varepsilon)D$ vertices in X is at most $\varepsilon n^2 p$ given that D_0 is large enough. Thus,

$$M_0 \geq \left(e_{G'}(Y) - \varepsilon n^2 p\right)(1 - \varepsilon)D \geq \left(\frac{2}{3} - 11\varepsilon\right)\frac{Dn^2 p}{2}.$$

To obtain a bound on M from M_0, we can subtract the number of triangles $xy_1 y_2$ as above such that either y_1 or y_2 is not in N'_x. Since $|N_x| \leq (1 + \varepsilon)np$ by Proposition 4.1 (i), we have

$$|N_x \setminus N'_x| = |N_x| - |N'_x| \leq (1 + \varepsilon)np - (2/3 - 3\varepsilon)np = (1/3 + 4\varepsilon)np.$$

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Thus, if ε is small enough, by Proposition 4.1 (iii) we have,

$$M \geq M_0 - \sum_{x \in X} \left(e_{G'}(N_x \setminus N'_x, N'_x) + e_{G'}(N_x \setminus N'_x) \right)$$

$$\geq M_0 - \sum_{x \in X} (1 + \varepsilon) \left(\left(\frac{1}{3} + 4\varepsilon \right) \frac{2}{3} n^2 p^3 + \left(\frac{1}{3} + 4\varepsilon \right)^2 \frac{n^2 p^3}{2} \right)$$

$$\geq \left(\frac{2}{3} - 11\varepsilon \right) \frac{Dn^2 p}{2} - \sum_{x \in X} \left(\frac{5}{9} + 10\varepsilon \right) \frac{n^2 p^3}{2} = \left(\frac{1}{9} - 21\varepsilon \right) \frac{Dn^2 p}{2}.$$

Therefore there exists a triangle as claimed, provided that ε is sufficiently small. $\qquad\square$

The following proposition establishes the fact that it is necessary to have $\Omega(p^{-2})$ vertices not covered by triangles. Its proof closely follows the argument from [17, Proposition 6.3].

Proposition 4.6. *Let $\varepsilon > 0$. There exists a positive constant $C(\varepsilon)$ such that if $Cn^{-1/2} \leq p \ll 1$, then $G(n, p)$ a.a.s. contains a spanning subgraph of minimum degree at least $(1 - \varepsilon)np$ such that $\Omega(p^{-2})$ of its vertices are not contained in a triangle.*

Proof. Let C be a constant satisfying $C \geq 2$ and $e^{C^2/15} \geq 8e/\varepsilon$. If $p \geq (\log n/n)^{1/2}$, then by Proposition 4.1, a.a.s. $\delta(G(n, p)) \geq (1 - \varepsilon/4)np$ and each pair of vertices of $G(n, p)$ has at most $2np^2$ common neighbours. If $Cn^{-1/2} \leq p < (\log n/n)^{1/2}$, then still a.a.s. $\delta(G(n, p)) \geq (1 - \varepsilon/4)np$, but $G(n, p)$ may contain some edges whose endpoints have more than $2np^2$ common neighbours. Let H be the subgraph consisting of all such edges, and let v be an arbitrary vertex. By Chernoff's inequality, the probability that v and some other vertex have more than $2np^2$ common neighbours is at most $e^{-np^2/15}$. Therefore,

$$P(\deg_H(v) \geq (\varepsilon/4)np) \leq \binom{n}{(\varepsilon/4)np} \left(p \cdot e^{-np^2/15} \right)^{(\varepsilon/4)np} \leq \left(\frac{4enp}{\varepsilon np} \cdot e^{-C^2/15} \right)^{(\varepsilon/4)np} = o(n^{-1}),$$

so a.a.s. $\Delta(H) < (\varepsilon/4)np$. Finally, let $G = G(n, p) - H$. Clearly, the endpoints of every edge of G have at most $2np^2$ common neighbours. Moreover, by Proposition 4.1 (i), we may assume that $\delta(G) > (1 - \varepsilon/2)np$.

Let X be an arbitrary fixed set of $(\varepsilon/4)p^{-2}$ vertices of G and let $W = \{v \notin X : \deg(v, X) \geq 2|X|p\}$. By Chernoff's inequality, the probability that a vertex v belongs to W is $e^{-\Omega(p^{-1})}$ and these events are independent for different vertices. Since $p \gg e^{-\Omega(p^{-1})}$, Chernoff's inequality implies that a.a.s. $|W| \leq (\varepsilon/4)np$. Moreover, since our assumption on p implies that $|X| \leq (\varepsilon/8)n$, we can apply Chernoff's inequality and deduce that a.a.s. $\deg(u, X) \leq (\varepsilon/4)np$ for every vertex u.

Let G' be the subgraph of G obtained by deleting all edges within X, all edges between X and W, and deleting edges incident to any $y \notin X \cup W$ according to the following rule – for every triangle xyz in G with $x \in X$ and $z \notin X \cup W$, remove the edge yz. It is quite easy to see that no vertex of X is contained in a triangle in G'. Let us now estimate $\delta(G')$. Since a vertex $u \in X \cup W$ lost only edges connecting it to X and W, we have

$$\deg_{G'}(u) > (1 - \varepsilon/2)np - \deg(u, X) - \deg(u, W) \geq (1 - \varepsilon/2)np - \deg(u, X) - |W| \geq (1 - \varepsilon)np.$$

Since a vertex $y \notin X \cup W$ is incident to at most $(\varepsilon/2)p^{-1}$ vertices $x \in X$ and it has at most $2np^2$ common neighbours with each such x, we then have

$$\deg_{G'}(y) > (1 - \varepsilon/2)np - (\varepsilon/4)p^{-1} \cdot 2np^2 \geq (1 - \varepsilon)np.$$

Thus G' has the required properties. □

We end this section with two propositions whose proofs are farily standard and are omitted. Proposition 4.7 asserts that in a typical random graph $G(n,p)$, the reduced graph of a regular partition of a subgraph $G \subset G(n,p)$ inherits the minimum degree condition that we impose on G. The final proposition, Proposition 4.8 can be proved using Lemma 2.6, and asserts that every regular triple in a random graph is typical.

Proposition 4.7. *Let $\gamma > 0$ and $p \gg n^{-1}$. There exist $\varepsilon_0(\gamma)$ and $\delta_0(\gamma)$ such that if $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$, then the following holds asymptotically almost surely. Given a subgraph G of $G(n,p)$, let $(V_i)_{i=1}^k$ be an (ε, p)-regular partition of G such that $|V_i| \leq \varepsilon n$ for all i and every part forms an (ε, p)-regular pair with at least $(1 - \varepsilon)k$ other parts. Let R be its (δ, ε, p)-reduced graph. If G has minimum degree at least $(2/3 + \gamma)np$, then R has minimum degree at least $(2/3 + \gamma/2)k$.*

Proposition 4.8. *Let $p \gg n^{-1/2}$. For all positive ε', δ, and ξ, there exists a constant $\varepsilon_0(\varepsilon', \delta)$ such that a.a.s. in $G(n,p)$, every copy of a graph from $\mathcal{G}(K_3, (n_1, n_2, n_3), (d_{12}, d_{23}, d_{31}), (\varepsilon, p))$ is ε'-typical provided that $\varepsilon \leq \varepsilon_0$, $d_{12}, d_{23}, d_{31} \geq \delta$, and $n_1, n_2, n_3 \geq \xi n$.*

5   Obtaining balanced super-regular triples

In Section 3.1, we mentioned that the process of absorbing exceptional vertices into a triangle packing may cause some regular triples in our graph to become slightly unbalanced. The following lemma describes a greedy procedure that finds a small triangle packing which restores the balance in each of these triples.

Lemma 5.1. *Let $p \gg n^{-1/2}$. For all positive reals δ, ε', and γ, there exists an $\varepsilon_0(\delta, \varepsilon', \gamma)$ such that if $\varepsilon < \varepsilon_0$, then the following holds asymptotically almost surely. Let G be a subgraph of $G(n,p)$ and let $V_1, \ldots, V_{3k}$ be disjoint subsets of $V(G)$ satisfying $|V_i| \in [(1-\varepsilon)m, (1+\varepsilon)m]$ for some $m = \Omega(n)$. Let R be a graph on the vertex set $[3k]$ of minimum degree at least $(2 + \gamma)k$ such that $\{3t - 2, 3t - 1, 3t\}$ forms a triangle for all $t \in [k]$ and assume that $(V_i)_{i=1}^{3k}$ is (δ, ε, p)-regular over R.*
    Then there exist subsets B and S of $V(G)$ such that the following holds.

