On the probability of nonexistence in binomial subsets

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Abstract

Let Γ be a hypergraph with vertex set Ω, let $p\colon \Omega \to [0,1]$, and let Ω_p be a random set formed by including every $\omega \in \Omega$ independently with probability $p(\omega)$. We investigate the general question of deriving fine (asymptotic) estimates for the probability that Ω_p is an independent set in Γ, which is an omnipresent problem in probabilistic combinatorics. Our main result provides a sequence of lower and upper bounds on this quantity, each of which can be evaluated explicitly. Under certain natural conditions, we obtain an explicit closed formula that is asymptotic to this probability. We demonstrate the applicability of our results with two concrete examples: subgraph containment in random graphs and arithmetic progressions in random subsets of the integers.

1 Introduction

Let Γ be a hypergraph with vertex set Ω and, given $p\colon \Omega \to [0,1]$, let Ω_p be a random subset of Ω formed by including every $\omega \in \Omega$ independently with probability $p(\omega)$. What is the probability that Ω_p is an independent set in Γ? This very general question arises in many different settings.

Example 1. Let F be a graph, let $n \in \mathbf{N}$, and let $\Omega = E(K_n) = \binom{[n]}{2}$ be the edge set of the complete graph with vertex set $[n] := \{1, \ldots, n\}$. Let Γ be the collection of the edge sets of all copies of F in K_n. Fix some $p \in [0,1]$ and let $p(\omega) = p$ for every $\omega \in \Omega$. Then we are asking for the probability that the random graph $G_{n,p}$ is F-*free*, that is, it does not contain F as a (not necessarily induced) subgraph.

Example 2. An arithmetic progression of length $r \in \mathbf{N}$ (an r-*AP* for short) is a subset of the integers of the form $\{a + kb\colon k \in [r]\}$. Let $\Omega = [n]$, let Γ be the set of all r-APs in $[n]$, and let $p(\omega) = p$ for all $\omega \in \Omega$. Then we are asking for the probability that the random subset $[n]_p$ is r-AP-free.

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Example 3. Let Ω be a finite set of points in the plane. Include a triple $\{i, j, k\}$ in Γ if the points i, j, k lie on a common line. Now we are asking for the probability that the random subset Ω_p of points is in general position.

It is not hard to find other natural examples that provide further motivation for studying this general question. We make the following definition for the sake of brevity.

Definition 4. An *increasing family* is a triple (Ω, Γ, p), where Ω is a finite set, Γ is a collection of non-empty subsets of Ω, and p is a map from Ω to $(0, 1)$.[1]

Given an increasing family (Ω, Γ, p), we shall fix an (arbitrary) ordering of the elements of Γ as $\gamma_1, \ldots, \gamma_N$. We then let X_i be the indicator variable of the event that $\gamma_i \subseteq \Omega_p$ and we set $X = X_1 + \cdots + X_N$. Thus, X counts the number of sets in Γ that are fully contained in Ω_p and our goal is to compute the probability that $X = 0$. Of course the notations Ω_p, γ_i, X_i, X, and N all depend on a given increasing family (Ω, Γ, p), but we shall always suppress this dependence as it will be clear from the context.

Most of the time, we will be interested in *sequences* $(\Omega_n, \Gamma_n, p_n)$ of increasing families, indexed by a parameter n that tends to infinity, and ask:

What are the asymptotics of the probability $\mathbf{P}[X = 0]$ *as* $n \to \infty$*?*

This question can also be viewed as a computational problem: we want to derive closed formulas that are asymptotic to $\mathbf{P}[X = 0]$, at least for various ranges of the parameter p.

1.1 The Harris and Janson inequalities

The main reason why computing $\mathbf{P}[X = 0]$ is challenging is that the variables $X_1, \ldots, X_N$ are usually not independent. However, this is not to say that there is no structure at all: every random variable X_i is a non-decreasing function on the product space $\{0, 1\}^{\Omega}$. An important inequality that applies in this case is the *Harris inequality*:

Theorem 5 (Harris inequality [10])**.** *Let X and Y be random variables defined on a product probability space over $\{0, 1\}^{\Omega}$. If X and Y are both non-decreasing (or non-increasing), then*

$$\mathbf{E}[XY] \geq \mathbf{E}[X]\,\mathbf{E}[Y].$$

If X is non-decreasing and Y is non-increasing, then

$$\mathbf{E}[XY] \leq \mathbf{E}[X]\,\mathbf{E}[Y].$$

In our setting, for every $I \subseteq [N]$, the random variable $\prod_{i \in I}(1 - X_i)$ is non-increasing, so we easily deduce from Harris' inequality that

$$\mathbf{P}[X = 0] = \mathbf{E}\left[\prod_{i=1}^{N}(1 - X_i)\right] \geq \prod_{i=1}^{N}(1 - \mathbf{E}[X_i]). \tag{1}$$

Note that (1) would be true with equality if $X_1, \ldots, X_N$ were independent. An upper bound on $\mathbf{P}[X = 0]$ is given by *Janson's inequality*, which states that the reverse of (1) holds up to

[1] For technical reasons, we shall exclude the case that $p(\omega) \in \{0, 1\}$. That case can always be addressed by changing Γ or by a continuity argument.

a multiplicative error term that is an explicit function of the pairwise dependencies between the X_i. More formally, for indices $i, j \in [N]$, we write $i \sim j$ if $i \neq j$ and $\gamma_i \cap \gamma_j \neq \emptyset$. Further, we define the sum of joint moments

$$\Delta_2 = \sum_{i \sim j} \mathbf{E}[X_i X_j].$$

Theorem 6 (Janson's inequality [2, 15]). *For every increasing family,*

$$\mathbf{P}[X = 0] \leq \exp(-\mathbf{E}[X] + \Delta_2). \tag{2}$$

To compare this with (1), we will now assume that the individual probabilities of $\gamma_i \subseteq \Omega_p$ are not too large, say $\mathbf{E}[X_i] \leq 1 - \varepsilon$ for some $\varepsilon > 0$. In this case, we may use the fact that $1 - x \geq \exp(-x - x^2/(1-x)) \geq \exp(-x - x^2/\varepsilon)$ for $x \in [0, 1-\varepsilon]$ to obtain from (1)

$$\mathbf{P}[X = 0] \geq \prod_{i \in [N]} (1 - \mathbf{E}[X_i]) \geq \exp(-\mathbf{E}[X] - \delta_1/\varepsilon), \tag{3}$$

where

$$\delta_1 = \sum_{i \in [N]} \mathbf{E}[X_i]^2. \tag{4}$$

Combining this with (2), we get

$$\mathbf{P}[X = 0] = \exp\big(-\mathbf{E}[X] + O(\delta_1 + \Delta_2)\big). \tag{5}$$

If $\delta_1 + \Delta_2 = o(1)$, then (5) gives the correct asymptotics of $\mathbf{P}[X = 0]$. The condition $\Delta_2 = o(1)$ requires that the pairwise correlations among the X_i vanish asymptotically in a well-defined sense. This rather strict requirement is not satisfied in many natural settings, including the ones presented in Examples 1–3 for certain choices of p. It is therefore an important question to obtain better approximations of $\mathbf{P}[X = 0]$ in cases when the pairwise dependencies among the X_i are not negligible. This is the starting point of our investigations.

1.2 Triangles in random graphs

Even though our results can and will be phrased in the general framework of increasing families and are thus widely applicable, we believe that it is useful to keep in mind the following well studied instance of the problem that will serve as a guiding example.

Example 7. Suppose that X denotes the number of triangles in $G_{n,p}$, as in Example 1 with $F = K_3$. Since each triangle has three edges, we have $\mathbf{E}[X_i] = p^3$ for all i. Thus $\mathbf{E}[X] = \binom{n}{3}p^3$ and $\delta_1 = O(n^3p^6)$. Moreover, we have $\Delta_2 = O(n^4p^5)$, because if two distinct triangles intersect, then their union is the graph with 4 vertices and 5 edges. Thus (5) implies that as long as $p = o(n^{-4/5})$, we have

$$\mathbf{P}[X = 0] = \exp\big(-n^3p^3/6 + o(1)\big).$$

This result was already obtained by Erdős and Rényi [8] under the much stronger assumption that $p = O(n^{-1})$. The assumption on p was later weakened by Frieze [9] to $p = O(n^{-1+c})$ for some small constant $c > 0$. Extending the above result, Wormald [24] and later Stark and Wormald [22] obtained asymptotic expressions for $\mathbf{P}[X = 0]$ even when $p = \omega(n^{-4/5})$

and thus (5) no longer gives an asymptotic bound. For example, it was shown in [22] that if $p = o(n^{-7/11})$, then

$$\mathbf{P}[X = 0] = \exp\Big(- \frac{n^3p^3}{6} + \frac{n^4p^5}{4} - \frac{7n^5p^7}{12} + \frac{n^2p^3}{2} - \frac{3n^4p^6}{8} + \frac{27n^6p^9}{16} + o(1)\Big).$$

One goal of the present paper is to give a simple interpretation of the individual terms in this formula. Indeed, we will formulate a general result from which the above formula may be obtained by a few short calculations. More precisely, we will prove a generalization of (5) that takes into account the k-wise dependencies between the X_i for all $k \geq 2$.

1.3 Joint cumulants, clusters, dependency graphs

Let $A = \{Z_1, \dots, Z_m\}$ be a finite set of real-valued random variables. The *joint moment* of the variables in A is

$$\Delta(A) := \mathbf{E}[Z_1 \cdots Z_m]. \tag{6}$$

The *joint cumulant* of the variables in A is

$$\kappa(A) := \sum_{\pi \in \Pi(A)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{P \in \pi} \Delta(P), \tag{7}$$

where $\Pi(A)$ denotes the set of all partitions of A into non-empty sets. In particular,

$$\begin{aligned}
\kappa(\{X\}) &= \mathbf{E}[X], \\
\kappa(\{X, Y\}) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y], \\
\kappa(\{X, Y, Z\}) &= \mathbf{E}[XYZ] - \mathbf{E}[X]\,\mathbf{E}[YZ] - \mathbf{E}[Y]\,\mathbf{E}[XZ] - \mathbf{E}[Z]\,\mathbf{E}[XY] + 2\,\mathbf{E}[X]\,\mathbf{E}[Y]\,\mathbf{E}[Z].
\end{aligned}$$

The joint cumulant $\kappa(A)$ can be regarded as a measure of the mutual dependence of the variables in A. For example, $\kappa(\{X, Y\})$ is simply the covariance of X and Y. In particular, $\kappa(\{X, Y\}) = 0$ if X and Y are independent. More generally, the following holds.

Proposition 8. *Let A be a finite set of real-valued random variables. If A can be partitioned into two subsets A_1 and A_2 such that all variables in A_1 are independent of all variables in A_2, then $\kappa(A) = 0$.*

In fact, Proposition 8 remains valid when one replaces the independence assumption with the weaker assumption that $\Delta(B_1 \cup B_2) = \Delta(B_1)\Delta(B_2)$ for all $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. An elegant proof of Proposition 8 can be found in [1]. The proposition motivates the definition of the following notion.

Definition 9 (decomposable, cluster)**.** A set A of random variables is *decomposable* if there exists a partition $A = A_1 \cup A_2$ such that the variables in A_1 are independent of the variables in A_2. A non-decomposable set is also called a *cluster*.

For an increasing family (Ω, Γ, p), it is natural to define the *dependency graph* G_Γ as the graph on the vertex set $[N]$ whose edges are all pairs $\{i, j\}$ such that $\gamma_i \cap \gamma_j \neq \emptyset$. We write $\mathcal{C}_k$ for the collection of all k-element subsets $V \subseteq [N]$ such that $G_\Gamma[V]$ is connected. Since we have assumed that $p(\omega) \notin \{0, 1\}$ for all $\omega \in \Omega$, a set of variables $\{X_i : i \in V\}$ forms a cluster if and only if $V \in \mathcal{C}_{|V|}$. In particular, we have $\kappa(\{X_i : i \in V\}) = 0$ whenever $V \notin \mathcal{C}_{|V|}$. Set

$$\kappa_k = \sum_{V \in \mathcal{C}_k} \kappa(\{X_i : i \in V\}) \quad \text{and} \quad \Delta_k = \sum_{V \in \mathcal{C}_k} \Delta(\{X_i : i \in V\}). \tag{8}$$

Note that this definition is consistent with the definition of Δ_2 given above. Moreover, it follows from (7) and the Harris inequality that $|\kappa_k| \le C_k \Delta_k$ for some C_k depending only on k.

1.4 The main result

Let (Ω, Γ, p) be an increasing family. Given a subset $V \subseteq [N]$, we write

$$\partial(V) := N_{G_\Gamma}(V) \setminus V$$

for the external neighbourhood of V in the dependency graph, and let

$$\lambda(V) := \sum_{i \in \partial(V)} \mathbf{E}[X_i \mid \prod_{j \in V} X_j = 1]$$

be the expected number of external neighbours i of V such that $\gamma_i \subseteq \Omega_p$, conditioned on $\gamma_j \subseteq \Omega_p$ for all $j \in V$. Then, for $k \in \mathbf{N}$ we define

$$\Lambda_k := \max\big\{\lambda(V) \colon V \subseteq [N], 1 \le |V| \le k\big\}.$$

We say that a sequence of increasing families is *sparse* if $\max\{\mathbf{P}[X_i = 1] \colon i \in [N]\} = o(1)$ and that it is *subcritical* if $\Lambda_k = O(1)$ for every constant $k \in \mathbf{N}$.

In other words, a sequence $(\Omega_n, \Gamma_n, p_n)_{n \in \mathbf{N}}$ is sparse if

$$\lim_{n \to \infty} \max_{\gamma \in \Gamma_n} \mathbf{P}[\gamma \subseteq (\Omega_n)_{p_n}] = 0$$

and it is subcritical if for every every k,

$$\limsup_{n \to \infty} \max_{V \subseteq \Gamma_n, |V| \le k} \mathbf{E}\left[\left|\left\{\gamma \in \Gamma_n \colon \gamma \cap \bigcup V \neq \emptyset,\ \gamma \setminus \bigcup V \subseteq (\Omega_n)_{p_n}\right\}\right|\right] < \infty.$$

Our main result is the following.

Theorem 10. *Suppose that $(\Omega_n, \Gamma_n, p_n)_{n \in \mathbf{N}}$ is a sparse and subcritical sequence of increasing families and let X denote the number of edges of Γ_n that are fully contained in $(\Omega_n)_{p_n}$. Then for every $k \in \mathbf{N}$,*

$$\mathbf{P}[X = 0] = \exp\big(-\kappa_1 + \kappa_2 - \kappa_3 + \cdots + (-1)^k \kappa_k + O(\delta_1 + \Delta_{k+1})\big)$$

as $n \to \infty$, where δ_1, $\kappa_1, \ldots, \kappa_k$, and Δ_{k+1} are defined as above.