(i) $|B| \leq 4k$,
(ii) $G[S]$ contains a perfect triangle packing,
(iii) $|V_i \cap (B \cup S)| \leq \varepsilon' m$ for all $i \in [3k]$, and
(iv) $V_i \setminus (B \cup S)$ have equal sizes for all i.

Proof. Let $\varepsilon_1 = \varepsilon_{2.7}(\frac{1}{2}, \delta)$ and $\varepsilon_0 = \min\{\frac{\varepsilon'}{4(3/\gamma + 1)}, \varepsilon_{4.8}(\varepsilon_1, \frac{\delta}{2}), \frac{\delta}{2}, \varepsilon_1\}$. Assume that $\varepsilon \in (0, \varepsilon_0)$ is given.
    Let $C_t = \{3t - 2, 3t - 1, 3t\}$ for $t \in [k]$ be triangles of the graph R. For each vertex $i \in [3k]$, call an index $t \in [k]$ *i-rich* or *rich with respect to i* if i is adjacent to all three vertices of C_t, and assume that there are g_i i-rich indices. Then by the minimum degree condition on R, we have

$$3g_i + 2(k - g_i) \geq (2 + \gamma)k,$$

which is equivalent to $g_i \geq \gamma k$. Thus for each vertex $i \in [3k]$ of R, we can assign an i-rich index $t \in [k]$ to it so that every index in $[k]$ is chosen by at most $(3k)/(\gamma k) = 3/\gamma$ vertices.

Consider the following process that adjusts the parts one by one. Throughout the process, we will maintain sets $B \subset V(G)$ and $Z_i \subset V_i$ for each $i \in [3k]$; they are empty at the beginning. Call a triangle C_t balanced if the sets $V_{3t-2} \setminus Z_{3t-2}$, $V_{3t-1} \setminus Z_{3t-1}$, and $V_{3t} \setminus Z_{3t}$ have equal cardinalities. Assume that the triangles $C_1, \ldots, C_{t-1}$ are already balanced and we are trying to balance the triangle C_t. Without loss of generality, we may assume that $|V_{3t-i} \setminus Z_{3t-i}| - |V_{3t} \setminus Z_{3t}| = x_i m$ and $0 \le x_i \le 2\varepsilon$ for $i \in \{1,2\}$. Thus we have to remove $x_i m$ vertices from V_{3t-i} for $i \in \{1,2\}$ in order to make C_t balanced. By moving at most 2 arbitrary vertices from each set V_{3t-1} and V_{3t-2} to B and also to Z_{3t-1} and Z_{3t-2}, respectively, we may assume that both $x_1 m$ and $x_2 m$ are divisible by 3. First consider the set V_{3t-1} and let s be the rich index with respect to $3t-1$ which we have chosen above. As we will later establish, for every $i \in [3k]$, $|V_i \setminus Z_i| \ge m/2$ throughout the process. Therefore by Proposition 2.1, the triple $(V_{3t-1} \setminus Z_{3t-1}, V_{3s-j} \setminus Z_{3s-j}, V_{3s-k} \setminus Z_{3s-k})$ inherits the regularity of $(V_{3t-1}, V_{3s-j}, V_{3s-k})$ and is always $(2\varepsilon, p)$-regular of density at least $\delta/2$ for every pair $\{j, k\} \subset \{0,1,2\}$. By Proposition 4.8, a.a.s. it must also be ε_1-typical. Thus by Proposition 2.7 we can find $x_1 n/3$ triangles across this triple. Do this for each pair $\{j, k\}$ and update the sets Z_{3t-1}, Z_{3s-2}, Z_{3s-1}, and Z_{3s} by placing all the vertices of these triangles into corresponding parts. Note that even though the sizes of the sets in C_s have decreased, the number by which they decreased is the same for all three of them and thus after performing the same procedure for V_{3t-2}, the triangles $C_1, \ldots, C_t$ will be balanced.

Note that in the end, $|B| \le 4k$. Moreover, throughout the process, by the restriction that every index is the chosen rich index for at most $3/\gamma$ other indices, we always have, $|Z_i| \le 2\varepsilon m \cdot (3/\gamma + 1) \le \min\{\varepsilon' m, m/2\}$ as claimed. Define $S = \left(\cup_{i=1}^{3k} Z_i \right) \setminus B$ and we have the sets B and S as claimed. $\square$

Below is the main theorem of this section. It says that we can partition our graph into balanced super-regular triples, a collection of vertex-disjoint triangles, and a set of at most $O(p^{-2})$ exceptional vertices. We would like to remark that the upper bound imposed on the sizes of the common neighbourhoods in (v) will come in handy in the proof of Theorem 6.6, where we show that the triples $(W_{3t-2}, W_{3t-1}, W_{3t})$ are close to being strong-super-regular, see Proposition 2.13.

Theorem 5.2. *For an arbitrary γ, there exist $\delta(\gamma)$ and $\varepsilon_0(\gamma)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist constants $C(\varepsilon), D(\varepsilon)$, and $\xi(\varepsilon)$ satisfying the following. If $p \ge C(\log n/n)^{1/2}$, then a.a.s. for every spanning subgraph $G' \subset G(n,p)$ with $\delta(G') \ge (2/3 + \gamma)np$, there exist a further subgraph $G'' \subset G'$ and a partition of $V(G)$ into sets B, S, and $(W_i)_{i=1}^{3k}$, where $k \le D$, such that*

(i) $|B| \le D p^{-2}$.
(ii) $G'[S]$ contains a perfect triangle packing.
(iii) $(W_{3t-2}, W_{3t-1}, W_{3t})$ is a (δ, ε, p)-super-regular triple in G'' for all $t \in [k]$.
(iv) $|W_{3t-2}| = |W_{3t-1}| = |W_{3t}| \ge \xi n$ for all $t \in [k]$.
(v) In the graph G'', for all $t \in [k]$, the endpoints of every edge in (W_{3t-2}, W_{3t-1}) have at most $4|W_{3t}|p^2$ common neighbours in W_{3t} and a similar statement holds for other choices of indices.

Proof. Given a γ, let $\delta = \delta_{4.7}(\gamma)$ and $\varepsilon_0 = \varepsilon_{4.7}(\gamma)$. Moreover, for a given $\varepsilon \in (0, \varepsilon_0)$, let $\varepsilon_1 \le \min\{\varepsilon/2, \varepsilon_{2.10}(\varepsilon, \delta), \delta/2\}$, $\varepsilon_2 \le \min\{\varepsilon_{4.5}^2/36^2, \varepsilon_1^2/400, \varepsilon_{5.1}(\delta, \varepsilon_1/3, \gamma/2)\}$, $\varepsilon_3 \le (1/27) \min\{\varepsilon_{4.8}(\varepsilon_2, \delta)^3, \varepsilon_2\}$.
Let $K = K_{2.3}(\varepsilon_3, 2, 1/\varepsilon_3)$, $\eta = \min\{\eta_{2.3}(\varepsilon_3, 2, 1/\varepsilon_3), 1\}$, $\xi = \varepsilon_3/(4K)$, $C = \max\{C_{4.1}(\eta), C_{4.5}\}$, and $D = \max\{3D_{4.5}, 18KD_{4.2(ii)}(1/3, \xi), 24K\}$.
Proposition 4.1 (iii) implies that $G(n,p)$ is a.a.s. $(\eta, 2, p)$-upper-uniform. Thus we can apply the regularity lemma, Theorem 2.3, to obtain an (ε_3, p)-regular partition $V_0, V_1, \ldots, V_{3k}$ of the graph G',

where each part forms a regular pair with at least $(3 - 3\varepsilon_3)k$ other parts. Let $m = |V_i|$, note that $m \geq n/(2K)$, and let R be the reduced graph with parameter δ. Since G' has minimum degree at least $(2/3 + \gamma)np$, by Proposition 4.7, a.a.s. the reduced graph has minimum degree at least $(2 + \gamma/2)k$. Thus by Theorem 2.16, we may assume that $(V_{3t-2}, V_{3t-1}, V_{3t})$ forms an (ε_3, p)-regular triple of density at least δ for all $t \in [k]$. By Proposition 4.4, a.a.s. there are at most $(\varepsilon_3/2)pm^2$ edges in (V_{3t-1}, V_{3t-2}) whose endpoints have more than $2|V_{3t}|p^2$ common neighbours in V_{3t}. Similar estimate holds for the edges in (V_{3t-2}, V_{3t}) and (V_{3t-1}, V_{3t}). Delete all such edges for all $t \in [k]$ to obtain the subgraph G''. Then in the graph G'', each triple $(V_{3t-2}, V_{3t-1}, V_{3t})$ is $(3\varepsilon_3^{1/3}, p)$-regular by Proposition 2.2.

By Proposition 4.8, we may assume that every $(3\varepsilon_3^{1/3}, p)$-regular triple of density at least δ is ε_2-typical. Thus for each index i, if we let $X_i \subset V_i$ be the collection of non ε_2-typical vertices, then $|X_i| \leq \varepsilon_2|V_i|$. Furthermore, for each $t \in [k]$, add to X_{3t} the collection of those vertices $v \in V_{3t}$ such that $\deg(v, V_{3t-j}) \neq (1 \pm \varepsilon_2)d_{3t,3t-j}p|V_{3t-j}|$ for some $j \in \{1, 2\}$ and define X_{3t-1} and X_{3t-2} accordingly (there are at most $4\varepsilon_2 n$ such vertices by regularity). By adding arbitrary vertices to X_i if necessary, we may assume that $|X_i| = 5\varepsilon_2|V_i|$. Move all the vertices in X_i from V_i to V_0 and denote the resulting partition by $(V_i')_{i=0}^{3k}$. We then have $|V_0'| \leq |V_0| + \sum_i |X_i| \leq 6\varepsilon_2 n$.

Consider the following process of finding triangles that absorbs the vertices in V_0'. Let T be the empty set; we will update it throughout the process. Apply Lemma 4.5 to find a triangle which hits V_0' but not T and move all the vertices of this triangle into T. If $|T \cap V_i'| \geq \sqrt{\varepsilon_2}|V_i'| - 3$ for some index $i \in [3k]$, then move all the vertices of V_i' into T. This way, we will have

$$|T| \leq 3|V_0'| \cdot (1/\sqrt{\varepsilon_2}) + 3|V_0'| \leq 36\sqrt{\varepsilon_2}n \leq \varepsilon_{4.5}n$$

throughout the process. Terminate the process when we cannot find such triangles anymore. Then, a.a.s. we must have $|V_0'| \leq (D/3)p^{-2}$. Let B_0 be the collection of all the remaining vertices of V_0', and S_0 be the set of vertices in the copies of the triangles that we found. Let $V_i'' = V_i' \setminus S_0 = V_i \setminus (B_0 \cup S_0)$ and note that for all i, since $|V_i \cap (B_0 \cup S_0)| \leq |X_i| + \sqrt{\varepsilon_2}|V_i| \leq (\varepsilon_1/6)|V_i|$, then $|V_i''| \geq (1 - \varepsilon_1/6)|V_i|$.

By Proposition 2.1, $(V_{3t-2}'', V_{3t-1}'', V_{3t}'')$ forms a $(2\varepsilon_3, p)$-regular triple of density at least $\delta - \varepsilon_3 \geq \delta/2$ for all $t \in [k]$. Apply Lemma 5.1 to $(V_i'')_{i=1}^{3k}$ to obtain sets B_1 and S_1. Observe that $|B_0 \cup B_1| \leq (D/3)p^{-2} + 4K \leq (2D/3)p^{-2}$ and $G[S_0 \cup S_1]$ contains a perfect triangle packing. Also, most crucially, if we let $BS = B_0 \cup S_0 \cup B_1 \cup S_1$ and $W_i = V_i \setminus BS = V_i'' \setminus (B_1 \cup S_1)$, then all W_i have equal sizes and moreover, $|W_i| \geq (1 - \varepsilon_1/6 - \varepsilon_1/3)m \geq (1 - \varepsilon_1/2)m$.

We will remove some vertices from each set W_i to make the triples $(W_{3t-2}, W_{3t-1}, W_{3t})$ super-regular for all $t \in [k]$. Since $\xi n \leq |X_i| \leq |BS \cap V_i| \leq (\varepsilon_1/2)m$ for all i, by Proposition 4.2 (ii), there are at most $(D/(18K))p^{-1}$ vertices which have more than $(2\varepsilon_1/3)pm$ neighbours in $BS \cap V_i$ for each fixed $i \in [3k]$. Let Y_1 be the collection of such vertices for the set $BS \cap V_2$ and $BS \cap V_3$ which lie in V_1 and similarly define Y_2, Y_3. By placing arbitrary vertices into $Y_1, Y_2,$ or Y_3 as necessary, we may assume that $|Y_1| = |Y_2| = |Y_3| \leq (D/(9k))p^{-1}$. Consider the set $W_i' = W_i \setminus Y_i$ for $i \in \{1, 2, 3\}$. Then since $|Y_i| + |BS \cap V_i| \leq (D/(9K))p^{-1} + (\varepsilon_1/2)m \leq \varepsilon_1 m$, in total we removed at most $\varepsilon_1 m$ vertices from each part of (V_1, V_2, V_3) to obtain (W_1', W_2', W_3'). By the definition of the sets X_i at the beginning, all the vertices in W_1' were ε_1-typical in the triple (V_1, V_2, V_3), and by the choice of Y_1, they have at most $(2\varepsilon_1/3)pm + |Y_2| \leq \varepsilon_1 pm$ neighbours in the deleted portion in V_2 (similar for V_3). Thus by Proposition 2.10, all the vertices in W_1' are ε-typical in the triple (W_1', W_2', W_3'). Also, since all the vertices of V_1 not in X_1 had $(1 \pm \varepsilon_1)d_{12}p|V_2|$ neighbours in V_2, they will still have $(1 \pm 2\varepsilon_1)d_{12}p|V_2|$ neighbours in W_2' and similar for other choices of indices. Moreover, the triple (W_1', W_2', W_3') inherits the regularity of (V_1, V_2, V_3) and is $(2\varepsilon_3, p)$-regular of density at least $\delta - \varepsilon_3 > \delta/2$, see Proposition 2.1.

Thus by the fact $2\varepsilon_1 < \varepsilon$ and $2\varepsilon_3 < \varepsilon$, (W_1', W_2', W_3') is $(\delta/2, \varepsilon, p)$-super-regular. Repeat the above process for all other triples. Let B be the union of B_0, B_1, and Y_i for all $i \in [3k]$ as above so that $|B| \leq (2D/3)p^{-2} + (D/3)p^{-1} = Dp^{-1}$ and $S = S_0 \cup S_1$. We also have the bound $|W_i'| \geq |V_i|/2 \geq \xi n$ for all i. Moreover, (v) will hold since in G'' all the edges between W_2' and W_3' have at most $2|V_i|p^2$ common neighbours in V_1, and therefore at most $4|W_1'|p^2$ in W_1' (similar for other indices). $\qquad\square$

6 Obtaining balanced strong-super-regular triples

In the previous section, we managed to decompose the graph into balanced super-regular triples, a triangle packing, and a small set of exceptional vertices. In this section, we will show how by slightly enlarging the triangle packing and the exceptional set, we can make these triples strong-super-regular.

Our main tool, which will also be used in the next section, is the following lemma, which constructs small quasi-random matchings in super-regular triples. For the application in this section, in Theorem 6.6 below, V_i''s will be the sets of non-good vertices in each part of a regular partition of the host graph. We want to find vertex disjoint triangles that cover these sets of non-good vertices. As an intermediate step, we construct random matchings M_{ij}' which later can be coupled with the non-good vertices in order to construct vertex-disjoint triangles. See the discussion in Section 3.2 for more detailed description. We would like to remark that even though the stronger assumption (A1) implies the weaker assumption (A2), we state both of them, as (A1) is much simpler and in one of the two applications of Lemma 6.1, we can verify that this stronger condition is satisfied. Also note that the statement of this lemma holds not only for strong-super-regular triples coming from subgraphs of random graphs, but also for general strong-super-regular triples.