In the applications considered in this paper, it will always be the case that $\kappa_k = \Delta_k + o(\Delta_k)$ for every fixed k. For example, this is automatically so if $\max\{p(\omega) \colon \omega \in \Omega_n\} = o(1)$, as can be seen from definition (7). In such cases, the first-order behaviour of κ_k is thus given by Δ_k. However, this does *not* mean that we can then replace κ_i by Δ_i in the formula for $\mathbf{P}[X = 0]$ given by Theorem 10, because the lower-order terms in the κ_i can be non-negligible, see, e.g., the proof of Corollary 15.

The fact that $\kappa_1 = \mathbf{E}[X]$ shows that the case $k = 1$ of Theorem 10 gives (a slight weakening of) Janson's inequality (5). Unlike Janson's inequality, our Theorem 10 requires the additional assumptions of sparsity and subcriticality. Whereas the sparsity condition is rather natural[2],

[2]Even if all the X_i are independent, but $\max_i \mathbf{E}[X_i] = \Omega(1)$, then it is not true that $\mathbf{P}[X = 0] = \exp(-\kappa_1 + o(1))$ for the simple reason that $1 - x \neq e^{-x + o(1)}$ unless $x = o(1)$.

the latter condition is hard to motivate and perhaps not necessary. As we will see further below, subcriticality implies that $\Delta_{k+1} = O(\Delta_k)$ for all constant k, which gives at least an indication of the type of assumption that is involved.

We shall derive Theorem 10 from a more general result, Theorem 11 below. Even though the former is sufficiently general to handle all applications considered in this paper, the latter has the advantage that it can be applied in certain non-sparse settings. Its disadvantage lies in the fact that the error terms are somewhat less transparent.

For a set of random variables A, we define

$$\delta(A) := \Delta(A) \cdot \max\{\mathbf{E}[X] \colon X \in A\}.$$

and for $k \in \mathbf{N}$ we set

$$\delta_k := \sum_{V \in \mathcal{C}_k} \delta(\{X_i \colon i \in V\}) \quad \text{and} \quad \rho_k := \max_{\substack{V \subseteq [N] \\ 1 \le |V| \le k}} \mathbf{P}[X_i = 1 \text{ for some } i \in V \cup \partial(V)].$$

Observe that the above definition of δ_k generalises (4).

Theorem 11. *Let (Ω, Γ, p) be an increasing family and let $k \in \mathbf{N}$. Assume that there is some $\varepsilon > 0$ such that $\mathbf{E}[X_i] \le 1 - \varepsilon$ for all $i \in [N]$ and $\rho_{k+1} \le 1 - \varepsilon$. Then there exists $K = K(k, \varepsilon)$ such that*

$$\left| \log \mathbf{P}[X = 0] + \kappa_1 - \kappa_2 + \kappa_3 - \cdots + (-1)^{k+1} \kappa_k \right| \le K \cdot (\delta_{1,K} + \Delta_{k+1,K}), \tag{9}$$

where

$$\delta_{1,K} = \sum_{r=1}^{K} \delta_r \qquad \text{and} \qquad \Delta_{k+1,K} = \sum_{r=k+1}^{K} \Delta_r.$$

We will show that Theorem 11 implies Theorem 10 in Section 2. The proof of Theorem 11, which is the main part of this paper, will be presented in Section 3.

1.5 Applications and examples

1.5.1 Random hypergraphs

A fundamental question studied by the random graphs community, raised already in the seminar paper of Erdős and Rényi [8], is to determine the probability that $G_{n,p}$ contains no copies of a given 'forbidden' graph F. The classical result of Bollobás [5], proved independently by Karoński and Ruciński [16], determines this probability asymptotically for every strictly balanced[3] F, but only for p such that the expected number of copies of F in $G_{n,p}$ is constant. (In the case when F is a tree or a cycle, this was done earlier by Erdős and Rényi [8] and in the case when F is a complete graph, by Schürger [21].) It was later proved by Frieze [9] that the same estimate remains valid as long as the expected number of copies of F in $G_{n,p}$ is $o(n^{\varepsilon})$ for some positive constant ε that depends only on F. Prior to this work and the work of Stark and Wormald [22], the strongest result of this form (i.e., determining the probability of being F-free asymptotically) for a general graph F followed from Harris' and Janson's inequalities, see (5). Finally, we remark that for several special graphs F, the probability that $G_{n,p}$ (or $G_{n,m}$) is F-free can be computed very precisely either when $p = 1/2$ or, in some

[3] A graph F is strictly balanced if $e_F/v_F > e_H/v_H$ for every proper nonempty subgraph H of F.

cases, even for all sufficiently large $p = o(1)$ (or $m = o(n^2)$) using the known precise structural characterisations of F-free graphs, see [4, 11, 17, 18].

We consider the following natural generalisation of this question. Let $G^{(r)}_{n,p}$ denote the random r-uniform hypergraph (r-graph for short) on n vertices containing every possible edge (r-element subset of the vertices) with probability p, independently of other edges. (In particular, $G^{(2)}_{n,p}$ is simply the binomial random graph $G_{n,p}$.) Given a family $\mathcal{F} = \{F_1, \dots, F_t\}$ of r-graphs, what is the probability that $G^{(r)}_{n,p}$ is *$\mathcal{F}$-free*, that is, it simultaneously avoids all copies of all r-graphs in $\mathcal{F}$? We will assume that the r-graphs in $\mathcal{F}$ are pairwise non-isomorphic and that they do not have isolated vertices; in any case, removing duplicates from $\mathcal{F}$ or isolated vertices from a hypergraph in $\mathcal{F}$ does not affect the probability that we are interested in.

We now define $(\Omega_n, \Gamma_n, p_n)$ similarly as we did in Example 1. That is, we let $\Omega_n = \binom{[n]}{r}$ be the edge set of $K^{(r)}_n$, the complete r-graph with vertex set $[n]$, let Γ_n be the collection of edge sets of subhypergraphs of $K^{(r)}_n$ that are isomorphic to one of the r-graphs in $\mathcal{F}$, and let p_n be a sequence of probabilities (which, however, we interpret as constant functions on Ω_n). Then $(\Omega_n, \Gamma_n, p_n)$ is a sequence of increasing families and $\mathbf{P}[X = 0]$ is the probability that $G^{(r)}_{n,p_n}$ is $\mathcal{F}$-free. Using Theorem 10, we can get the correct asymptotics for this probability in a range of p_n.

For an r-graph F, define

$$m_*(F) := \min\left\{\frac{e_F - e_H}{v_F - v_H} : H \subseteq F \text{ with } v_H < v_F \text{ and } e_H > 0\right\},$$

where v_K and e_K denote, respectively, the numbers of vertices and edges in an r-graph K. For a family $\mathcal{F}$ of r-graphs, we then set

$$m_*(\mathcal{F}) := \min\{m_*(F) \colon F \in \mathcal{F}\} \quad \text{and} \quad d(\mathcal{F}) := \min\{e_F/v_F \colon F \in \mathcal{F}\}.$$

Corollary 12. *Let $\mathcal{F}$ be a finite family of r-uniform hypergraphs, each containing at least two edges, and assume that $(p_n)_{n\in\mathbf{N}}$ satisfies*

$$np_n^{m_*(\mathcal{F})} = o(1) \qquad \text{and} \qquad np_n^{2d(\mathcal{F})} = o(1). \tag{10}$$

Then for every $k \in \mathbf{N}$, as $n \to \infty$,

$$\mathbf{P}\left[G^{(r)}_{n,p} \text{ is } \mathcal{F}\text{-free}\right] = \exp\big(- \kappa_1 + \kappa_2 - \dots + (-1)^k\kappa_k + O(\Delta_{k+1}) + o(1)\big).$$

In Corollary 12, the first condition on p_n in (10) ensures that the associated sequence of increasing families is sparse and subcritical (thus allowing the application of Theorem 10), whereas the second condition ensures that $\delta_1 = o(1)$. These two conditions can be simplified under certain natural assumptions on the family $\mathcal{F}$. Recall that the *r-density* of an r-graph F with at least two edges is

$$m_r(F) := \max\left\{\frac{e_H - 1}{v_H - r} : H \subseteq F \text{ with } e_H > 1\right\}$$

and that F is *r-balanced* if the maximum above is achieved with $H = F$, that is, if $m_r(F) = (e_F - 1)/(v_F - r)$. Observe that for every F with at least two edges, we have

$$m_r(F) \geq \frac{e_F - 1}{v_F - r} \geq m_*(F).$$

We claim that if F is r-balanced, then in fact $m_r(F) = m_*(F)$. Indeed, writing $\alpha_K = (e_K - 1)/(v_K - r)$, we see that for every $H \subseteq F$ with $v_H < v_F$ and $e_H > 1$,

$$\frac{e_F - e_H}{v_F - v_H} = \frac{(e_F - 1) - (e_H - 1)}{(v_F - r) - (v_H - r)} = \frac{\alpha_F(v_F - r) - \alpha_H(v_H - r)}{(v_F - r) - (v_H - r)} \geq m_r(F),$$

since $m_r(F) = \alpha_F \geq \alpha_H$ (as F is r-balanced) and this inequality continues to hold if $e_H = 1$. Thus $m_*(F) \geq m_r(F)$. Moreover, if $r = 2$, then the second condition in (10) follows from the first condition, since $2e_F/v_F \geq (e_F - 1)/(v_F - 2)$ for every graph F and consequently $m_*(\mathcal{F}) \leq 2d(\mathcal{F})$ for every family of graphs $\mathcal{F}$.

Corollary 13. *Let $\mathcal{F}$ be a finite family of 2-balanced graphs and assume that $(p_n)_{n\in\mathbf{N}}$ satisfies $p_n = o(n^{-1/m_2(F)})$ for every $F \in \mathcal{F}$. Then for every $k \in \mathbf{N}$, as $n \to \infty$,*

$$\mathbf{P}\left[G_{n,p} \text{ is } \mathcal{F}\text{-free}\right] = \exp\big(-\kappa_1 + \kappa_2 - \cdots + (-1)^k\kappa_k + O(\Delta_{k+1}) + o(1)\big).$$

Suppose that $\mathcal{F}$ is a finite family of 2-balanced graphs, let $m_2(\mathcal{F}) = \min_{F\in\mathcal{F}} m_2(F)$, and fix an arbitrary positive ε. If we replace the assumption of Corollary 13 with the stronger assumption that $p = O(n^{-1/m_2(\mathcal{F})-\varepsilon})$, the corollary gives an asymptotic formula for the probability that $G_{n,p}$ is $\mathcal{F}$-free. (Moreover, it is not hard to see that this formula is $\exp(f(n,p))$ for some bivariate polynomial f with rational coefficients.) This is an immediate consequence of the fact that $\Delta_{k+1} = o(1)$ whenever k is sufficiently large as a function of ε and $\mathcal{F}$, which we shall now verify. To this end, suppose that C is a collection of k copies of graphs from $\mathcal{F}$ in K_n that form a cluster in the sense of Definition 9 and let G be the union of these k subgraphs of K_n. Since C is a cluster, one can order its elements as $F_1, \ldots, F_k$ such that F_{i+1} intersects $F_1 \cup \cdots \cup F_i$ for each $i \in [k-1]$. As this intersection is clearly a subgraph of F_{i+1}, one can show (using induction on i) that

$$\frac{e_G - 1}{v_G - 2} \geq m_*(\mathcal{F}) = \min_{F\in\mathcal{F}} m_2(F).$$

On the other hand, there are functions $v, V \colon \mathbf{N} \to \mathbf{N}$ depending only on $\mathcal{F}$ such that $v(k) \to \infty$ as $k \to \infty$ and $v(k) \leq v_G \leq V(k)$ for every G that is obtained from a cluster of k copies of $\mathcal{F}$ in a complete graph of an arbitrary order. Consequently,

$$\Delta_{k+1} \leq pn^2 \sum_{i=v(k+1)}^{V(k+1)} 2^{\binom{i}{2}} \left(p^{m_2(\mathcal{F})}n\right)^{i-2} = O\big(n^{2-\varepsilon\cdot m_2(\mathcal{F})\cdot(v(k+1)-2)}\big)$$

and thus $\Delta_{k+1} = o(1)$ whenever k is sufficiently large as a function of $\mathcal{F}$ and ε.

Of course, neither Corollary 12 nor Corollary 13 would be particularly useful if one could not compute the values κ_k for at least several small integers k. We perform these calculations for two special cases.

Corollary 14. *If $p = o(n^{-4/5})$, then the probability that $G_{n,p}$ is simultaneously K_3-free and C_4-free is asymptotically*

$$\exp\Big(-\frac{n^3p^3}{6} - \frac{n^4p^4}{8} + \frac{n^6p^7}{4} + \frac{n^5p^6}{2}\Big).$$

Corollary 15. *If $p = o(n^{-7/11})$, then the probability that $G_{n,p}$ is triangle-free is asymptotically*

$$\exp\Big(-\frac{n^3p^3}{6} + \frac{n^4p^5}{4} - \frac{7n^5p^7}{12} + \frac{n^2p^3}{2} - \frac{3n^4p^6}{8} + \frac{27n^6p^9}{16}\Big).$$

Corollary 15 was obtained independently by Stark and Wormald [22], who also proved a similar result in $G_{n,m}$, the uniform random graph with n vertices and m edges. It extends a result of Wormald [24] that applies to a smaller range of p. However, the derivation of Corollary 15 from Theorem 10 is very short compared to the proofs in [22] and [24].

1.5.2 Arithmetic progressions

As a second application, we will estimate the probability that a binomial random subset of $[n]$ is r-AP-free, i.e., does not contain any arithmetic progression of length r. Given a sequence $(p_n)_{n\in\mathbf{N}}$ of probabilities, we define a sequence of increasing families $(\Omega_n, \Gamma_n, p_n)$ by setting $\Omega_n = [n]$ and letting Γ_n be the set of all r-APs contained in $[n]$ (and, again, considering p_n as a constant function on Ω_n). Then $\mathbf{P}[X = 0]$ is the probability that the random subset $[n]_{p_n}$ is r-AP-free.