Lemma 6.1. *For all positive δ and η with $\eta < 1/140$, there exist $\varepsilon(\delta)$ and $C(\delta, \eta)$ such that the following holds. Let (V_1, V_2, V_3) be a (δ, ε, p)-super-regular triple with $m = |V_1| = |V_2| = |V_3|$ and $p \geq C(\log m/m)^{1/2}$. For each i and j, let $d_{ij}p$ be the density of (V_i, V_j), let $q_{ij} = \eta/(d_{ij}pm)$, and let E_{ij} be a subgraph of (V_i, V_j) with $|E_{ij}| \leq \eta d_{ij}pm^2$. Form a set M_{ij}' by selecting every edge in $(V_i, V_j) \setminus E_{ij}$ independently with probability q_{ij} and let $M_{ij} \subset M_{ij}'$ be the set of all selected edges in (V_i, V_j) that are not incident to any other edge in $M_{12}' \cup M_{13}' \cup M_{23}'$. Moreover, for each i, let Q_i be the set of all vertices in V_i that are covered by some edge in $M_{12} \cup M_{13} \cup M_{23}$. Assume that for each i, j, and k, there is a set V_i' such that*

(A1) the neighbourhood of every $v \in V_i'$ contains at most $\eta d_{ij}d_{ik}d_{jk}p^3 m^2$ edges of E_{jk} or

(A2) for every $v \in V_i'$, every subgraph (N_j'', N_k'') of $(N(v) \cap V_j, N(v) \cap V_k)$ such that $\deg(w, N_k'') \leq 2d_{ik}d_{jk}p^2 m$ for all $w \in N_j''$ and $\deg(w, N_j'') \leq 2d_{ij}d_{jk}p^2 m$ for all $w \in N_k''$ contains at most $\eta d_{ij}d_{ik}d_{jk}p^3 m^2$ edges of E_{jk}.

Then $M_{12} \cup M_{13} \cup M_{23}$ is a matching and with probability tending to 1 as m tends to infinity, for each i, j, and k,

(M1) $(\eta/2)m \leq |M_{ij}| \leq 2\eta m$,

(M2) every $v \in V_i$ has at most $3\eta d_{ij}pm$ neighbours in Q_j,

(M3) the neighbourhood of every $v \in V_i'$ contains at least $(\eta/2)\delta^2 p^2 m$ edges of M_{jk}, and

(M4) the endpoints of each η-good edge in (V_j, V_k) have at least $(1 - 4\eta)d_{ij}d_{ik}p^2m$ common neighbours in $V_i \setminus Q_i$.

Proof. Let $\varepsilon = \min\{1/100, \delta/50\}$. Fix i, j, and k with $\{i, j, k\} = \{1, 2, 3\}$. For each vertex $v \in V_i$, let $N_j = N(v) \cap V_j$ and $N_k = N(v) \cap V_k$. By construction, $M_{12} \cup M_{13} \cup M_{23}$ is a matching.

Claim 6.2. *With probability* $1 - o(1)$, $(\eta/2)m \leq |M_{ij}| \leq 2\eta m$.

Proof. By our assumption on $|E_{ij}|$, there are at least $(1 - \eta)d_{ij}pm^2$ edges in $(V_i, V_j) \setminus E_{ij}$, so $\mathbb{E}[|M'_{ij}|] \geq (1 - \eta)d_{ij}pm^2q_{ij} = (1 - \eta)\eta m$, and Chernoff's inequality implies that $|M'_{ij}| \geq (3\eta/4)m$ with probability $1 - o(1)$. In order to estimate $|M_{ij}|$, note that $|M'_{ij}| - |M_{ij}|$ is at most the number of vertices in $V_i \cup V_j$ that are incident to an edge of M'_{ij} and some other edge in $M'_{12} \cup M'_{13} \cup M'_{23}$. Let $\mathcal{A}_w$ denote the event that w is such a "bad" vertex. Since (V_1, V_2, V_3) is (δ, ε, p)-super-regular, $\deg(v, V_j) \leq (1 + \varepsilon)d_{ij}pm$ and $\deg(v, V_k) \leq (1 + \varepsilon)d_{ik}pm$ for every $v \in V_i$. Hence, if $w \in V_i$, then

$$P(\mathcal{A}_w) \leq \deg(w, V_j)q_{ij} \cdot (\deg(w, V_j)q_{ij} + \deg(w, V_k)q_{ik}) \leq (1 + \varepsilon)^2 2\eta^2$$

and the expected number of such "bad" vertices in V_i is at most $(1 + \varepsilon)^2 2\eta^2 m$. The events $\{\mathcal{A}_w \colon w \in V_i\}$ are mutually independent, so by Chernoff's inequality, with probability at least $1 - o(m^{-1})$, there are at most $3\eta^2 m$ "bad" vertices in V_i and similarly, there are at most $3\eta^2 m$ "bad" vertices in V_j. Hence, $|M_{ij}| \geq (3/4 - 6\eta)\eta m \geq (\eta/2)m$ with probability $1 - o(1)$. Finally, since the number of edges in $(V_i, V_j) \setminus E_{ij}$ is at most $d_{ij}pm^2$, we have $\mathbb{E}[|M'_{ij}|] \leq d_{ij}pm^2q_{ij} = \eta m$, and Chernoff's inequality implies that $|M_{ij}| \leq |M'_{ij}| \leq 2\eta m$ with probability $1 - o(1)$. $\qquad\square$

Claim 6.3. *For each fixed vertex* v, *with probability* $1 - o(m^{-1})$, *we have* $\deg(v, Q_j) \leq 3\eta d_{ij}pm$.

Proof. Let Q'_i be the set of vertices in V_i that are covered by some edge in $M'_{ij} \cup M'_{ik}$ and note that $Q'_i \supset Q_i$ (similarly define Q'_j and Q'_k). For a vertex $w \in V_j$, let $\mathcal{B}_w$ denote the event that $w \in Q'_j$. Since (V_1, V_2, V_3) is (δ, ε, p)-super-regular, $\deg(w, V_i) \leq (1 + \varepsilon)d_{ij}pm$ and $\deg(w, V_k) \leq (1 + \varepsilon)d_{jk}pm$. Hence,

$$P(\mathcal{B}_w) \leq \deg(w, V_j)q_{ij} + \deg(w, V_k)q_{ik} \leq 2(1 + \varepsilon)\eta.$$

The events $\{\mathcal{B}_w \colon w \in V_j\}$ are mutually independent and $|N_j| \geq (\delta/2)pm$, so by Chernoff's inequality, $|N_j \cap Q'_j| \leq (5\eta/2)|N_j|$ with probability at least $1 - e^{-c\eta\delta pm}$ for some absolute positive constant c. It follows that

$$\deg(v, Q_j) \leq \deg(v, Q'_j) = |N_j \cap Q'_j| \leq (5\eta/2)(1 + \varepsilon)d_{ij}pm \leq 3\eta d_{ij}pm$$

with probability $1 - o(m^{-1})$. $\qquad\square$

Claim 6.4. *For each fixed* $v \in V'_i$, *with probability* $1 - o(m^{-1})$, *the pair* (N_j, N_k) *contains at least* $(\eta/2)d_{ij}d_{ik}p^2m$ *edges of* M_{jk}.

Proof. Without loss of generality, we may assume that $(i, j, k) = (1, 2, 3)$. Since v is ε-typical, $(1 + \varepsilon^2)(1 + 2\varepsilon/\delta) \leq 2$, and $5\varepsilon + 2\varepsilon/\delta \leq 1/7$, Proposition 2.14 implies that there are sets $N''_2 \subset N_2$ and $N''_3 \subset N_3$ such that (N''_2, N''_3) contains at least $(6/7)d_{12}d_{13}d_{23}p^3m^2$ edges, no vertex in N''_2 has more than $2d_{13}d_{23}p^2m$ neighbours in N''_3, and vice versa, no vertex in N''_3 has more than $2d_{12}d_{23}p^2m$

neighbours in N_2''. Since at most $\eta d_{12}d_{13}d_{23}p^3m^2$ edges among (N_2'', N_3'') belong to E_{23} by either (A1) or (A2), it follows that

$$\mathbb{E}[|M_{23}' \cap (N_2'', N_3'')|] \geq (5/7)d_{12}d_{13}d_{23}p^3m^2q_{23} = (5/7)d_{12}d_{13}\eta p^2m.$$

Since $|M_{23}' \cap (N_2'', N_3'')|$ is a sum of independent indicator random variables, Chernoff's inequality implies that for some absolute constant c,

$$P\left(|M_{23}' \cap (N_2'', N_3'')| \geq (4/7)d_{12}d_{13}\eta p^2m\right) \geq 1 - e^{-c\eta\delta^2p^2m} \geq 1 - 1/m^2,$$

provided that C is sufficiently large.

In order to estimate $|M_{23} \cap (N_2'', N_3'')|$, note that $|M_{23}' \cap (N_2'', N_3'')| - |M_{23} \cap (N_2'', N_3'')|$ is at most the number of vertices in $N_2'' \cup N_3''$ that are incident to an edge in $M_{23}' \cap (N_2'', N_3'')$ and some other edge in $M_{12}' \cup M_{13}' \cup M_{23}'$. Let $\mathcal{C}_w$ denote the event that w is such a "bad" vertex. If $w \in N_2''$, then

$$P(\mathcal{C}_w) \leq \deg(w, N_3'')q_{23} \cdot (\deg(w, V_3)q_{23} + \deg(w, V_1)q_{12})$$
$$\leq 2d_{13}d_{23}p^2mq_{23} \cdot ((1+\varepsilon)d_{23}pmq_{23} + (1+\varepsilon)d_{12}pmq_{12}) = (1+\varepsilon)4\eta^2d_{13}\eta^2p.$$

Since $|N_2''| \leq |N_2| \leq (1+\varepsilon)d_{12}pm$, the expected number of such "bad" vertices in N_2'' is at most $(1+\varepsilon)^24\eta^2d_{12}d_{23}p^2m$. The events $\{\mathcal{C}_w : w \in N_2''\}$ are mutually independent, so by Chernoff's inequality, for some absolute constant c, with probability at least $1 - e^{-c\delta^2\eta^2p^2m}$, there are at most $5\eta^2d_{12}d_{13}p^2m$ "bad" vertices in N_2'' and similarly, there are at most $5\eta^2d_{12}d_{13}p^2m$ "bad" vertices in N_3''. Hence, with probability $1 - o(m^{-1})$,

$$|M_{23} \cap (N_2'', N_3'')| \geq (4/7 - 10\eta)d_{12}d_{13}\eta p^2m \geq (\eta/2)d_{12}d_{13}p^2m,$$

provided that C is sufficiently large. $\qquad\square$

Claim 6.5. *With probability* $1 - o(1)$*, the endpoints of every η-good edge in (V_j, V_k) have at least* $(1 - 4\eta)d_{ij}d_{ik}p^2m$ *common neighbours in $V_i \setminus Q_i$.*

Proof. For an arbitrary vertex $v \in V_i$, let $\mathcal{D}_v$ denote the event that $v \in Q_i'$. Clearly,

$$P(\mathcal{D}_v) \leq \deg(v, V_j)q_{ij} + \deg(v, V_k)q_{ik} \leq 2(1+\varepsilon)\eta.$$

Fix some η-good edge in (V_j, V_k) and let $A \subset V_i$ be the set of common neighbours of its endpoints. Then $\mathbb{E}[|A \cap Q_i'|] = \sum_{v \in A} P(\mathcal{D}_v) \leq |A| \cdot 2(1+\varepsilon)\eta$. Moreover by definition, $|A| \geq (1-\eta)d_{ij}d_{ik}p^2m \geq (1/2)\delta^2p^2m$. Since the events $\mathcal{D}_v$ are mutually independent, Chernoff's inequality implies that

$$P(|A \cap Q_i'| \geq 3\eta|A|) \leq e^{-c\delta^2\eta p^2m}$$

for some absolute positive constant c. Hence, if C is sufficiently large, then with probability at least $1 - 1/m^3$,

$$|A \setminus Q_i| \geq |A \setminus Q_i'| \geq (1 - 3\eta)|A| \geq (1 - 3\eta)(1 - \eta)d_{ij}d_{ik}p^2m \geq (1 - 4\eta)d_{ij}d_{ik}p^2m.$$

Since there are at most m^2 good edges, the claim is proved. $\qquad\square$

Finally, note that Claims 6.2–6.5 imply that (M1)–(M4) are satisfied with probability $1 - o(1)$ (one needs to apply the union bound over all choices of vertices in order to deduce (M2) and (M3) from 6.3 and 6.4). $\qquad\square$

Below is the main theorem of this section. It says that we can partition our graph into balanced strong-super-regular triples, a collection of vertex-disjoint triangles, and a set of at most $O(p^{-2})$ exceptional vertices. In the next section, we will prove that each of those strong-super-regular triples contains a triangle-factor.

Theorem 6.6. *For an arbitrary positive γ, there exists a positive δ such that for all ε, there exist constants C, D, and ξ that satisfy the following. If $p \geq C(\log n/n)^{1/2}$, then a.a.s. every $G \subset G(n,p)$ with $\delta(G) \geq (2/3 + \gamma)np$ contains a subgraph $G' \subset G$ whose vertex set can be partitioned into sets B, S, and $(W'_i)_{i=1}^{3k}$, where $k \leq D$, such that*

(i) $|B| \leq Dp^{-2}$,
(ii) $G[S]$ contains a perfect triangle packing,
(iii) $(W'_{3t-2}, W'_{3t-1}, W'_{3t})$ is a (δ, ε, p)-strong-super-regular in G' for all $t \in [k]$, and
(iv) $|W'_{3t-2}| = |W'_{3t-1}| = |W'_{3t}| \geq \xi n$ for all $t \in [k]$.

Proof. Let $\delta = \min\{\delta_{5.2}(\gamma)/2, 3/4\}$. Without loss of generality, we may assume that $\varepsilon \leq 2/\delta$. Furthermore, let $\varepsilon_3 = \varepsilon_{2.10}(\varepsilon, 2\delta)$, $\varepsilon_1 = \min\{\varepsilon\delta^2/1180, \varepsilon_3\delta^2/40\}$, $\varepsilon_2 = \min\{\varepsilon_{2.13}(\varepsilon_1, 2\delta), \varepsilon_{5.2}(\gamma), \varepsilon_{6.1}(\delta), \varepsilon/2\}$, and $\xi = \xi_{5.2}(\varepsilon_2)/2$. Let $\eta = 12\varepsilon_1/\delta^2$. Moreover, let

$$D = (6/\xi)\max\{D_{5.2(\varepsilon_2)}, D_{4.2(ii)}(1/4, \varepsilon_1\xi), D_{4.4}(\varepsilon_1\varepsilon_2\delta^3\xi^2/4, 1/4)\}$$

and $C = \max\{C_{5.2}(\varepsilon_2), C_{4.3}(3\varepsilon_1\xi)\}$.

By Theorem 5.2, there exists a further subgraph G' of G whose vertex set can be partitioned into sets B_0, S_0, and $(W_i)_{i=1}^{3k}$ such that $|B_0| \leq (D/2)p^{-2}$, $G[S_0]$ contains a perfect triangle packing, and for all $t \in \{1, \ldots, k\}$, the triple $(W_{3t-2}, W_{3t-1}, W_{3t})$ is $(2\delta, \varepsilon_2, p)$-super-regular in G' and satisfies $|W_{3t-2}| = |W_{3t-1}| = |W_{3t}|$. Moreover, the endpoints of no edge in (W_{3t-2}, W_{3t-1}) have more than $4|W_{3t}|p^2$ common neighbours in W_{3t} (and a similar statement holds for other choices of indices). We will show that each such triple contains a slightly smaller (δ, ε, p)-strong-super-regular triple in such a way that all but at most $O(p^{-1})$ leftover vertices can be covered by vertex-disjoint triangles. Obviously, this will imply the assertion of the theorem.

Without loss of generality, we will only consider the triple (W_1, W_2, W_3). For the sake of brevity, let $m = |W_1| = |W_2| = |W_3|$ and note that $m \geq 2\xi n$. Without loss of generality, we can condition on the event that $G(n, p)$ satisfies the assertions of

- Proposition 4.2 (ii) with $\rho = 1/4$ and $\xi = \varepsilon_1 m/n$,

- Proposition 4.4 with $\varepsilon = \varepsilon_1\varepsilon_2\delta^3 m^2/(4n^2)$, and $\rho = 1/4$, and

- Proposition 4.3 with $\xi = 6\varepsilon_1 m/(2n)$.