Corollary 16. *Let $r \geq 3$ and assume that $p_n = o(n^{-1/(r-1)})$. Then for $k \in \mathbf{N}$, as $n \to \infty$,*

$$\mathbf{P}\left[[n]_p \text{ is } r\text{-AP-free}\right] = \exp\big(-\kappa_1 + \kappa_2 - \kappa_3 + \cdots + (-1)^k\kappa_k + O(\Delta_{k+1}) + o(1)\big).$$

The assumption on p_n simply makes sure that the family is subcritical and that $\delta_1 = o(1)$. Observe that every pair of integers lies in at most $\binom{r}{2}$ many r-APs. If follows that for every $\ell \geq 2$, there are $O(n^\ell)$ subsets of $[n]$ with at most $\ell(r-1)$ elements that are unions of r-APs contained in $[n]$ and that a set of i integers contains at most $\binom{i}{2}\binom{r}{2}$ many r-APs. Consequently, if $p = O(n^{-1/(r-1)-\varepsilon})$ for some positive ε, then $\Delta_{k+1} = o(1)$ whenever k is sufficiently large as a function of r and ε. In particular, for such p, Corollary 16 gives an asymptotic formula for the probability that $[n]_p$ is r-AP-free. To give a concrete example, we perform the calculations for $r = 3$ and $k = 2$.

Corollary 17. *If $p = o(n^{-4/7})$, then the probability that $[n]_p$ is 3-AP-free is asymptotically*

$$\exp\Big(-\frac{n^2p^3}{4} + \frac{7n^3p^5}{12}\Big).$$

1.6 Related work and open problems

Janson's inequality was first proved (by Svante Janson himself) during the 1987 conference on random graphs in Poznań, in response to Bollobás' announcement of his estimate [6] for the chromatic number of random graphs, which requires a strong upper bound on the probability that a random graph contains no large cliques. A related estimate was found, during the same conference, by Łuczak. Janson's original proof was based on the analysis of the moment-generating function of X whereas Łuczak's proof used martingales. Both of these arguments can be found in [14]. Our proof of Theorem 11 is inspired by a subsequent proof of Janson's inequality that was found soon afterwards by Boppana and Spencer [7]; it uses only the Harris inequality. Somewhat later, Janson [12] showed that his proof actually gives bounds for the whole lower tail, and not just for the probability $\mathbf{P}[X = 0]$. Around the same time, Suen [23] proved a correlation inequality that is very similar to Janson's. Suen's inequality gives a slightly weaker estimate (which was later sharpened by Janson [13]), but is applicable in a much more general context. Another generalisation of Janson's inequality was obtained recently by Riordan and Warnke [19].

In [24], Wormald proved that if $p = o(n^{-2/3})$, then

$$\mathbf{P}[G_{n,p} \text{ is } K_3\text{-free}] = \exp\Big(-\frac{n^3p^3}{6} + \frac{n^4p^5}{4} - \frac{7n^5p^7}{12} + o(1)\Big), \tag{11}$$

whereas for $G_{n,m}$ with $m = d\binom{n}{2}$ and $d = o(n^{-2/3})$, we have

$$\mathbf{P}[G_{n,m} \text{ is } K_3\text{-free}] = \exp\Big(-\frac{n^3d^3}{6} + o(1)\Big).$$

These results were strengthened recently by Stark and Wormald [22], who obtained the bound in Corollary 15 (which implies (11)) and also the bound

$$\mathbf{P}[G_{n,m} \text{ is } K_3\text{-free}] = \exp\Big(-\frac{n^3d^3}{6} + \frac{n^2d^3}{2} - \frac{n^4d^6}{8} + o(1)\Big),$$

where $m = d\binom{n}{2}$, which holds when $d = o(n^{-7/11})$. In fact, they were able to obtain a more general result, which states that in the range where Corollary 12 is applicable, the probability that $G_{n,p}$ or $G_{n,m}$ is F-free is approximated by the exponential of the first few terms of a power series in n and p (resp. d) whose terms depend only on F. However, the way in which these terms are computed is rather implicit. In contrast, in the setting of binomial random subsets, such as $G_{n,p}$, our Theorem 10 explains what these terms are.

While our results (and our methods) apply only to binomial subsets (e.g., $G_{n,p}$ and not $G_{n,m}$), the results for $G_{n,p}$ could conceivably be transferred to $G_{n,m}$ using the identity

$$\mathbf{P}[G_{n,m} \text{ is } F\text{-free}] = \frac{\mathbf{P}[G_{n,p} \text{ is } F\text{-free}] \cdot \mathbf{P}[e(G_{n,p}) = m \mid G_{n,p} \text{ is } F\text{-free}]}{\mathbf{P}[e(G_{n,p}) = m]}. \tag{12}$$

It was shown by Stark and Wormald [22] that the conditional probability in the right-hand side of (12) can be computed explicitly, for a carefully chosen p of the same order of magnitude as d. However, this is not at all an easy task.

It would be interesting to establish a similar relationship in the more abstract and general setting of hypergraphs. If this was possible, Theorem 10 could be used to count independent sets of a given (sufficiently small) cardinality in general hypergraphs. In some sense, this would complement the counting results that can be obtained with the so-called hypergraph container method developed by Balogh, Morris, and Samotij [3] and by Saxton and Thomason [20]. Whereas the container method applies to somewhat large independent sets, which exhibit a "global" structure, our Theorem 10 would yield estimates on the number of smaller independent sets that only exhibit "local" structure. In particular, the container method can be used to estimate the probability that $G_{n,p}$ is F-free whenever $p = \omega(n^{-1/m_2(F)})$ for every nonbipartite graph F. For p in this range, $G_{n,p}$ conditioned on being F-free is approximately $(\chi(F) - 1)$-partite with very high probability. On the other hand, our method (and the method of [22]) applies whenever $p = o(n^{-1/m_2(F)})$, provided that F is 2-balanced. For p in this range, the edges of $G_{n,p}$ conditioned on being F-free are still distributed very uniformly with probability very close to one.

2 Proof of Theorem 10

In this section, we will show that Theorem 11 implies Theorem 10. To do so, we start with the following lemma, which also clarifies the definition of Λ_k.

Lemma 18. *Every increasing family satisfies the following for every $k \geq 1$:*

$$\Delta_{k+1}/\Delta_k \leq \Lambda_k \quad \textit{and} \quad \delta_{k+1}/\delta_k \leq \Lambda_k.$$

Proof. For every $V \in \mathcal{C}_{k+1}$ there exist at least two distinct $i \in V$ such that $V \setminus \{i\} \in \mathcal{C}_k$. Indeed, every connected graph with at least two vertices has at least two non-cut vertices. Therefore for each $V \in \mathcal{C}_{k+1}$ we can make a canonical choice of a set $V^- \subset V$ such that $V^- \in \mathcal{C}_k$ and

$$\max\{\mathbf{E}[X_i] \colon i \in V\} = \max\{\mathbf{E}[X_i] \colon i \in V^-\}. \tag{13}$$

Denoting by i_V the unique element in $V \setminus V^-$, we have $i_V \in \partial(V^-)$, since $G_\Gamma[V]$ is connected. Moreover,

$$\Delta(\{X_i \colon i \in V\}) = \Delta(\{X_i \colon i \in V^-\}) \cdot \mathbf{E}[X_{i_V} \mid \prod_{i \in V^-} X_i = 1]$$

and, analogously,

$$\delta(\{X_i \colon i \in V\}) = \delta(\{X_i \colon i \in V^-\}) \cdot \mathbf{E}[X_{i_V} \mid \prod_{i \in V^-} X_i = 1].$$

It follows that

$$\begin{aligned}\Delta_{k+1} &\leq \sum_{V^- \in \mathcal{C}_k} \Delta(\{X_i \colon i \in V^-\}) \sum_{j \in \partial(V^-)} \mathbf{E}[X_j \mid \prod_{i \in V^-} X_i = 1] \\ &= \sum_{V^- \in \mathcal{C}_k} \Delta(\{X_i \colon i \in V^-\}) \cdot \lambda(V^-) \leq \Delta_k \cdot \Lambda_k\end{aligned}$$

and, analogously, $\delta_{k+1} = \delta_k \cdot \Lambda_k$. □

Proof of Theorem 10 from Theorem 11. Let $(\Omega_n, \Gamma_n, p_n)$ be a sparse and subcritical sequence of increasing families. Let $\varepsilon > 0$ be arbitrary, fix $k \in \mathbf{N}$, and let $K = (k, \varepsilon)$ be such that Theorem 11 holds with k and ε. We verify that $(\Omega_n, \Gamma_n, p_n)$ satisfies the assumptions of Theorem 11 for all sufficiently large n. First, since the sequence is sparse, $\mathbf{E}[X_i] \leq 1 - \varepsilon$ for all $i \in [N]$ and all sufficiently large n. Next, fix some $V \subseteq [N]$ of size at most $k+1$. Then for sufficiently large n, we have $\sum_{i \in V} \mathbf{E}[X_i] \leq (1-\varepsilon)/2$ (by sparsity) and

$$\sum_{i \in \partial(V)} \mathbf{E}[X_i] \leq \lambda(V) \cdot \mathbf{P}[\prod_{i \in V} X_i = 1] \leq \lambda(V) \cdot \max\{\mathbf{E}[X_i] \colon i \in [N]\} \leq (1-\varepsilon)/2,$$

(using sparsity and subcriticality), which implies that, by the union bound,

$$\rho_{k+1} = \max_{\substack{V \subseteq [N] \\ 1 \leq |V| \leq k+1}} \mathbf{P}[X_i = 1 \text{ for some } i \in V \cup \partial(V)] \leq 1 - \varepsilon.$$

Therefore Theorem 11 yields

$$|\log \mathbf{P}[X=0] + \kappa_1 - \kappa_2 + \cdots + (-1)^{k+1}\kappa_k| \leq K \cdot (\delta_{1,K} + \Delta_{k+1,K}) \tag{14}$$

for sufficiently large n. It remains to show that the right-hand side of (14) is $O(\Delta_{k+1} + \delta_1)$. By Lemma 18 and since $\Lambda_K = O(1)$, we see that

$$K \cdot \delta_{1,K} = K \cdot \sum_{r=1}^{K} \delta_r = O(\delta_1) \quad \text{and} \quad K \cdot \Delta_{k+1,K} = K \cdot \sum_{r=k+1}^{K} \Delta_r = O(\Delta_{k+1}),$$

which completes the proof. □

3 Proof of Theorem 11

Let (Ω, Γ, p) be an increasing family. We start the proof by establishing some notational conventions. Given a subset $V \subseteq [N]$, we use the abbreviations

$$X_V := \prod_{i \in V} X_i \quad \text{and} \quad \overline{X}_V := \prod_{i \in V} (1 - X_i).$$

Note that these are the indicator variables for the events "$\gamma_i \subseteq \Omega_p$ for all $i \in V$" and "$\gamma_i \not\subseteq \Omega_p$ for all $i \in V$", respectively. Besides being positively correlated, the variables X_V satisfy the FKG lattice condition

$$\mathbf{E}[X_U]\,\mathbf{E}[X_V] \leq \mathbf{E}[X_{U \cup V}]\,\mathbf{E}[X_{U \cap V}] \quad \text{for all} \quad U, V \subseteq [N]. \tag{15}$$

To see that this is true, rewrite (15) using $\mathbf{E}[X_W] = \prod_{\omega \in \bigcup_{i \in W} \gamma_i} p(\omega)$, take logarithms of both sides, and note that

$$\begin{aligned}
\sum_{\omega \in \bigcup_{i \in U \cup V} \gamma_i} \log p(\omega) &= \sum_{\omega \in \bigcup_{i \in U} \gamma_i} \log p(\omega) + \sum_{\omega \in \bigcup_{i \in V} \gamma_i} \log p(\omega) - \sum_{\omega \in (\bigcup_{i \in U} \gamma_i) \cap (\bigcup_{i \in V} \gamma_i)} \log p(\omega) \\
&\geq \sum_{\omega \in \bigcup_{i \in U} \gamma_i} \log p(\omega) + \sum_{\omega \in \bigcup_{i \in V} \gamma_i} \log p(\omega) - \sum_{\omega \in \bigcup_{i \in U \cap V} \gamma_i} \log p(\omega),
\end{aligned}$$

since $\log p(\omega) < 0$ for all ω and $\bigcup_{i \in U \cap V} \gamma_i \subseteq \left(\bigcup_{i \in U} \gamma_i\right) \cap \left(\bigcup_{i \in V} \gamma_i\right)$. We will also use the notation

$$\mu_\pi := \prod_{P \in \pi} \mathbf{E}[X_P]$$

whenever π is a set of subsets of $[N]$ (usually a partition of some subset of $[N]$). Thus for a non-empty subset $V \subseteq [N]$, the value

$$\kappa(V) := \sum_{\pi \in \Pi(V)} (-1)^{|\pi|-1} (|\pi| - 1)! \mu_\pi \tag{16}$$

is the joint cumulant of $\{X_i \colon i \in V\}$. For the sake of brevity, we will from now on write $\kappa(V)$ instead of $\kappa(\{X_i \colon i \in V\})$. Recall that for a non-empty subset $V \subseteq [N]$, we denote by $\partial(V)$ the external neighbourhood of V in the dependency graph, that is,

$$\partial(V) = N_{G_\Gamma}(V) \setminus V.$$

We define

$$\rho_V := \mathbf{P}[X_i = 1 \text{ for some } i \in V \cup \partial(V)], \tag{17}$$

so that $\rho_{k+1} = \max\{\rho_V \colon V \subseteq [N], 1 \leq |V| \leq k+1\}$. Moreover, we set

$$I(V) := [N] \setminus (V \cup \partial(V)).$$

Neglecting the distinction between an index i and the variable X_i, we may say that $\partial(V)$ contains the variables outside of V that are dependent on V and $I(V)$ contains those that are independent of V. Recall also that $\mathcal{C}_i$ is the collection of all i-element sets $A \subseteq [N]$ such that $G_\Gamma[A]$ is connected. We will also write $\mathcal{C}_i(\ell)$ for the subset of $\mathcal{C}_i$ comprising all $A \in \mathcal{C}_i$ with $\max A = \ell$.