For each i, let $X_i \subset W_i$ be the collection of vertices that are not ε_1-good. By Proposition 2.13, $|X_i| \leq \varepsilon_1 m$ and we may assume that $|X_i| = \varepsilon_1 m$. We perform the following cleaning-up procedure. While constantly updating the sets X_1, X_2, and X_3, repeat the following. If there exists an i and a vertex $v \in W_i \setminus X_i$ such that either

(A) $|N(v) \cap X_j| \geq 4\varepsilon_1 pm$ for some j or

(B) the neighbourhood of v contains more than $\varepsilon_2 d_{12}d_{13}d_{23}p^3 m^2$ edges whose endpoints have more than $5\varepsilon_1 p^2 m$ common neighbours in X_i,

then move v to X_i.

Claim 6.7. *The cleaning-up procedure finishes with $\varepsilon_1 m \leq |X_i| \leq 3\varepsilon_1 m$ for all i.*

Proof. Suppose that at some point in time, $|X_i| > 3\varepsilon_1 m$ for some i, and consider the earliest such moment. Without loss of generality, we may assume that $i = 1$. Clearly, $3\varepsilon_1 m < |X_1| \leq 4\varepsilon_1 m$ and $|X_j| \leq 3\varepsilon_1 m$ if $j \neq 1$. Since at the beginning, every X_1 contained at most $\varepsilon_1 m$ vertices, W_1 contains either $\varepsilon_1 m$ vertices satisfying (A) or $\varepsilon_1 m$ vertices satisfying (B). The former is impossible, since $|X_j| \leq 3\varepsilon_1 m$ for $j \neq 1$ and we assumed that $G(n,p)$ satisfies the assertion of Proposition 4.2 (ii) with $\xi = \varepsilon_1 m/n$. Since the endpoints of each edge in (W_2, W_3) have at most $4p^2 m$ common neighbours in W_1, the latter would imply that (W_2, W_3) contains

$$(\varepsilon_1 m) \cdot (\varepsilon_2 d_{12}d_{13}d_{23}p^3 m^2)/(4p^2 m) \geq (\varepsilon_1 \varepsilon_2 \delta^3/4)pm^2$$

edges whose endpoints have more than $5\varepsilon_1 p^2 m$ common neighbours in X_1. Since $|X_1| \leq 4\varepsilon_1 m$, this is impossible by our assumption that $G(n,p)$ satisfies the assertion of Proposition 4.4 with $\varepsilon = \varepsilon_1 \varepsilon_2 \delta^3 m^2/(4n^2)$, and $\rho = 1/4$. $\qquad\square$

It is not hard to check that $(W_1 \setminus X_1, W_2 \setminus X_2, W_3 \setminus X_3)$ is $(\delta/2, \varepsilon', p)$-strong-super-regular. Unfortunately, this conclusion does not help us at the moment as we first need to absorb $X_1 \cup X_2 \cup X_3$ into vertex-disjoint triangles and in the process of absorbing those vertices, we may use some vertices from the triple $(W_1 \setminus X_1, W_2 \setminus X_2, W_3 \setminus X_3)$.

For every i, let $Y_i \subset X_i$ be the set of vertices in X_i that have more than $4\varepsilon_1 pm$ neighbours in X_j for some j with $j \neq i$. Since $|X_j| \leq 3\varepsilon_1 m$ and we assumed that $G(n,p)$ satisfies the assertion of Proposition 4.2 (ii) with $\xi = \varepsilon_1 m/n$, then $|Y_i| \leq Dp^{-1}/(6k)$. By adding arbitrary vertices of X_i to Y_i, we can guarantee that $|Y_1| = |Y_2| = |Y_3|$. For every i and j, let $E_{ij} = (X_i, W_j) \cup (W_i, X_j)$. Since (W_i, W_j) is (ε_2, p)-regular and $|X_i| \geq \varepsilon_1 |W_i| \geq \varepsilon_2 |W_i|$, we have

$$|E_{ij}| \leq (d_{ij} + \varepsilon)p(|X_i||W_j| + |W_i||X_j|) \leq (1 + \varepsilon/\delta)6\varepsilon_1 d_{ij}pm^2 \leq \eta d_{ij}pm^2.$$

Fix a vertex $v \in W_i \setminus Y_i$. We check that (A2) in Lemma 6.1 is satisfied. Let (N_j'', N_k'') be as in (A2) in Lemma 6.1. Since $|N_j'' \cap X_j| \leq |N(v) \cap X_j| \leq 4\varepsilon_1 pm$ and similarly, $|N_k'' \cap X_k| \leq 4\varepsilon_1 pm$, we have

$$|E_{jk} \cap (N_j'', N_k'')| \leq 4\varepsilon_1 pm(2d_{ij}d_{jk}p^2 m + 2d_{ik}d_{jk}p^2 m) \leq (16\varepsilon_1/\delta)d_{ij}d_{ik}d_{jk}p^3 m^2 \leq \eta d_{ij}d_{ik}d_{jk}p^3 m^2.$$

Lemma 6.1 implies that a.a.s. for each i and j, there exists an $M_{ij} \subset (W_i \setminus X_i, W_j \setminus X_j)$ such that (M1)–(M4) in Lemma 6.1 are satisfied with $V_i = W_i$ and $V_i' = W_i \setminus Y_i$ for each i. Let Q_i be defined as in Lemma 6.1.

Claim 6.8. *The sets $X_1 \setminus Y_1$, $X_2 \setminus Y_2$, and $X_3 \setminus Y_3$ can be covered by vertex-disjoint triangles that use only vertices in $Q_1 \cup X_1$, $Q_2 \cup X_2$, and $Q_3 \cup X_3$.*

Proof. Since $M_{12} \cup M_{13} \cup M_{23}$ is a matching whose edges are not incident to any vertex in $X_1 \cup X_2 \cup X_3$, it suffices to show that for each i, j, and k, the vertices of $X_i \setminus Y_i$ can be paired with some $|X_i \setminus Y_i|$ edges of M_{jk} to form vertex-disjoint triangles.

Let H be the bipartite graph on the vertex set $(X_i \setminus Y_i) \cup M_{jk}$, where a vertex $w \in X_i \setminus Y_i$ is adjacent to an edge $\{u, v\} \in M_{jk}$ if and only if $\{u, v, w\}$ is a triangle in (W_1, W_2, W_3). Clearly, it suffices to prove that H contains a matching that covers $X_i \setminus Y_i$. We check that Hall's condition holds in H. Fix an arbitrary non-empty set $S \subset X_i \setminus Y_i$. If $|N_H(S)| \leq |S|$, then there would be an x with $1 \leq x = |S| \leq |X_i| \leq 3\varepsilon_1 m$ such that $G(n, p)$ contains some x independent edges and x vertices, each of which is adjacent to both ends of at least $(\eta/2)\delta^2 p^2 m$ of those edges, see (M3) in Lemma 6.1. This would contradict our assumption that $G(n, p)$ satisfies the assertion of Proposition 4.3 with $\xi = \eta\delta^2 m/(2n) = 6\varepsilon_1 m/(2n)$. Hence, $|N_H(S)| > |S|$ for all non-empty $S \subset X_i \setminus Y_i$. $\qquad\square$

Fix any such triangle packing and for each i, let $X_i' = X_i \cup T_i$, where $T_i \subset Q_i$ is the set of vertices in $W_i \setminus X_i$ that are covered by the triangle packing. Note that $|T_i| = |X_j \setminus Y_j| + |X_k \setminus Y_k|$. Let $W_i' = W_i \setminus X_i'$. Since for each i, $|X_i'| = |X_i| + |T_i| = |X_1 \setminus Y_1| + |X_2 \setminus Y_2| + |X_3 \setminus Y_3| + |Y_i|$ and $|Y_1| = |Y_2| = |Y_3|$, the sets W_1', W_2', and W_3' have the same number of elements. Denote this number by m' and note that $m' \geq m - 9\varepsilon_1 m \geq m/2 \geq \xi n$.

Claim 6.9. *The triple (W_1', W_2', W_3') is (ε, δ, p)-strong-super-regular.*

Proof. Since (W_1, W_2, W_3) is $(\varepsilon/2, p)$-regular with density at least $2\delta p$ and $m' \geq m/2$, Proposition 2.1 implies that (W_1', W_2', W_3') is (ε, p)-regular with density at least δp. Fix an index i, recall that $|X_i'| \leq 9\varepsilon_1 m \leq \varepsilon_3 m$, and let v be an arbitrary vertex in W_i'. Without loss of generality, we may assume that $i = 1$. Since $v \notin X_1$, (A) implies that $\deg(v, X_j) \leq 4\varepsilon_1 pm$ for every j. Moreover, (M2) in Lemma 6.1 implies that $\deg(v, Q_j) \leq 3\eta d_{ij} pm$. Hence,

$$\deg(v, X_j') \leq \deg(v, X_j) + \deg(v, T_j) \leq \deg(v, X_j) + \deg(v, Q_j) \leq \varepsilon_3 pm,$$

and by Proposition 2.10, v becomes ε-typical in (W_1', W_2', W_3'). It remains to show that v is also ε-good. Since $v \notin X_i$, it was ε_1-good in (W_1, W_2, W_3) and it satisfies (B). Hence, the endpoints of all but at most $(\varepsilon_1 + \varepsilon_2)d_{12}d_{13}d_{23}p^3 m^2$ edges in the neighbourhood of v have at least $(1 - \varepsilon_1 - 5\varepsilon_1/\delta^2)d_{12}d_{13}p^2 m$ common neighbours in $W_1 \setminus X_1$. Moreover by (M4), they have at most $4\eta d_{12}d_{13}p^2 m$ common neighbours in Q_1. Since $1 - \varepsilon_1 - 5\varepsilon_1/\delta^2 - 4\eta \geq (1 + \varepsilon_2/\delta)^2(1 - \varepsilon)$, each such edge is ε-good in the new triple. It follows that v is ε-good. $\qquad\square$

Finally, let $B = B_0 \cup \bigcup_{i=1}^{3k} Y_i$ and let $S = S_0 \bigcup_{i=1}^{3k}(T_i \cup (X_i \setminus Y_i))$. Clearly, the sets B, S, and $(W_i')_{i=1}^{3k}$ partition the vertex set of G,

$$|B| \leq |B_0| + \sum_{i=1}^{3k} |Y_i| \leq Dp^{-2}/2 + 3k \cdot Dp^{-1}/(6k) \leq Dp^{-2}/2,$$

and $G[S]$ contains a perfect triangle packing. Finally, by Claim 6.9, for each $t \in [k]$, the triple $(W_{3t-2}', W_{3t-1}', W_{3t}')$ is (ε, δ, p)-strong-super-regular and satisfies $|W_{3t-2}'| = |W_{3t-1}'| = |W_{3t}'| \geq \xi n$. $\qquad\square$

7 Perfect triangle packing in strong-super-regular triples

In the previous section, we managed to decompose the graph into balanced strong-super-regular triples, a triangle packing, and a small set of exceptional vertices. In this section, we will show how to find a triangle-factor in each of those triples. We start this section by showing how to construct sets of "buffer" vertices and edges that will allow us to complete an almost-spanning triangle packing into a triangle-factor.

Lemma 7.1. *For all positive constants δ, ξ, and η with $\eta \leq 1/140$, there exist constants $C(\delta, \eta, \xi)$ and $\varepsilon(\delta, \eta)$ such that if $p \geq C(\log n/n)^{1/2}$, then $G(n, p)$ a.a.s. satisfies the following. Let (W_1, W_2, W_3) be a (δ, ε, p)-strong-super-regular triple in a subgraph of $G(n, p)$ such that $|W_1| = |W_2| = |W_3| \geq \xi n$. Then there exist edge sets M_{12}, M_{13}, M_{23} and vertex sets X_1, X_2, X_3 with the following properties:*

(P1) $M_{12} \cup M_{13} \cup M_{23}$ *is a matching.*

(P2) *For all j and k, $M_{jk} \subset (W_j, W_k)$ and $(\eta/2)|W_j| \leq |M_{jk}| \leq 2\eta|W_j|$.*

(P3) *For all i, $|X_i| \leq (\eta/4)|W_i|$ and $X_i \subset W_i \setminus Q_i$, where Q_i is the set of vertices in W_i that are covered by some edge in $M_{ij} \cup M_{ik}$.*

(P4) *For all i, j, and k, if $Z_i \subset W_i$ has size $|M_{jk}|$ and contains X_i, then the subgraph of (W_1, W_2, W_3) induced by Z_i and M_{jk} contains a triangle-factor.*

Proof. For the sake of brevity, let $m = |W_1| = |W_2| = |W_3|$. Without loss of generality, we may assume that $\delta \leq 1$. Let $\alpha = \delta^2/24$ and let β be a positive constant satisfying $\beta \log(e/\beta) < \alpha\eta/30$. Moreover, let $\varepsilon = \min\{\delta/2, \eta/4, \varepsilon_{6.1}(\delta), \beta \cdot (\varepsilon_0)_{4.8}(\alpha\delta/16, \delta/2), \alpha\beta\delta/16\}$ and let $\varepsilon' = 4\varepsilon/\delta$. Finally, let C be sufficiently large so that $C(\log n/n)^{1/2} \geq C_{6.1}(\log m/m)^{1/2}$ and without loss of generality we may assume that $G(n, p)$ satisfies the assertion of Proposition 4.3 with $\xi = \eta\delta^2 m/(2n)$ and $\xi = \eta\delta^2 m/(12n)$, and Proposition 4.8 with $\varepsilon'_{4.8} = \alpha\delta/16$ and $\delta_{4.8} = \delta/2$.

For all i and j, let $q_{ij} = \eta/(d_{ij}mp)$ and select each ε'-good edge of (W_i, W_j) independently with probability q_{ij}. Let M'_{ij} be the set of all selected edges in (W_i, W_j) and let $M_{ij} \subset M'_{ij}$ be the set of all those edges that are not incident to any other selected edge. By Proposition 2.12, (W_i, W_j) contains at most $\eta d_{ij}pm^2$ edges that are not ε'-good. Since each $v \in W_i$ is ε-good, its neighbourhood contains at most $\eta d_{12}d_{13}d_{23}p^3m^2$ edges that are not ε'-good. Therefore, Lemma 6.1 applies with E_{ij} being the set of non-ε'-good edges in (W_i, W_j) and $V_i = V'_i = W_i$.

Claim 7.2. *With probability $1 - o(1)$, every set $Y_i \subset W_i$ of size βm satisfies the following. All but at most $\alpha\eta m$ edges of M_{jk} belong to the neighbourhood of some vertex of Y_i.*

Proof. Fix a $Y_i \subset W_i$ of size βm. By Proposition 2.1, the triple (Y_i, W_j, W_k) is $(\varepsilon/\beta, p)$-regular, and the densities of all three of its parts are at least $\delta/2$. Moreover, since we assumed that $G(n, p)$ satisfies the assertion of Proposition 4.8, (Y_i, W_j, W_k) is $\alpha\delta/16$-typical and by our assumption on ε, it is $(\alpha\delta/16, p)$-regular. By Proposition 2.12, all but at most $(\alpha/2)d_{23}pm^2$ edges between W_j and W_k are $\alpha/2$-good, so in particular all but at most $(\alpha/2)d_{23}pm^2$ edges in (W_j, W_k) belong to the neighbourhood of some vertex in Y_i. Hence the expected number of edges chosen among those "bad" edges is at most $(\alpha/2)d_{23}pm^2q_{23} = (\alpha\eta/2)m$. Chernoff's inequality implies that the probability that more than $\alpha\eta m$

of those edges are chosen to M'_{jk} is at most $e^{-\alpha\eta m/30}$. Since there are $\binom{m}{\beta m}$ βm-subsets of W_i,

$$\binom{m}{\beta m} \leq \left(\frac{em}{\beta m}\right)^{\beta m} = e^{\beta\log(e/\beta)m},$$

and $\alpha\eta/30 > \beta\log(e/\beta)$, the probability that all sets Y_i have the claimed property is $1 - o(1)$. $\qquad \square$

Lemma 6.1 and Claim 7.2 imply that there exist M_{12}, M_{13}, and M_{23} such that $M_{12} \cup M_{13} \cup M_{23}$ is a matching and for all i, j, and k (properties 1, 2, and 3 follow from (M1), (M3), and (M4) of Lemma 6.1, respectively, whereas property 4 follows from Claim 7.2):

1. $(\eta/2)m \leq |M_{jk}| \leq 2\eta m$,

2. the neighbourhood of every vertex in V_i contains at least $(\eta/2)\delta^2 p^2 m$ edges of M_{jk},

3. the endpoints of each edge of M_{jk} have at least $(1/2)\delta^2 p^2 m$ common neighbours in $W_i \setminus Q_i$,

4. for every set $Y_i \subset W_i$ of size βm, all but at most $\alpha\eta m$ edges of M_{jk} belong to the neighbourhood of some vertex of Y_i.