Assume that there is $\varepsilon > 0$ such that $\mathbf{E}[X_i] \leq 1 - \varepsilon$ for all $i \in [N]$. Then we need to show that for every $k \in \mathbf{N}$ such that $\rho_{k+1} \leq 1 - \varepsilon$, there is some $K = K(k, \varepsilon)$ such that

$$\Big| \log \mathbf{P}[X = 0] + \sum_{i \in [k]} (-1)^{i+1} \kappa_i \Big| \leq K \cdot (\delta_{1,K} + \Delta_{k+1,K}),$$

where

$$\delta_{1,K} = \sum_{i=1}^{K} \delta_i \quad \text{and} \quad \Delta_{k+1,K} = \sum_{i=k+1}^{K} \Delta_i.$$

To do so, we first write out the probability that $X = 0$ using the chain rule:

$$\mathbf{P}[X = 0] = \prod_{\ell \in [N]} \mathbf{P}[X_\ell = 0 \mid \overline{X}_{[\ell-1]} = 1] = \prod_{\ell \in [N]} \left(1 - \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]\right).$$

Note that by the Harris inequality, $\mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] \leq \mathbf{E}[X_\ell] \leq 1 - \varepsilon$. Taking logarithms of both sides of the above equality and using the fact that $|\log(1-x)+x| \leq x^2/\varepsilon$ for $x \in [0, 1-\varepsilon]$, we get

$$\Big| \log \mathbf{P}[X = 0] + \sum_{\ell \in [N]} \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] \Big| \leq \sum_{\ell \in [N]} \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]^2/\varepsilon.$$

Hence, using again $\mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] \leq \mathbf{E}[X_\ell]$,

$$\Big| \log \mathbf{P}[X = 0] + \sum_{\ell \in [N]} \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] \Big| \leq \sum_{\ell \in [N]} \mathbf{E}[X_\ell]^2/\varepsilon = \delta_1/\varepsilon. \tag{18}$$

Thus, our main goal becomes estimating the sum

$$\sum_{\ell \in [N]} \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1]. \tag{19}$$

We shall do this by approximating (19) by an expression involving the quantities

$$q(V, S) := \frac{(-1)^{|V|-1}\, \mathbf{E}[X_V]}{\mathbf{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1]}, \tag{20}$$

This ratio is well-defined for all $V, S \subseteq [N]$ because

$$\mathbf{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1] \geq \mathbf{E}[\overline{X}_{S \setminus I(V)}] > 0,$$

which is a consequence of the Harris inequality and the assumption that $p(\omega) < 1$ for all $\omega \in \Omega$. The relationship between (19) and (20) is made precise in the following lemma:

Lemma 19. *Let $k \in \mathbf{N}$ be such that $\rho_{k+1} \leq 1 - \varepsilon$. Then*

$$\Big| \sum_{\ell \in [N]} \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} q(V, [\ell - 1]) \Big| \leq \Delta_{k+1}/\varepsilon.$$

We postpone the proof of Lemma 19 to Section 3.1 and instead show how it implies the assertion of the theorem. Before we do this, we need several additional definitions.

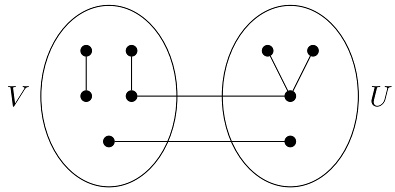

Figure 1: The set U attaches to V, i.e., $U \hookrightarrow V$, but not vice-versa.

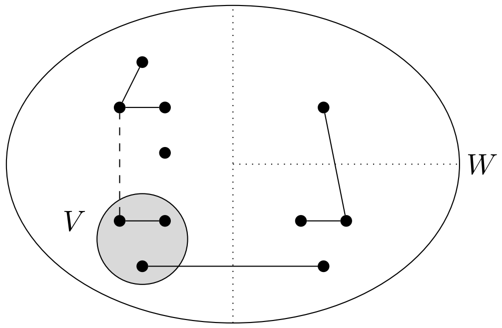

Figure 2: A partition in $\Pi^{\mathsf{C}}_V(W)$. Note that V is the union of components of the subgraph induced by the part containing it. If the dashed edge were in G_Γ, then the partition would no longer be in $\Pi^{\mathsf{C}}_V(W)$.

Definition 20 (Attachment)**.** Given subsets $U, V \subseteq [N]$, let us say that U *attaches* to V, in symbols $U \hookrightarrow V$, if every connected component of $G_\Gamma[U \cup V]$ contains a vertex of V (see Figure 1).

We state the following simple facts for future reference:

(i) We have $\emptyset \hookrightarrow V$ for every $V \subseteq [N]$.

(ii) If $i \in \partial(V)$, then $\{i\} \hookrightarrow V$.

(iii) If $U \hookrightarrow V$ and $U' \hookrightarrow V$ then also $U \cup U' \hookrightarrow V$.

(iv) If $V \in \mathcal{C}_{|V|}$ and $U \hookrightarrow V$, then $U \cup V \in \mathcal{C}_{|U \cup V|}$.

Definition 21. Suppose that $\emptyset \neq V \subseteq W \subseteq [N]$. We define

$$\Pi^{\mathsf{C}}_V(W) \subseteq \Pi(W)$$

to be the set of all partitions π of W that contain a part $P \in \pi$ such that $V \subseteq P$ and V is the union of connected components of $G_\Gamma[P]$ (see Figure 2).

Next, for $\emptyset \neq V \subseteq W \subseteq [N]$, we define

$$\kappa_V(W) := \sum_{\pi \in \Pi^{\mathsf{C}}_V(W)} (-1)^{|\pi|-1}(|\pi|-1)!\mu_\pi. \tag{21}$$

Note that this is very similar to the definition (16) of $\kappa(W)$, except that we sum over $\Pi^{\mathsf{C}}_V(W)$ instead of $\Pi(W)$. For $i \geq 0$ and all $V, S \subseteq [N]$ where $V \neq \emptyset$, we set

$$\kappa^{(k)}_V(S) := \sum_{\substack{V \subseteq W \subseteq V \cup S \\ W \hookrightarrow V \\ |W| \leq k}} (-1)^{|W|-1}\kappa_V(W). \tag{22}$$

Undoubtedly this is a very complicated definition. However, it serves as a convenient 'bridge' between $q(V,[\ell-1])$ and the values κ_i, as shown by the following two lemmas:

Lemma 22. *Let $k \in \mathbf{N}$ be such that $\rho_{k+1} \le 1-\varepsilon$. Then there is some $K = K(k,\varepsilon)$ such that*

$$\Big| \sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} \big(q(V,[\ell-1]) - \kappa_V^{(k)}([\ell-1])\big)\Big| \le K \cdot (\delta_{1,K} + \Delta_{k+1,K}).$$

Lemma 23. *For every $k \in \mathbf{N}$, we have*

$$\sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} \kappa_V^{(k)}([\ell-1]) = \sum_{i \in [k]} (-1)^{i+1} \kappa_i.$$

We claim that Theorem 11 is an easy consequence of Lemmas 19, 22, and 23. Indeed, let $k \in \mathbf{N}$ and assume that $\rho_{k+1} \le 1-\varepsilon$. If follows from (18), the above three lemmas, and the triangle inequality that

$$\Big| \log \mathbf{P}[X=0] + \sum_{i \in [k]} (-1)^{i+1} \kappa_i \Big| \le \delta_1/\varepsilon + \Delta_{k+1}/\varepsilon + K' \cdot (\delta_{1,K'} + \Delta_{k+1,K'})$$

for some $K' = K'(k,\varepsilon)$. The assertion of the theorem now follows simply by observing that the right-hand side above is at most $K \cdot (\delta_{1,K} + \Delta_{k+1,K})$ for $K = K' + 1/\varepsilon$.

3.1 Proof of Lemma 19

We derive Lemma 19 from the following auxiliary lemma, which will also be used in the proof of Lemma 22.

Lemma 24. *Assume that $V, S \subseteq [N]$ are disjoint. Then for every integer $k \ge 0$,*

$$(-1)^k \cdot \mathbf{E}[X_V \mid \overline{X}_S = 1] \le (-1)^{k+|V|-1} \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ |U| \le k}} q(V \cup U, S). \tag{23}$$

Proof. We claim that it suffices to prove that for every integer $k \ge 0$,

$$(-1)^k \cdot \mathbf{E}\left[X_V \overline{X}_S\right] \le \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 0 \le |U| \le k}} (-1)^{k+|U|}\, \mathbf{E}\left[X_{V \cup U}\right] \mathbf{E}\left[\overline{X}_{S \cap I(V \cup U)}\right]. \tag{24}$$

Indeed, (24) implies (23) because

$$\mathbf{E}\left[\overline{X}_{S \cap I(V \cup U)}\right] = \mathbf{P}\left[\overline{X}_S = 1\right] \cdot \mathbf{E}\left[\overline{X}_{S \setminus I(V \cup U)} \mid \overline{X}_{S \cap I(V \cup U)} = 1\right]^{-1}$$

and because definition (20) gives

$$q(V \cup U, S) = \frac{(-1)^{|V|+|U|-1}\, \mathbf{E}[X_{V \cup U}]}{\mathbf{E}[\overline{X}_{S \setminus I(V \cup U)} \mid \overline{X}_{S \cap I(V \cup U)} = 1]}.$$

We prove (24) by induction on k. When $k=0$, this inequality simplifies to

$$\mathbf{E}[X_V \overline{X}_S] \le \mathbf{E}[X_V]\, \mathbf{E}[\overline{X}_{S \cap I(V)}],$$

which holds because $\overline{X}_S \leq \overline{X}_{S\cap I(V)}$ and because X_V and $X_{S\cap I(V)}$ are independent. Assume now that $k \geq 1$ and that (24) holds for all k' with $0 \leq k' < k$. It follows from the Bonferroni inequalities that

$$(-1)^k \cdot \overline{X}_{S\cap\partial(V)} \leq (-1)^k \cdot \sum_{\substack{U'\subseteq S\cap\partial(V)\\ |U'|\leq k}} (-1)^{|U'|} X_{U'}. \tag{25}$$

Since S and V are disjoint and $\partial(V) \cup V = V^c$, then multiplying (25) through by $X_V \overline{X}_{S\cap I(V)}$ and taking expectations yields

$$(-1)^k \cdot \mathbf{E}[X_V \overline{X}_S] \leq \sum_{\substack{U'\subseteq S\cap\partial(V)\\ |U'|\leq k}} (-1)^{k+|U'|}\, \mathbf{E}[X_{V\cup U'} \overline{X}_{S\cap I(V)}] \tag{26}$$

Observe that for every $U' \subseteq S \cap \partial(V)$, the sets $V \cup U'$ and $S \cap I(V)$ are disjoint. In particular, if U' is non-empty, then we may appeal to the induction hypothesis (with $k \leftarrow k - |U'|$) to bound each term in the right-hand side of (26) as follows. As $S \cap I(V) \cap I(V \cup U' \cup U'') = S \cap I(V \cup U' \cup U'')$, then

$$\begin{aligned}(-1)^{k+|U'|} &\cdot \mathbf{E}[X_{V\cup U'} \overline{X}_{S\cap I(V)}] \\ &\leq \sum_{\substack{U''\subseteq S\cap I(V)\\ U''\hookrightarrow V\cup U'\\ 0\leq|U''|\leq k-|U'|}} (-1)^{k+|U'|+|U''|}\, \mathbf{E}[X_{V\cup U'\cup U''}]\, \mathbf{E}[\overline{X}_{S\cap I(V\cup U'\cup U'')}]. \end{aligned} \tag{27}$$

Finally, observe that every non-empty $U \subseteq S$ such that $U \hookrightarrow V$ can be partitioned into a non-empty $U' \subseteq S \cap \partial(V)$ and an $U'' \subseteq S \cap I(V)$ such that $U'' \hookrightarrow (V \cup U')$ in a unique way. Indeed, one sets $U' = U \cap \partial(V)$ and $U'' = U \setminus U'$; this is the only such partition. Since $\emptyset \hookrightarrow V$ by definition, then bounding each term in (26) that corresponds to a non-empty U' using (27) and rearranging the sum gives (24). $\square$

Proof of Lemma 19. Fix an $\ell \in [N]$ and an integer k such that $\rho_{k+1} \leq 1 - \varepsilon$. Invoking Lemma 24 with $V = \{\ell\}$ and $S = [\ell - 1]$ twice, first with $k \leftarrow k - 1$ and then with $k \leftarrow k$, to get both an upper and a lower bound on $\mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]}]$, we obtain

$$\Bigg| \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{\substack{U\subseteq[\ell-1], U\hookrightarrow\{\ell\}\\ |U|\leq k-1}} q(U \cup \{\ell\}, [\ell - 1]) \Bigg| \leq \Bigg| \sum_{\substack{U\subseteq[\ell-1], U\hookrightarrow\{\ell\}\\ |U|=k}} q(U \cup \{\ell\}, [\ell - 1]) \Bigg|.$$

Since the sets $U \cup \{\ell\}$ with $U \subseteq [\ell - 1]$, $U \hookrightarrow \{\ell\}$, and $|U| = i - 1$ are precisely the elements of $\mathcal{C}_i(\ell)$, we can rewrite the above inequality as

$$\Bigg| \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{i\in[k]} \sum_{V\in\mathcal{C}_i(\ell)} q(V, [\ell - 1]) \Bigg| \leq \sum_{V\in\mathcal{C}_{k+1}(\ell)} |q(V, [\ell - 1])|. \tag{28}$$

It follows from definition (20) and Harris' inequality that

$$\begin{aligned} |q(V, S)| &= \frac{\mathbf{E}[X_V]}{\mathbf{E}[\overline{X}_{S\setminus I(V)} \mid \overline{X}_{S\cap I(V)} = 1]} \\ &= \frac{\mathbf{E}[X_V]}{1 - \mathbf{P}[X_i = 1 \text{ for some } i \in S \setminus I(V) \mid \overline{X}_{S\cap I(V)} = 1]} \leq \frac{\mathbf{E}[X_V]}{1 - \rho_V}, \end{aligned}$$

Since $\rho_V \le \rho_{k+1} \le 1-\varepsilon$ for all V with $|V| = k+1$, summing (28) over all $\ell \in [N]$ yields

$$\Big| \sum_{\ell \in [N]} \mathbf{E}[X_\ell \mid \overline{X}_{[\ell-1]} = 1] - \sum_{\ell \in [N]} \sum_{i \in [k]} \sum_{V \in \mathcal{C}_i(\ell)} q(V, [\ell-1]) \Big| \le \Delta_{k+1}/\varepsilon,$$

which is precisely the assertion of the lemma. □

3.2 Proof of Lemma 22 – preliminaries

The goal of this subsection is to prove a recursive formula for $\kappa_V(W)$, Lemma 28 below, which will be used in the proof of Lemma 22.

Definition 25. Suppose that $\emptyset \ne V \subseteq W \subseteq [N]$. Define sets $\Pi_V(W)$ and $\Pi_V^{\hookrightarrow}(W)$ as follows:

1. $\Pi_V(W)$ is the set of all partitions of W that contain V as a part;
2. $\Pi_V^{\hookrightarrow}(W)$ is the set of all partitions $\pi \in \Pi_V(W)$ such that $U \hookrightarrow V$ for all $U \in \pi \setminus \{V\}$.

Since by now we have defined several different classes of partitions of a set W, now is a good moment to pause and convince ourselves that

$$\Pi_V^{\hookrightarrow}(W) \subseteq \Pi_V(W) \subseteq \Pi_V^{\mathsf{C}}(W) \subseteq \Pi(W).$$

As a first step towards the promised recursive formula, we give an alternative expression for $\kappa_V(W)$.

Definition 26 (Degree of a part in a partition)**.** For a partition π of a subset of $[N]$ and any part $P \in \pi$, let $d_\pi(P)$ denote the number of parts $P' \in \pi \setminus \{P\}$ such that G_Γ contains an edge between P' and P. We call $d_\pi(P)$ the *degree* of P in π.

Lemma 27. *If $\emptyset \ne V \subseteq W \subseteq [N]$, then*

$$\kappa_V(W) = \sum_{\pi \in \Pi_V(W)} (-1)^{|\pi|-1} \chi_V(\pi) \mu_\pi,$$

where

$$\chi_V(\pi) = \begin{cases} 1 & \text{if } |\pi| = 1 \\ d_\pi(V)(|\pi|-2)! & \text{if } |\pi| \ge 2. \end{cases}$$

Proof. Given a $\pi \in \Pi_V^{\mathsf{C}}(W)$, let P denote the part of π containing V. Define a map $f\colon \Pi_V^{\mathsf{C}}(W) \to \Pi_V(W)$ as follows. If $P = V$, then let $f(\pi) = \pi$. Otherwise, let $f(\pi)$ be the partition obtained from π by splitting P into V and $P \setminus V$. Clearly,

$$\kappa_V(W) = \sum_{\pi \in \Pi_V^{\mathsf{C}}(W)} (-1)^{|\pi|-1}(|\pi|-1)!\mu_\pi = \sum_{\pi \in \Pi_V(W)} \sum_{\pi' \in f^{-1}(\pi)} (-1)^{|\pi'|-1}(|\pi'|-1)!\mu_{\pi'}.$$

Observe that every $\pi \in \Pi_V(W)$ has exactly $|\pi| - d_\pi(V)$ preimages via f. One of them is π itself and there are $|\pi| - 1 - d_\pi(V)$ additional partitions obtained from π by merging V with some other part $Q \in \pi$ such that G_γ contains no edges between V and Q. In particular, there is one preimage of size $|\pi|$ and there are $|\pi| - 1 - d_\pi(V)$ preimages of size $|\pi| - 1$. Furthermore,

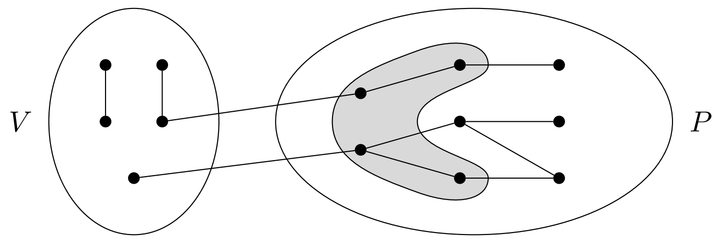

Figure 3: A set in $\mathrm{Cut}_V(P)$. Every element of $\mathrm{Cut}_V(P)$ is a cutset in $G_\Gamma(V \cup P)$ that disconnects V from P.

note that $\mu_{\pi'} = \mu_\pi$ for every $\pi' \in f^{-1}(\pi)$. Indeed, for every $Q \in \pi$ with no edges of G_Γ between Q and V, we have

$$\mathbf{E}[X_V] \cdot \mathbf{E}[X_Q] = \mathbf{E}[X_V X_Q] = \mathbf{E}[X_{V \cup Q}].$$

It follows that

$$\begin{aligned}\kappa_V(W) &= \sum_{\pi \in \Pi_V(W)} (-1)^{|\pi|-1} \Big((|\pi|-1)! - (|\pi| - 1 - d_\pi(V)) \cdot (|\pi|-2)! \Big) \mu_\pi \\ &= \sum_{\pi \in \Pi_V(W)} (-1)^{|\pi|-1} \chi_V(\pi) \mu_\pi,\end{aligned}$$

as claimed. $\square$

The following lemma is the main result of this subsection and the essential combinatorial ingredient of the proof of Lemma 22.

Lemma 28. *Suppose that $\emptyset \neq V \subseteq W \subseteq [N]$ and $W \hookrightarrow V$. For a set $P \subseteq W \setminus V$ such that $P \hookrightarrow V$, write $\mathrm{Cut}_V(P)$ for the collection of all sets C satisfying $\partial(V) \cap P \subseteq C \subseteq P$ and $C \hookrightarrow V$. Then*

$$\kappa_V(W) = \mathbf{E}[X_V] \sum_{\pi \in \Pi_V^{\hookrightarrow}(W)} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{\substack{P \in \pi \\ P \neq V}} \sum_{C \in \mathrm{Cut}_V(P)} \kappa_C(P). \tag{29}$$

Proof. Denote the right hand side of (29) by $r_V(W)$. We need to show $\kappa_V(W) = r_V(W)$. Let us first rewrite the inner sum in (29). To this end, fix some non-empty $P \subseteq W \setminus V$ such that $P \hookrightarrow V$. By the definition of $\kappa_C(P)$, see (21),

$$\sum_{C \in \mathrm{Cut}_V(P)} \kappa_C(P) = \sum_{C \in \mathrm{Cut}_V(P)} \sum_{\pi \in \Pi_C^{\mathsf{C}}(P)} (-1)^{|\pi|-1} (|\pi|-1)! \mu_\pi. \tag{30}$$

We may write this double sum more compactly as follows. For brevity, let $\partial_P(V) := \partial(V) \cap P$. Denote by $\tilde{\Pi}_V(P)$ the set of all partitions $\pi \in \Pi(P)$ such that some $Q \in \pi$ contains all neighbours of V in P, that is, such that $\partial_P(V) \subseteq Q$ for some $Q \in \pi$. We claim that

$$\sum_{C \in \mathrm{Cut}_V(P)} \kappa_C(P) = \sum_{\pi \in \tilde{\Pi}_V(P)} (-1)^{|\pi|-1} (|\pi|-1)! \mu_\pi. \tag{31}$$

Indeed, this follows from (30) because, letting

$$\mathcal{Q}(V,P) = \{(C,\pi)\colon C \in \mathrm{Cut}_V(P) \text{ and } \pi \in \Pi_C^{\mathsf{C}}(P)\},$$

the projection $p_2\colon \mathcal{Q}(V,P) \ni (C,\pi) \mapsto \pi \in \Pi(P)$ is a bijection between $\mathcal{Q}(V,P)$ and $\tilde{\Pi}_V(P)$. This is because for every $(C,\pi) \in \mathcal{Q}(V,P)$, C is the union of those connected components of $G_\Gamma(Q)$ that intersect $\partial_P(V)$. Furthermore, observe that the right-hand side of (31) is simply the joint cumulant of the set

$$P_V := \{X_i\colon i \in P \setminus \partial_P(V)\} \cup \{X_{\partial_P(V)}\},$$

which is obtained from P by replacing $\{X_i\colon i \in \partial_P(V)\}$ with the single variable $X_{\partial_P(V)}$. Therefore, it follows from (31) that

$$r_V(W) = \mathbf{E}[X_V] \sum_{\pi \in \Pi_V^{\hookrightarrow}(W)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{\substack{P\in\pi \\ P\neq V}} \kappa(P_V). \tag{32}$$

Let $\Pi'_V(W)$ be the set of all partitions in $\Pi_V(W)$ whose every part, except possibly V itself, contains a neighbour of V. We claim that the product in the right-hand side of (32) is zero for every $\pi \in \Pi'_V(W) \setminus \Pi_V^{\hookrightarrow}(W)$ and hence we may replace $\Pi_V^{\hookrightarrow}(W)$ with $\Pi'_V(W)$ in the range of summation in (32). Indeed, if $\pi \in \Pi'_V(W) \setminus \Pi_V^{\hookrightarrow}(W)$, then there is a $P \in \pi \setminus \{V\}$ such that $\partial_P(V) \neq \emptyset$ but $P \not\hookrightarrow V$. In particular, some connected component of $G_\Gamma[P]$ is disjoint from $\partial_P(V)$ and hence $\kappa(P_V) = 0$. Expanding $\kappa(P_V)$ again, we obtain

$$r_V(W) = \mathbf{E}[X_V] \sum_{\pi \in \Pi'_V(W)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{\substack{P\in\pi \\ P\neq V}} \sum_{\pi' \in \tilde{\Pi}_V(P)} (-1)^{|\pi'|-1}(|\pi'|-1)!\mu_{\pi'}. \tag{33}$$

Let us write $\mathcal{P}$ to denote the set of all pairs $(\pi,\pi^*) \in \Pi'_V(W) \times \Pi_V(W)$ obtained as follows. Choose an arbitrary partition $\pi \in \Pi'_V(W)$ and refine every $P \in \pi \setminus \{V\}$ by replacing it by some $\pi_P \in \tilde{\Pi}_V(P)$, so that $\partial_P(V)$ is contained in a single part of π_P; finally, let π^* be the resulting partition of W.

Suppose that $(\pi,\pi^*) \in \mathcal{P}$. Enumerate the parts of π as $V, P_1, \ldots, P_t$ and suppose that π^* was obtained from π by refining each P_j into $i_j + 1$ parts, so that $|\pi^*| = t + 1 + i_1 + \ldots + i_t$. Then, letting

$$f(\pi,\pi^*) = f_t(i_1,\ldots,i_t) := (-1)^t t! \prod_{i\in[t]} (-1)^{i_j} i_j! = (-1)^{|\pi^*|-1} t! \prod_{j\in[t]} i_j!,$$

we may rewrite (33) as

$$r_V(W) = \sum_{(\pi,\pi^*)\in\mathcal{P}} f(\pi,\pi^*)\mu_{\pi^*}. \tag{34}$$

Fix some $\pi^* \in \Pi_V(W)$ and note that π^* contains $d_{\pi^*}(V)$ parts other than V that intersect $\partial(V)$. Write $s := |\pi^*|$, $t := d_{\pi^*}(V)$, and $\pi^* = \{V, P_1^*, \ldots, P_{s-1}^*\}$ so that $P_1^*, \ldots, P_t^*$ are the parts intersecting $\partial(V)$. Fix an arbitrary permutation σ of $[s-1]$ such that $\sigma(1) \in [t]$. Such a σ can be used to define a π such that $(\pi,\pi^*) \in \mathcal{P}$ in the following way. Consider the sequence $P_\sigma^* := (P_{\sigma(1)}^*, \ldots, P_{\sigma(s-1)}^*)$. For every $i \in [t]$, let P_i be the union of P_i^* and all the P_j^*, with $j \in [s-1] \setminus [t]$, for which P_i^* is the right-most element among $P_1^*, \ldots, P_t^*$ that is to the left of P_j^* in P_σ^*. (Since $\sigma(1) \in [t]$, then each P_j^* with $j \in [s-1] \setminus [t]$ has one of $P_1^*, \ldots, P_t^*$ left of it.) A moment's thought reveals that each partition π with $(\pi, \pi^*) \in \mathcal{P}$ is obtained this way from exactly $|f(\pi, \pi^*)|$ permutations σ. It follows that

$$
\begin{aligned}
r_V(W) &= \sum_{\pi^* \in \Pi_V(W)} (-1)^{|\pi^*|-1} \mu_{\pi^*} \sum_{\substack{\pi \in \Pi'_V(W) \\ (\pi,\pi^*) \in \mathcal{P}}} |f(\pi, \pi^*)| \\
&= \sum_{\pi^* \in \Pi_V(W)} (-1)^{|\pi^*|-1} \mu_{\pi^*} \cdot |\{\sigma \in \mathrm{Sym}(|\pi^*|-1) \colon \sigma(1) \in \{1, \dots, d_{\pi^*}(V)\}\}| \\
&= \sum_{\pi^* \in \Pi_V(W)} (-1)^{|\pi^*|-1} \mu_{\pi^*} \cdot \chi_V(\pi^*),
\end{aligned}
$$

where $\chi_V(\pi^*)$ is as defined in Lemma 27. By Lemma 27, we conclude that $r_V(W) = \kappa_V(W)$, as required. □

3.3 Proof of Lemma 22

For $V, S \subseteq [N]$ and $k \in \mathbf{N}$ such that $0 \le |V| \le k$, we define

$$
\tilde{\kappa}_V^{(k)}(S) := (-1)^{|V|-1} \,\mathbf{E}[X_V] \sum_{0 \le i \le k-|V|} \Big(\sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} \kappa_U^{(k-|V|)}(S \cap I(V)) \Big)^i \tag{35}
$$

and

$$
q^{(k)}(V, S) := (-1)^{|V|-1} \,\mathbf{E}[X_V] \sum_{0 \le i \le k-|V|} \Big(\sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k-|V|}} q(U, S \cap I(V)) \Big)^i. \tag{36}
$$

Our proof of Lemma 22 consists of three steps. First, in Lemma 29, we show that $q(V, S) \approx q^{(k)}(V, S)$. Second, in Lemma 30, we show that $\kappa_V^{(k)}(S) \approx \tilde{\kappa}_V^{(k)}(S)$. Finally, the fact that $q^{(k)}(V, S)$ and $\tilde{\kappa}_V^{(k)}(S)$ satisfy similar recurrences (given the above approximate equalities) allows us to prove that also $q(V, S) \approx \kappa_V^{(k)}(S)$. Lemma 22 then follows easily. The precise definition of '$\approx$' above will be expressed by the following quantities. For integers k and K satisfying $1 \le k \le K$,

$$
\Delta_k(V) := \sum_{\substack{U \hookrightarrow V \\ |U \cup V| = k}} \mathbf{E}[X_{U \cup V}] \qquad \text{and} \qquad \Delta_{k,K}(V) = \sum_{j=k}^{K} \Delta_j(V). \tag{37}
$$

and

$$
\delta_{k,K}(V) := \sum_{\substack{U \hookrightarrow V \\ k \le |U \cup V| \le K}} \mathbf{E}[X_{U \cup V}] \max\{\mathbf{E}[X_i] \colon i \in U \cup V\}. \tag{38}
$$

Lemma 29. *Let $\varepsilon > 0$ and $k \in \mathbf{N}$ be such that $\rho_k \le 1 - \varepsilon$. There exists $K = K(k, \varepsilon)$ such that for all $V, S \subseteq [N]$ with $1 \le |V| \le k$,*

$$
|q(V, S) - q^{(k)}(V, S)| \le K \cdot \big(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\big).
$$

Proof. Fix V and S as in the statement of the lemma and set

$$x := \mathbf{P}[X_i = 1 \text{ for some } i \in S \setminus I(V) \mid \overline{X}_{S \cap I(V)} = 1].$$

Then by definition

$$q(V, S) = \frac{(-1)^{|V|-1}\,\mathbf{E}[X_V]}{\mathbf{E}[\overline{X}_{S \setminus I(V)} \mid \overline{X}_{S \cap I(V)} = 1]} = \frac{(-1)^{|V|-1}\,\mathbf{E}[X_V]}{1 - x}. \tag{39}$$