Fix any such M_{12}, M_{13}, and M_{23}. Next, for each i, let X_i be a random binomial subset of $W_i \setminus Q_i$, where each element is included with probability $\eta/5$. A simple application of Chernoff's inequality combined with Property 3 above shows that if C is sufficiently large, then with probability $1 - o(1)$, for all i, j, and k:

5. $|X_i| \leq (\eta/4)m$,

6. the endpoints of each edge of M_{jk} have at least $(\eta/12)\delta^2 p^2 m$ common neighbours in X_i.

Let X_1, X_2, and X_3 be arbitrary sets satisfying 5 and 6 and note that properties (P1)–(P3) are satisfied. It remains to show that (P4) is also satisfied.

Fix a $Z_i \subset W_i$ of size $|M_{jk}|$ such that $X_i \subset Z_i$. Let H be the bipartite graph on the vertex set $Z_i \cup M_{jk}$, where a vertex $w \in Z_i$ is adjacent to an edge $\{u, v\} \in M_{jk}$ if and only if $\{u, v, w\}$ is a triangle in (W_1, W_2, W_3). Clearly, it suffices to prove that H contains a perfect matching. We check that H satisfies the assumptions of Proposition 2.15 with $A = Z_i$, $B = M_{jk}$, and $L = \alpha\eta m$.

Fix an $S \subset Z_i$. If $0 < |S| \leq (\eta/4)\delta^2 m$, then $|N_H(S)| > |S|$ or otherwise there would be an $x \in [1, (\eta/4)\delta^2 m]$ such that $G(n, p)$ contains some x independent edges and x vertices, each of which is adjacent to both ends of at least $(\eta/2)\delta^2 p^2 m$ of those edges, see 2. This would contradict our assumption that $G(n, p)$ satisfies the assertion of Proposition 4.3 with $\xi = \eta\delta^2 m/(2n)$. On the other hand, if $|S| \geq (\eta/4)\delta^2 m \geq \beta m$, then by 4, $|M_{jk} \setminus N_H(S)| \leq \alpha\eta m$. Hence, $|N_H(S)| \geq |S|$ as long as $|Z_i \setminus S| \geq \alpha\eta m$

Finally, fix a $T \subset M_{jk}$ with $0 < |T| \leq \alpha\eta m = (\eta/24)\delta^2 m$. If $|N_H(T)| < |T|$, then there would be an $x \in [1, (\eta/24)\delta^2 m]$ such that $G(n, p)$ contains x vertices and x independent edges whose endpoints have at least $(\eta/12)\delta^2 p^2 m$ common neighbours among those x vertices (recall that $X_i \subset Z_i$), see 6. This would contradict our assumption that $G(n, p)$ satisfies the assertion of Proposition 4.3 with $\xi = \eta\delta^2 m/(12n)$. $\qquad \square$

With Lemma 7.1 at hand, without much effort we can prove Theorem 1.4 which says that a balanced strong-super-regular triple has a triangle-factor.

Proof of Theorem 1.4. Let $\eta = 1/140$, $\varepsilon_1 = \varepsilon_{2.7}(3\eta/2, \delta/2)$, and $\varepsilon = (\eta/4) \min\{\varepsilon_{7.1}(\delta, \eta), \varepsilon_1\}$. Let $C = C_{7.1}(\delta, \eta, \xi)$. Let (W_1, W_2, W_3) be a (δ, ε, p)-strong-super-regular triple and $m = |W_1| = |W_2| = |W_3|$. By Lemma 7.1, there exists a matching M_{12}, M_{13}, M_{23} and sets Q_1, Q_2, Q_3 and X_1, X_2, X_3 satisfying (P1), (P2), (P3), and (P4). Let $W_i' = W_i \setminus (Q_i \cup X_i)$ and note that

$$|W_1'| = |W_1| - |Q_1| - |X_1| = |W_1| - |M_{12}| - |M_{13}| - |X_1|.$$

Let $x = m - |M_{12}| - |M_{13}| - |M_{23}|$ and note that $(1 - 6\eta)m < x < (1 - 3\eta/2)m$. By applying Proposition 2.7, we can find x vertex-disjoint triangles inside the triple (W_1', W_2', W_3').

Note that the remaining $|W_1'| - x = |M_{23}| - |X_1|$ vertices in W_1' together with the set X_1, can be matched with the set M_{23} to construct $|M_{23}|$ vertex-disjoint triangles, by property (P4). Similarly, the remaining vertices in $W_2' \cup X_2$ and $W_3' \cup X_3$ can be matched with M_{13} and M_{12}, respectively. Therefore, we have found a perfect triangle packing of (W_1, W_2, W_3). □

Finally, we briefly summarize Sections 5, 6, and 7 in the proof of our main result, Theorem 1.3.

Proof of Theorem 1.3. Let $\delta = \delta_{6.6}(\gamma)$, $\varepsilon = \varepsilon_{1.4}(\delta)$, $\xi = \xi_{6.6}(\delta, \varepsilon)$ and $C = \max\{C_{6.6}(\varepsilon), C_{1.4}(\varepsilon, \xi)\}$, $D = D_{6.6}(\varepsilon)$. By Theorem 6.6, there exist set B, S, and $(W_i)_{i=1}^{3k}$ which satisfies (i) - (iv) of Theorem 6.6. Furthermore for each $t \in [k]$, by Theorem 1.4, each (δ, ε, p)-strong-super-regular triple $(W_{3t-2}, W_{3t-1}, W_{3t})$ contains a perfect triangle packing. Therefore all the vertices except B can be covered by vertex-disjoint triangles. Since $|B| \leq Dp^{-2}$, this completes the proof. □

8 Concluding Remarks

An immediate question we would like to ask is whether the assumption on p in Theorem 1.3 can be relaxed. Even though our argument breaks down (for a few reasons) if $p \ll (\log n/n)^{1/2}$, we believe that the conclusion of Theorem 1.3 still holds under the (weaker) assumption that $p \gg n^{-1/2}$. If this was true, it would completely resolve the problem of determining the local resilience of $G(n, p)$ with respect to the property of containing an almost spanning triangle packing.

We also believe that a similar argument can be used to obtain an extension of the theorem of Hajnal and Szemerédi [15] for larger cliques to the setting of sparse random graphs. Clearly, the edge probability p would have to be sufficiently large so that a corresponding form of Lemma 2.6 holds. However, in our opinion, the importance of such a result does not justify the technical complications one would have to face in order to prove it.

The more intriguing and interesting question comes from the attempt to embed general spanning or almost spanning graphs (by general we mean graphs that are not disjoint unions of a fixed graph) into sparse regular pairs. This gives rise to the following question.

Question. *Can we develop an embedding lemma for general graphs into regular pairs in random graphs for some $p = n^{-o(1)}$? How should the definition of strong-super-regularity be extended?*

It is quite likely that such an embedding lemma will provide another proof of the theorem of Böttcher, Kohayakawa, and Taraz [7] on embedding almost spanning subgraphs. However, one can hope for a better result where the graph we want to embed is smaller than the host graph by a sublinear number of vertices. To achieve this, one will most likely need to develop a tool similar to that of Theorem 6.6.

Another question can be asked regarding embedding of spanning subgraphs. Proposition 4.6 shows that as many as $\Omega(p^{-2})$ vertices have to be left out from the largest triangle packing. More generally, if every vertex of some graph H is contained in a triangle, then we cannot hope to embed H into a sparse host graph of the same order. However, this is no longer the case when H is bipartite. Thus we recall the following question posed by Böttcher, Kohayakawa, and Taraz [8].

Question. *Is it possible to have a perfect embedding for bipartite graphs?*

In fact, it might be true that what actually matters is not that the graph is bipartite, but the fact that there are enough vertices which are not contained in a copy of a triangle. See [17], where such a result is proved for dense graphs.

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