Since $0 \le x \le \rho_V$, by Harris' inequality, and $\rho_V \le \rho_k \le 1 - \varepsilon$, as $|V| \le k$, then (39) and the identity $(1-x)^{-1} = 1 + x + \ldots + x^{k-|V|} + x^{k-|V|+1}(1-x)^{-1}$ yield

$$\left| q(V, S) - (-1)^{|V|-1}\,\mathbf{E}[X_V] \cdot (1 + x + x^2 + \cdots + x^{k-|V|}) \right| \le \varepsilon^{-1}\,\mathbf{E}[X_V]\rho_V^{k-|V|+1}. \tag{40}$$

We now observe that

$$\mathbf{E}[X_V]\rho_V^{k-|V|+1} \le \mathbf{E}[X_V]\Big(\sum_{i \in V \cup \partial(V)} \mathbf{E}[X_i] \Big)^{k-|V|+1} = \mathbf{E}[X_V] \sum_{i_1, \ldots, i_{k-|V|+1}} \prod_{j=1}^{k-|V|+1} \mathbf{E}[X_{i_j}]$$

and note that if $i_1, \ldots, i_{k-|V|+1}$ are distinct elements of $\partial(V)$, then

$$\mathbf{E}[X_V] \prod_{j=1}^{k-|V|+1} \mathbf{E}[X_{i_j}] \le \mathbf{E}[X_{V \cup \{i_1, \ldots, i_{k-|V|+1}\}}]$$

by Harris' inequality; if, on the other hand, either $i_j \in V$ for some j or some two i_j are equal, then Harris' inequality and the fact that $|\,\mathbf{E}[X_i]| \le 1$ for each i imply the stronger bound

$$\mathbf{E}[X_V] \prod_{j=1}^{k-|V|+1} \mathbf{E}[X_{i_j}] \le \mathbf{E}[X_{V \cup \{i_1, \ldots, i_{k-|V|+1}\}}] \cdot \max\{\mathbf{E}[X_i] \colon i \in V \cup \{i_1, \ldots, i_{k-|V|+1}\}\}. \tag{41}$$

In particular, the right-hand side of (40) is bounded from above by

$$\varepsilon^{-1} \cdot (k - |V| + 1)! \cdot \Delta_{k+1}(V) + \varepsilon^{-1} \cdot k^{k-|V|+1} \cdot \delta_{1,k}(V),$$

which yields

$$\left| q(V, S) - (-1)^{|V|-1}\,\mathbf{E}[X_V] \cdot (1 + x + \cdots + x^{k-|V|}) \right| \le K_1 \cdot \big(\Delta_{k+1}(V) + \delta_{1,k}(V) \big) \tag{42}$$

for some constant K_1 that depends only on k and ε.

We claim that there is a constant $K_2 = K_2(k, \varepsilon)$ such that for all $i \in \{0, \ldots, k - |V|\}$,

$$\mathbf{E}[X_V] \cdot \left| x^i - \Big(\sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \le |U| \le k - |V|}} q(U, S \cap I(V)) \Big)^i \right| \le K_2 \cdot \big(\delta_{1,K_2}(V) + \Delta_{k+1,K_2}(V) \big). \tag{43}$$

Observe that (42) and (43) imply that

$$|q(V, S) - q^{(k)}(V, S)| \le K \cdot \big(\delta_{1,K}(V) + \Delta_{k+1,K}(V) \big)$$

for some $K = K(k, \varepsilon)$, giving the assertion of the lemma. It thus remains to prove (43).

We first consider the case $i = 1$. By the Bonferroni inequalities, for every positive j,

$$(-1)^{j-1} \cdot x \leq (-1)^{j-1} \cdot \sum_{\substack{U' \subseteq S \setminus I(V) \\ 1 \leq |U'| \leq j}} (-1)^{|U'|-1} \mathbf{E}[X_{U'} \mid \overline{X}_{S \cap I(V)} = 1].$$

Applying Lemma 24 with $k \leftarrow j - |U'|$, $V \leftarrow U'$, and $S \leftarrow S \cap I(V)$, we get that for each $U' \subseteq S \setminus I(V)$ with $1 \leq |U'| \leq j$,

$$(-1)^{|U'|-1} \mathbf{E}[X_{U'} \mid \overline{X}_{S \cap I(V)} = 1] \leq \sum_{\substack{U'' \subseteq S \cap I(V), U'' \hookrightarrow U' \\ 0 \leq |U''| \leq j - |U'|}} q(U' \cup U'', S \cap I(V)).$$

Next, observe that any non-empty $U \subseteq S$ with $U \hookrightarrow V$ of size at most j can be written uniquely as the disjoint union of U' and U'', where $U' \subseteq V \cup \partial(V)$ and $U'' \subseteq I(V)$ and $U'' \hookrightarrow U'$. The previous two equations then imply that

$$(-1)^{j-1} \cdot x \leq (-1)^{j-1} \cdot \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k - |V|}} q(U, S \cap I(V)). \tag{44}$$

Invoking (44) twice, first with $j \leftarrow k - |V|$ and then with $j \leftarrow k - |V| + 1$, to get both an upper and a lower bound on x, we obtain

$$\begin{aligned} \Bigg| x - \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k - |V|}} q(U, S \cap I(V)) \Bigg| &\leq \Bigg| \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ |U| = k - |V| + 1}} q(U, S \cap I(V)) \Bigg| \\ &\leq \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ |U| = k - |V| + 1}} \varepsilon^{-1} \mathbf{E}[X_U], \end{aligned} \tag{45}$$

where the last inequality uses the definition of $q(U, S \cap I_V)$ and the assumption that $\rho_k \leq 1 - \varepsilon$, see the discussion below (39).

Finally, we show how to deduce (43) from (45). Let

$$y := \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k - |V|}} q(U, S \cap I(V)),$$

so that the left-hand side of (43) is $\mathbf{E}[X_V] \cdot |x^i - y^i|$, and observe that, as in (45),

$$|y| \leq z := \sum_{\substack{U \hookrightarrow V \\ 1 \leq |U| \leq k - |V|}} \varepsilon^{-1} \mathbf{E}[X_U].$$

Fix an $i \in \{1, \ldots, k - |V|\}$. Since $|x| \leq 1$, then

$$|x^i - y^i| \leq |x - y| \cdot \sum_{j=0}^{i-1} |x^j y^{i-1-j}| \leq (1 + z)^{i-1} \cdot |x - y|,$$

which together with (45) implies that

$$\mathbf{E}[X_V] \cdot |x^i - y^i| \leq (1 + z)^{i-1} \mathbf{E}[X_V] \sum_{\substack{U \hookrightarrow V \\ |U| = k - |V| + 1}} \varepsilon^{-1} \mathbf{E}[X_U].$$

Note that for pairwise disjoint $U_1, \dots, U_j \subseteq [N]$, Harris' inequality gives

$$\prod_{\ell=1}^{j} \mathbf{E}[X_{U_\ell}] \leq \mathbf{E}[X_{U_1 \cup \ldots \cup U_j}]$$

and if $U_1, \dots, U_j \subseteq [N]$ are not pairwise disjoint, then the stronger FKG lattice condition (15) implies that

$$\prod_{\ell=1}^{j} \mathbf{E}[X_{U_\ell}] \leq \mathbf{E}[X_{U_1 \cup \ldots \cup U_j}] \cdot \max\{\mathbf{E}[X_i] \colon i \in U_1 \cup \ldots \cup U_j\}.$$

In particular, using a similar reasoning as for bounding the right-hand side of (40), we obtain

$$(1+z)^{i-1} \, \mathbf{E}[X_V] \sum_{\substack{U \hookrightarrow V \\ |U| = k - |V| + 1}} \varepsilon^{-1} \, \mathbf{E}[X_U] \leq K_4 \cdot \big(\delta_{1,ik}(V) + \Delta_{k+1,ik+1}(V)\big)$$

for sufficiently large $K_4 = K_4(k, \varepsilon)$. This shows (43) and hence the the lemma. □

Lemma 30. *For all $k \in \mathbf{N}$ there exists $K = K(k)$ such that the following holds. Suppose that $V, S \subseteq [N]$ where $1 \leq |V| \leq k$. Then*

$$|\kappa_V^{(k)}(S) - \tilde{\kappa}_V^{(k)}(S)| \leq K \cdot \big(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\big).$$

Proof. Fix k, S, and V as in the statement of the lemma and let

$$x := \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k - |V|}} \kappa_U^{(k-|V|)}(S \cap I(V)),$$

so that

$$\tilde{\kappa}_V^{(k)}(S) = (-1)^{|V|-1} \, \mathbf{E}[X_V](1 + x + x^2 + \cdots + x^{k-|V|}). \tag{46}$$

Recalling the definition (22), we may rewrite

$$x = \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k - |V|}} \; \sum_{\substack{U \subseteq W \subseteq U \cup (S \cap I(V)) \\ W \hookrightarrow U, |W| \leq k - |V|}} (-1)^{|W|-1} \kappa_U(W). \tag{47}$$

Recalling from the statement of Lemma 28 that

$$\mathrm{Cut}_V(W) = \{U \subseteq W \colon U \hookrightarrow V \text{ and } \partial(V) \cap W \subseteq U\},$$

we may switch the order of summation in (47) to obtain

$$x = \sum_{\substack{W \subseteq S, W \hookrightarrow V \\ 1 \leq |W| \leq k - |V|}} \; \sum_{U \in \mathrm{Cut}_V(W)} (-1)^{|W|-1} \kappa_U(W).$$

For the sake of brevity, write

$$f(W) := \sum_{U \in \mathrm{Cut}_V(W)} (-1)^{|W|-1} \kappa_U(W).$$

We may now rewrite (46) as

$$\tilde{\kappa}_V^{(k)}(S) = (-1)^{|V|-1}\,\mathbf{E}[X_V] \sum_{i=0}^{k-|V|} \sum_{\substack{W_1,\dots,W_i \subseteq S \\ W_1,\dots,W_i \hookrightarrow V \\ 1 \le |W_1|,\dots,|W_i| \le k-|V|}} f(W_1)\cdot \ldots \cdot f(W_i). \tag{48}$$

Consider first the total contribution $\tilde{\kappa}_1$ to the right-hand side of (48) of terms corresponding to $W_1,\dots,W_i \subseteq S \setminus V$ that are pairwise disjoint and whose union has size at most $k-|V|$. Each such term may be regarded as a partition of the set $W := V \cup W_1 \cup \ldots \cup W_i$ which satisfies $V \subseteq W \subseteq S$ and $|W| \le k$; this partition $\{V, W_1, \dots, W_i\}$ belongs to $\Pi_V^{\hookrightarrow}(W)$. Conversely, given a W with these properties, every partition $\pi \in \Pi_V^{\hookrightarrow}(W)$ corresponds to exactly $(|\pi|-1)!$ such terms; this is the number of ways to order the elements of $\pi \setminus \{V\}$ as $W_1,\dots,W_i$. Therefore,

$$\tilde{\kappa}_1 = (-1)^{|V|-1}\,\mathbf{E}[X_V] \sum_{\substack{V \subseteq W \subseteq V \cup S \\ W \hookrightarrow V, |W| \le k}} \sum_{\pi \in \Pi_V^{\hookrightarrow}(W)} (|\pi|-1)! \prod_{\substack{P \in \pi \\ P \ne V}} f(P).$$

In particular, Lemma 28 gives

$$\tilde{\kappa}_1 = (-1)^{|V|-1} \sum_{\substack{V \subseteq W \subseteq V \cup S \\ W \hookrightarrow V, |W| \le k}} (-1)^{|W|-|V|} \kappa_V(W) = \kappa_V^{(k)}(S).$$

Every term in the right-hand side of (48) corresponding to $W_1,\dots,W_i$ that is not included in $\tilde{\kappa}_1$ either satisfies $|V \cup W_1 \cup \ldots \cup W_i| > k$ or the sets $V, W_1, \dots, W_i$ are not pairwise disjoint. Let $\tilde{\kappa}_2 := \tilde{\kappa}_V^{(k)}(S)$ denote the total contribution of these terms. Since for every W,

$$|f(W)| \le \sum_{U \subseteq W} |\kappa_U(W)| \le \sum_{\pi \in \Pi(W)} |\pi|!\mu_\pi \le |W|^{|W|}\,\mathbf{E}[X_W],$$

there is a constant K_1 that depends only on k such that

$$|\tilde{\kappa}_2| \le K_1\,\mathbf{E}[X_V] \sum_{W_1,\dots,W_i} \prod_{j=1}^{i} \mathbf{E}[X_{W_j}],$$

where the sum ranges over all $i \le k-|V|$ and $W_1,\dots,W_i \subseteq S$, each of size at most $k-|V|$, such that either $|V \cup W_1 \cup \ldots \cup W_i| > k$ or the sets $V, W_1, \dots, W_i$ are not pairwise disjoint. An argument analogous to the one given at the end of the proof of Lemma 29, employing Harris' inequality and the stronger FKG lattice condition (15), gives

$$|\tilde{\kappa}_2| \le K \cdot \big(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\big)$$

for some K that depends only on k. □

Lemma 31. *Let $k \in \mathbf{N}$ be such that $\rho_k \le 1-\varepsilon$. Then there exists $K = K(k,\varepsilon)$ such that for all $V, S \subseteq [N]$ where $1 \le |V| \le k$, we have*

$$|q(V,S) - \kappa_V^{(k)}(S)| \le K \cdot \big(\delta_{1,K}(V) + \Delta_{k+1,K}(V)\big).$$

Proof. We prove the lemma by complete induction on k. To this end, let $k \geq 0$ and suppose that the statement holds for all $k' \in \mathbf{N}$ with $k' < k$. By the triangle inequality

$$|q(V,S) - \kappa_V^{(k)}(S)| \leq |q(V,S) - q^{(k)}(V,S)| + |q^{(k)}(V,S) - \tilde{\kappa}_V^{(k)}(S)| + |\tilde{\kappa}_V^{(k)}(S) - \kappa_V^{(k)}(S)|.$$

Lemmas 29 and 30 imply that

$$|q(V,S) - q^{(k)}(V,S)| + |\tilde{\kappa}_V^{(k)}(S) - \kappa_V^{(k)}(S)| \leq K_1 \cdot \big(\delta_{1,K_1}(V) + \Delta_{k+1,K_1}(V)\big)$$

for some sufficiently large $K_1 = K_1(k,\varepsilon)$ and thus it suffices to show that there is some $K_2 = K_2(k,\varepsilon)$ such that

$$|q^{(k)}(V,S) - \tilde{\kappa}_V^{(k)}(S)| \leq K_2 \cdot \big(\delta_{1,K_2}(V) + \Delta_{k+1,K_2}(V)\big). \tag{49}$$

To this end, observe first that since $k - |V| < k$, then the induction hypothesis states that there is a constant $K' = K'(k,\varepsilon)$ such that

$$\big|q(U, S \cap I(V)) - \kappa_U^{(k-|V|)}(S \cap I(V))\big| \leq K' \cdot \big(\delta_{1,K'}(U) + \Delta_{k-|V|+1,K'}(U)\big) \tag{50}$$

for all U such that $1 \leq |U| \leq k - |V|$. Let

$$x := \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k-|V|}} \kappa_U^{(k-|V|)}(S \cap I(V))$$

and, as in the proof of Lemma 29,

$$y := \sum_{\substack{U \subseteq S, U \hookrightarrow V \\ 1 \leq |U| \leq k-|V|}} q(U, S \cap I(V)).$$

Observe that

$$|y| \leq z := \sum_{\substack{U \hookrightarrow V \\ 1 \leq |U| \leq k-|V|}} \varepsilon^{-1}\, \mathbf{E}[X_U],$$

as in the proof of Lemma 29, and that (50) implies that

$$|x - y| \leq w := K' \cdot \sum_{\substack{U \hookrightarrow V \\ 1 \leq |U| \leq k-|V|}} \big(\delta_{1,K'}(U) + \Delta_{k-|V|+1,K'}(U)\big). \tag{51}$$

For any $i \geq 1$, we have

$$|x^i - y^i| \leq |x - y| \cdot \sum_{j=0}^{i-1} |x_j y^{i-1-j}| \leq |x - y| \cdot (|x| + |y|)^{i-1} \leq w(2z + w)^{i-1}.$$

It follows that

$$|q^{(k)}(V,S) - \tilde{\kappa}_V^{(k)}(S)| \leq \sum_{1 \leq i \leq k-|V|} \mathbf{E}[X_V] \cdot w(2z + w)^{i-1}. \tag{52}$$

Similarly as in the proofs of Lemmas 29 and 30, one sees that the FKG lattice condition (15) implies that the right hand side of (52) is bounded from above by $K_2 \cdot \big(\delta_{1,K_2}(V) + \Delta_{k+1,K_2}(V)\big)$, provided $K_2 = K_2(k,\varepsilon)$ is sufficiently large, as claimed. $\square$

Proof of Lemma 22. It follows from Lemma 31 that there is $K_1 = K_1(k, \varepsilon)$ such that

$$\Big|\sum_{\ell\in[N]}\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\big(q(V,[\ell-1]) - \kappa_V^{(k)}(S)\big)\Big| \le \sum_{\ell\in[N]}\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)} K_1\cdot\big(\delta_{1,K_1}(V) + \Delta_{k+1,K_1}(V)\big).$$

But if we choose K sufficiently large then the right-hand side is at most $K\cdot\big(\delta_{1,K} + \Delta_{k+1,K}\big)$, as required. $\square$

3.4 Proof of Lemma 23

Fix an integer k and an $\ell \in [N]$. Recalling (22), we rewrite the ℓth term of the sum from the statement of the lemma as follows:

$$\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\kappa_V^{(k)}([\ell-1]) = \sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\sum_{\substack{V\subseteq W\subseteq V\cup[\ell-1]\\ W\hookrightarrow V\\ |W|\le k}}(-1)^{|W|-1}\kappa_V(W).$$

It follows from Definition 20 that if V is connected then $W \hookrightarrow V$ if and only if W is connected. Therefore, changing the order of the last two sums in the right-hand side of the above identity yields

$$\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\kappa_V^{(k)}([\ell-1]) = \sum_{i\in[k]}\sum_{W\in\mathcal{C}_i(\ell)}\sum_{V\in\mathcal{C}_W}(-1)^{|W|-1}\kappa_V(W), \tag{53}$$

where $\mathcal{C}_W$ denotes the collection of all connected sets $V \subseteq W$ with $\max V = \max W$.

We claim that for each $W \in \mathcal{C}_i(\ell)$,

$$\kappa(W) = \sum_{V\in\mathcal{C}_W}\kappa_V(W). \tag{54}$$

Observe first that establishing this claim completes the proof of the lemma. Indeed, substituting (54) into (53) and summing over all ℓ gives

$$\begin{aligned}\sum_{\ell\in[N]}\sum_{i\in[k]}\sum_{V\in\mathcal{C}_i(\ell)}\kappa_V^{(k)}([\ell-1]) &= \sum_{i\in[k]}\sum_{\ell\in[N]}\sum_{W\in\mathcal{C}_i(\ell)}(-1)^{|W|-1}\kappa(W)\\ &= \sum_{i\in[k]}(-1)^{i-1}\sum_{W\in\mathcal{C}_i}\kappa(W) = \sum_{i\in[k]}(-1)^{i-1}\kappa_i.\end{aligned}$$

Therefore, we only need to prove the claim. To this end, fix a $W \in \mathcal{C}_i(\ell)$. Recalling (16) and (29), it clearly suffices to show that $\{\Pi_V^{\mathsf{C}}(W) : V \in \mathcal{C}_W\}$ is a partition of $\Pi(W)$. Obviously, $\Pi_V^{\mathsf{C}}(W) \subseteq \Pi(W)$ for each $V \in \mathcal{C}_W$. Conversely, given an arbitrary $\pi \in \Pi(W)$, let $P \in \pi$ be the part containing $\max W$ and let V be the connected component of $\max W$ in $G_\Gamma[P]$. Clearly, $V \in \mathcal{C}_W$ and $\pi \in \Pi_V^{\mathsf{C}}(W)$. Moreover, the connected component of $\max W$ in $G_\Gamma[P]$ is the only set V with this property, and so the sets $\Pi_V^{\mathsf{C}}(W)$ and $\Pi_U^{\mathsf{C}}(W)$ are disjoint for distinct $U, V \in \mathcal{C}_W$.

4 Proofs of the corollaries

To apply Theorem 10, all one needs to do is check the conditions (sparsity and subcriticality), compute the value δ_1, and compute as many values κ_k as one wants. The last part is typically the most labour-intensive. In this section, we carry out these calculations for the example applications mentioned in the introduction.

4.1 Random hypergraphs

We briefly recall the setup. We have an integer $r \geq 2$ and a set $\mathcal{F} = \{F_1, \dots, F_t\}$ of pairwise non-isomorphic r-uniform hypergraphs, each having at least two edges and no isolated vertices. We are interested in the asymptotic probability that $H_{n,p}^{(r)}$ avoids all copies of $F_1, \dots, F_t$. We encode this problem as an increasing family (Ω, Γ_n, p_n), as follows. We set $\Omega_n = \binom{[n]}{r}$. For each $i \in [t]$, we let $\Gamma_{i,n}$ be the collection of all edge sets of copies of F_i in $K_n^{(r)}$, and we set $\Gamma_n = \Gamma_{1,n} \cup \cdots \cup \Gamma_{t,n}$. Lastly, we may take p_n to be any sequence of probabilities, which we interpret as constant functions on Ω_n. Recall also the definitions:

$$m_*(F) = \min\left\{\frac{e_F - e_H}{v_F - v_H} \colon H \subseteq F \text{ with } v_H < v_F \text{ and } e_H > 0\right\}$$

and

$$m_*(\mathcal{F}) = \min\{m_*(F) \colon F \in \mathcal{F}\} \quad \text{and} \quad d(\mathcal{F}) = \min\{e_F/v_F \colon F \in \mathcal{F}\}.$$

Proof of Corollary 12. In light of Theorem 10 it is enough to show that

$$p_n = o\big(\min\{n^{-1/m_*(\mathcal{F})}, n^{-2d(\mathcal{F})}\}\big)$$

implies that $\delta_1 = o(1)$ and that $(\Omega_n, \Gamma_n, p_n)$ is sparse and subcritical. The sparsity condition follows from the assumption that $p_n = o(1)$. For δ_1, we have

$$\delta_1 = \sum_{i \in [N]} \mathbf{E}[X_i]^2 \leq \sum_{F \in \mathcal{F}} n^{v_F} p_n^{2e_F} = o(1),$$

since $p_n = o(n^{-v_F/2e_F})$ for each $F \in \mathcal{F}$. Last but not least, we verify the subcriticality condition. To this end, let $V \subseteq [N]$ be a set of size at most k and for $\gamma \subseteq \Omega_n$, write $\sigma_V(\gamma) := \gamma \setminus \bigcup_{i \in V} \gamma_i$. We may classify all $\gamma_i \in \Gamma_n$ with $i \in \partial(V)$ according to the isomorphism type of the r-graph spanned by the edges in $\gamma_i \setminus \sigma_V(\gamma_i)$, that is, $\gamma_i \cap \bigcup_{j \in V} \gamma_j$. Note that this is always the isomorphism type of a nonempty induced subhypergraph of some $F \in \mathcal{F}$. Accordingly,

$$\lambda(V) = \sum_{i \in \partial(V)} \mathbf{E}[X_i \mid \prod_{j \in V} X_j = 1] = \sum_{i \in \partial(V)} p_n^{|\sigma_V(\gamma_i)|} \leq \sum_{j \in [t]} \sum_{H} \big(k \max_{F \in \mathcal{F}} v_F\big)^{v_H} n^{v_{F_j} - v_H} p_n^{e_{F_j} - e_H},$$

where H goes over all induced subhypergraphs of F_j with at least one edge. Since for every $F \in \mathcal{F}$ and $H \subseteq F$ with $v_F < v_H$ and $e_H > 0$ we have $n^{v_F - v_H)} p_n^{e_F - e_H} = o(1)$, we see that if n is large enough, then $\lambda(V) \leq C_k$ for some constant C_k depending only on k and $\mathcal{F}$ (but not on V). It follows that Λ_k is bounded as $n \to \infty$. Hence $(\Omega_n, \Gamma_n, p_n)$ is subcritical and by Theorem 10 we have

$$\mathbf{P}[X = 0] = \exp\big(-\kappa_1 + \kappa_2 - \kappa_3 + \cdots + (-1)^k \kappa_k + O(\Delta_{k+1}) + o(1)\big)$$

for all constants k. $\square$

From now on, assume that $r = 2$. To prove Corollaries 14 and 15, we need to compute the quantities κ_k for small values of k. This can be done by the following general approach: We first enumerate all ‘isomorphism types’ of clusters in $\mathcal{C}_k$. Then we compute the joint cumulant for each isomorphism type. Finally we multiply each value with the size of the respective isomorphism class. This is made more precise as follows.

Definition 32. An *$\mathcal{F}$-complex* is a non-empty set of subgraphs of K_n, each of which is isomorphic to a graph in $\mathcal{F}$. An $\mathcal{F}$-complex C is *irreducible* if it cannot be written as the union of two $\mathcal{F}$-complexes C_1 and C_2 where every graph in C_1 is edge-disjoint from every graph in C_2. The set of all irreducible $\mathcal{F}$-complexes of cardinality k is denoted by $\mathcal{C}_k(\mathcal{F})$. The *graph G_C* of an $\mathcal{F}$-complex C is the subgraph of K_n formed by taking the union of (the edge sets of) the graphs in C.

Note that there is a natural bijection ϕ between the sets $A \subseteq [N]$ of size k and the $\mathcal{F}$-complexes of size i: ϕ maps $A = \{a_1, \ldots, a_k\}$ to the $\mathcal{F}$-complex $C = \{G_1, \ldots, G_k\}$, where G_i is the subgraph of K_n spanned by the edges in γ_{a_i} (recall that we have assumed that none of the graphs in $\mathcal{F}$ have isolated vertices). Note also that $\phi|_{\mathcal{C}_k}$ is a bijection between $\mathcal{C}_k$ and $\mathcal{C}_k(\mathcal{F})$. We can therefore write $\kappa(C)$ for an $\mathcal{F}$-complex C without ambiguity, obtaining

$$\kappa_k = \sum_{C \in \mathcal{C}_k(\mathcal{F})} \kappa(C).$$

Using (7) we can easily express $\kappa(C)$ in terms of G_C:

$$\kappa(C) = \sum_{\pi \in \Pi(C)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{C' \in \pi} p^{e_{G_{C'}}}. \tag{55}$$

Definition 33. Let C_1 and C_2 be $\mathcal{F}$-complexes. A map $f \colon V(G_{C_1}) \to V(G_{C_2})$ is an *isomorphism* from C_1 to C_2 if for every graph $H \in C_1$, the graph $f(H)$ belongs to C_2. We denote by $\mathrm{Aut}(C)$ the group of automorphisms of C, that is of isomorphisms from C to C.

It is easy to see that κ assigns equal values to isomorphic $\mathcal{F}$-complexes. The following simple lemma can then be used to compute the values κ_k. In the sequel, we will denote by $n^{\underline{i}}$ the falling factorial $n(n-1)\cdots(n-i+1)$.

Lemma 34. *Let $\mathcal{C}_k(\mathcal{F})/\cong$ be the set of isomorphism types of $\mathcal{F}$-complexes in $\mathcal{C}_k(\mathcal{F})$. Then*

$$\sum_{C \in \mathcal{C}_k(\mathcal{F})} \kappa(C) = \sum_{[C] \in \mathcal{C}_k(\mathcal{F})/\cong} \kappa(C) \cdot \frac{n^{\underline{v_{G_C}}}}{|\mathrm{Aut}(C)|}.$$

Proof. For each isomorphism type $[C]$, there are $n^{\underline{v_{G_C}}}$ ways to place the vertices of G_C into K_n, and then every element of $\mathcal{C}_k(\mathcal{F})$ isomorphic to C is counted once for every automorphism of C. □

Proof of Corollary 14. Suppose that $\mathcal{F} = \{K_3, C_4\}$ and that $p = o(n^{-4/5})$. Since both K_3 and C_4 are 2-balanced and

$$\min\big\{m_2(K_3), m_2(C_4)\big\} = \min\{2, 3/2\} \geq 5/4,$$

we can apply Corollary 13, which states that the probability that $G_{n,p}$ is simultaneously K_3-free and C_4-free is

$$\exp\big(-\kappa_1 + \kappa_2 - \kappa_3 + O(\Delta_4) + o(1)\big).$$

Figure 4 shows all seven non-isomorphic irreducible $\mathcal{F}$-complexes of size at most two. Using Lemma 34, the contribution to κ_k from a given $\mathcal{F}$-complex C of size k is

$$\kappa(C) \cdot \frac{n^{\underline{v_{G_C}}}}{|\mathrm{Aut}(C)|}.$$

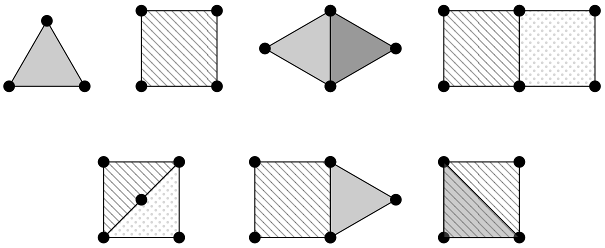

Figure 4: The irreducible $\{K_3, C_4\}$-complexes of size at most two.

For the complexes shown in Figure 4, we can easily calculate $|\operatorname{Aut}(C)|$ manually; going through the figure from the top left to the bottom right, we obtain the values

$$6, 8, 4, 4, 4, 2, 2.$$

Therefore we have

$$\kappa_1 = \frac{n^{\underline{3}} p^3}{6} + \frac{n^{\underline{4}} p^4}{8}$$

and

$$\begin{aligned}\kappa_2 &= \frac{n^{\underline{4}}(p^5 - p^6)}{4} + \frac{n^{\underline{6}}(p^7 - p^8)}{4} + \frac{n^{\underline{5}}(p^6 - p^8)}{4} + \frac{n^{\underline{5}}(p^6 - p^7)}{2} + \frac{n^{\underline{4}}(p^5 - p^7)}{2} \\ &= \frac{n^{\underline{6}} p^7}{4} + \frac{3n^{\underline{5}} p^6}{4} + o(1),\end{aligned}$$

since $p = o(n^{-4/5})$.

When calculating κ_3, we first observe that the graphs of the third $\mathcal{F}$-complex and the fifth $\mathcal{F}$-complex in Figure 4 each contain a C_4 that is not already part of the complex and that the graph of the bottom right $\mathcal{F}$-complex contains a triangle that is not a part of the complex. Let κ_3' denote the contribution of the two $\mathcal{F}$-complexes of size three that are obtained from one of these three complexes of size two by adding the 'extra' C_4 or K_3. Then

$$\kappa_3' = \frac{n^{\underline{4}}(p^5 - 2p^8 - p^9 + 2p^{10})}{4} + \frac{n^{\underline{5}}(p^6 - 3p^{10} + p^{12})}{4} = \frac{n^{\underline{5}} p^6}{4} + o(1).$$

On the other hand, the contribution of every other $\mathcal{F}$-complex of to κ_3 is at most in the order of $(p + np^2 + n^2p^3) \cdot \kappa_2$, because, except in the two cases mentioned above, the graph of a complex of size three is obtained from the graph of a complex of size two by adding either a new edge, or a new vertex and two new edges, or two new vertices and three new edges. Using the assumption $p = o(n^{-4/5})$, we get

$$(p + np^2 + n^2p^3) \cdot \kappa_2 = O(n^6p^8 + n^5p^7 + n^7p^9 + n^8p^{10}) = o(1),$$

and therefore

$$\kappa_3 = \frac{n^{\underline{5}} p^6}{4} + o(1).$$

Similar considerations show that

$$\Delta_4 \le O\big((p + np^2 + n^2p^3) \cdot \kappa_3'\big) + O\big((1 + p + np^2 + n^2p^3) \cdot (\kappa_3 - \kappa_3')\big) = o(1).$$

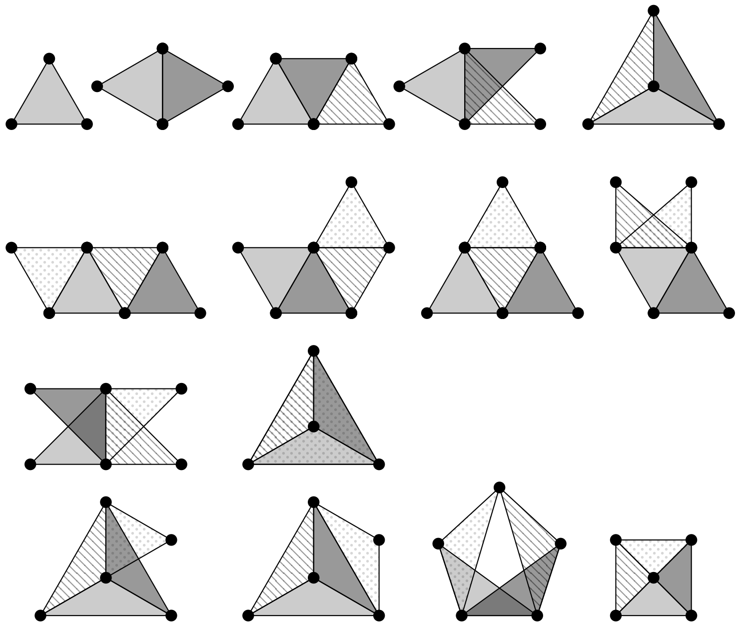

Figure 5: The irreducible $\{K_3\}$-complexes of size at most four. The four complexes in the bottom row are negligible when $p = o(n^{-7/11})$.

Since our assumption on p implies that $\max\{\kappa_1, \kappa_2, \kappa_3\} = o(n)$, we can replace the falling factorials $n^{\underline{i}}$ in $\kappa_1, \kappa_2, \kappa_3$ with powers n^i with only an additive error of $o(1)$. Thus the probability that $G_{n,p}$, with $p = o(n^{-4/5})$, is simultaneously triangle-free and C_4-free is asymptotically

$$\exp\Big(- \frac{n^3p^3}{6} - \frac{n^4p^4}{8} + \frac{n^6p^7}{4} + \frac{n^5p^6}{2} \Big),$$

as claimed. □

Proof of Corollary 15. Suppose that $\mathcal{F} = \{K_3\}$ and that $p = o(n^{-7/11})$. Since K_3 is 2-balanced and $m_2(K_3) = 2 \geq 11/7$, we can apply Corollary 13, which tells us that the probability that $G_{n,p}$ is triangle-free is

$$\exp\big(- \kappa_1 + \kappa_2 - \kappa_3 + \kappa_4 + O(\kappa_5) + o(1)\big).$$

In Figure 5 we see representations of all isomorphism types of irreducible $\mathcal{F}$-complexes of size up to four. Generating a similar list of complexes of size five would most likely require the help of a computer.

By Lemma 34, the contribution to κ_k from the isomorphism type of an $\mathcal{F}$-complex C of size k is

$$\kappa(C) \cdot \frac{n^{\underline{v_{G_C}}}}{|\operatorname{Aut}(C)|}.$$

For the complexes shown in Figure 5, it is not too difficult to calculate $|\operatorname{Aut}(C)|$ by hand. In fact, since the automorphism group of K_3 comprises all 3! permutations of $V(K_3)$, automorphisms of $\{K_3\}$-complexes are simply automorphisms of the 3-uniform hypergraphs involved[4].

[4] But for general $\mathcal{F}$, it is wrong to think of an $\mathcal{F}$-complex isomorphism as a hypergraph isomorphism.

For example, the leftmost $\mathcal{F}$-complex in the second row has exactly two automorphisms: the trivial one, and the unique automorphism exchanging the vertices belonging to exactly one triangle. Under our assumptions on p, we have $\kappa_k = \Delta_k + o(1)$ for $k \in \{3, 4\}$. This is the case because $|\kappa_k - \Delta_k| = O(p\Delta_k)$ and

$$p\Delta_3 \leq O(n^5p^8 + n^4p^7) = o(1) \qquad \text{and} \qquad p\Delta_4 \leq p \cdot O(1 + p + p^2 n) \cdot \Delta_3 = o(1),$$

see Figure 5.

Now we just work through the figure row by row (from the top left to the bottom right) and in this order, we compute (using the first row)

$$\kappa_1 = \frac{n^{\underline{3}}p^3}{6},$$
$$\kappa_2 = \frac{n^{\underline{4}}(p^5 - p^6)}{4},$$
$$\kappa_3 = \Delta_3 + o(1) = \frac{n^{\underline{5}}p^7}{2} + \frac{n^{\underline{5}}p^7}{12} + \frac{n^{\underline{4}}p^6}{6} + o(1),$$

and (using the other rows)

$$\kappa_4 = \Delta_4 + o(1) = \frac{n^{\underline{6}}p^9}{2} + \frac{n^{\underline{6}}p^9}{2} + \frac{n^{\underline{6}}p^9}{6} + \frac{n^{\underline{6}}p^9}{2} + \frac{n^{\underline{6}}p^9}{48} + \frac{n^{\underline{4}}p^6}{24} + O(n^5p^8) + o(1).$$

The term $O(n^5p^8)$ represents the contribution of the four complexes in the bottom row of Figure 5, which is $o(1)$, as $p = o(n^{-7/11})$. Finally, we have

$$\Delta_5 = O(p\Delta_4 + np^2\Delta_4 + n^5p^8 + n^5p^9) = O(n^4p^7 + n^5p^8 + n^6p^{10} + n^7p^{11}) = o(1),$$

since the graph of an $\mathcal{F}$-complex of size five must be obtained by adding either a new edge or a new vertex and two new edges to one of the graphs in Figure 5, or else it must be isomorphic to one of the first three graphs in the bottom row of Figure 5 (as the graphs of the remaining complexes of size four contain only triangles that are already in the complex).

Finally, $\kappa_1 = n^{\underline{3}}p^3/6 = (n^3 - 3n^2)p^3/6 + o(1)$ and, since $\max\{\kappa_2, \kappa_3, \kappa_4\} = o(n)$, we may replace the falling factorials $n^{\underline{i}}$ in the remaining expressions by n^i. Adding up the terms in $-\kappa_1 + \kappa_2 - \kappa_3 + \kappa_4$, we obtain that the probability that $G_{n,p}$ with $p = o(n^{-7/11})$ is triangle-free is asymptotically

$$\exp\Big(-\frac{n^3p^3}{6} + \frac{n^4p^5}{4} - \frac{7n^5p^7}{12} + \frac{n^2p^3}{2} - \frac{3n^4p^6}{8} + \frac{27n^6p^9}{16}\Big),$$

as claimed. $\square$

4.2 Arithmetic progressions

As explained in the introduction, we let $\Omega_n = [n]$ and let Γ_n be the set of r-APs in $[n]$. We let p_n be a sequence of probabilities. This defines $(\Omega_n, \Gamma_n, p_n)$, where, as before, we regard p_n as both a real number and a constant function on Ω_n.

Proof of Corollary 16. Suppose that $p_n = o(n^{-1/(r-1)})$. Then $(\Omega_n, \Gamma_n, p_n)$ is clearly sparse. Now the corollary will follow from Theorem 10 provided that $\delta_1 = o(1)$ and that $(\Omega_n, \Gamma_n, p_n)$ is subcritical. We verify that these two conditions are satisfied.

To this end, observe first that any two distinct numbers $a, b \in [n]$ are contained in at most $\binom{r}{2}$ many arithmetic progressions of length r. In particular, $|N| \leq \binom{n}{2}\binom{r}{2}$ and so

$$\delta_1 = O(n^2 p^{2r}) = o(1).$$

Also, given any $V \subseteq [N]$ of size at most k, we have

$$\lambda(V) = \sum_{i \in \partial(V)} \mathbf{E}[X_i \mid \prod_{j \in V} X_j = 1] = O(1 + np^{r-1}) = O(1),$$

since each r-AP intersecting the set $\bigcup_{j \in V} \gamma_j$ either intersects this set in at least two elements (there are only constantly many such choices, so their contribution is $O(1)$) or in exactly one element (contributing $O(np^{r-1})$). We conclude that $\Lambda_k = O(1)$, completing the proof. $\square$

Proof of Corollary 17. Assume that $p = o(n^{-4/7})$. Then by Corollary 16 with $r = 3$ and $k = 2$,

$$\mathbf{P}[X = 0] = \exp\big(-\kappa_1 + \kappa_2 + O(\Delta_3) + o(1)\big),$$

It remains to calculate κ_1, κ_2, and Δ_3. For $i \in [n]$, the number of 3-APs containing i is

$$f(i) = \frac{n}{2} + \min\{i, n-i\} + O(1),$$

where $\min\{i, n-i\}$ counts the 3-APs that have i as their midpoint, and $n/2$ counts the others. Thus the total number of 3-APs in $[n]$ is

$$\frac{1}{3}\sum_{i=1}^{n} f(i) = \frac{n^2}{4} + O(n),$$

and therefore (using $np^3 = o(1)$)

$$\kappa_1 = \frac{n^2 p^3}{4} + o(1).$$

If $\{X_i, X_j\}$ is a cluster of size two, then $|\gamma_i \cap \gamma_j|$ is either 1 or 2. The number of pairs γ_i, γ_j intersecting in two elements is at most $\binom{n}{2}\binom{3}{2}^2$, so the contribution of these pairs to κ_2 is $O(n^2p^4)$, which is $o(1)$ by our assumption on p. The number of pairs $\{\gamma_i, \gamma_j\}$ with $i \neq j$ and $|\gamma_i \cap \gamma_j| \geq 1$ is precisely $\sum_{i=1}^{n}\binom{f(i)}{2}$ and hence the number M of pairs with $|\gamma_i \cap \gamma_j| = 1$ satisfies

$$M = \sum_{i=1}^{n}\binom{f(i)}{2} + O(n^2) = \frac{1}{2}\sum_{i=1}^{n} f(i)^2 + O(n^2).$$

Since

$$\begin{aligned}\sum_{i=1}^{n} f(i)^2 &= \sum_{i=1}^{n}\big(n/2 + \min\{n-i, i\}\big)^2 + O(n^2) = 2\sum_{i=1}^{\lfloor n/2 \rfloor}(n/2 + i)^2 + O(n^2)\\ &= 2\left(\frac{n^3}{3} - \frac{(n/2)^3}{3}\right) + O(n^2) = \frac{7n^3}{12} + O(n^2),\end{aligned}$$

using $n^2p^4 = o(1)$ we get that

$$\kappa_2 = M(p^5 - p^6) + O\big(n^2(p^4 - p^6)\big) = \frac{17n^3p^5}{24} + o(1).$$

Lastly, we claim that $\Delta_3 = O(n^4p^7) = o(1)$. Indeed, let $\mathcal{C}_3^*$ be the family of all $\{i,j,k\} \in \mathcal{C}_3$ such that $|\gamma_i \cup \gamma_j \cup \gamma_k| < 7$. Since any two distinct numbers are contained in at most three 3-APs, a simple case analysis shows that

$$\sum_{V \in \mathcal{C}_3^*} \Delta(\{X_i : i \in V\}) = O(n^2p^5 + n^3p^6) = o(1).$$

On the other hand, $\Delta(\{X_i : i \in V\}) = p^7$ for every $V \in \mathcal{C}_3 \setminus \mathcal{C}_3^*$. Thus,

$$\Delta_3 \le |\mathcal{C}_3|p^7 + \sum_{V \in \mathcal{C}_3^*} \Delta(\{X_i : i \in V\} = O(n^4p^7 + n^2p^4 + n^3p^6) = o(1)$$

and we conclude that the probability that $[n]_p$ contains no 3-AP is asymptotically

$$\exp\Big(-\frac{n^2p^3}{4} + \frac{7n^3p^5}{12}\Big),$$

as claimed. $\square$

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