EXPONENTIAL DECAY OF LOOP LENGTHS IN THE LOOP $O(n)$ MODEL WITH LARGE n

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Abstract. The loop $O(n)$ model is a model for a random collection of non-intersecting loops on the hexagonal lattice, which is believed to be in the same universality class as the spin $O(n)$ model. It has been conjectured that both the spin and the loop $O(n)$ models exhibit exponential decay of correlations when $n > 2$. We verify this for the loop $O(n)$ model with large parameter n, showing that long loops are exponentially unlikely to occur, uniformly in the edge weight x. Our proof provides further detail on the structure of typical configurations in this regime. Putting appropriate boundary conditions, when nx^6 is sufficiently small, the model is in a dilute, disordered phase in which each vertex is unlikely to be surrounded by any loops, whereas when nx^6 is sufficiently large, the model is in a dense, ordered phase which is a small perturbation of one of the three ground states.

1. Introduction

After the introduction of the Ising model [19] and Ising's conjecture that it does not undergo a phase transition, physicists tried to find natural generalizations of the model with richer behavior. In [12], Heller and Kramers described the classical version of the celebrated quantum Heisenberg model where spins are vectors in the (two-dimensional) unit sphere in dimension three. Later, Stanley introduced the *spin $O(n)$ model* by allowing spins to take values in higher-dimensional spheres [26]. We refer the interested reader to [7] for a history of the subject.

Formally, a configuration of the spin $O(n)$ model on a finite graph G is an assignment $\sigma \in \Omega := (\sqrt{n} \cdot \mathbb{S}^{n-1})^{V(G)}$ of spins to each vertex of G, where $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere and the choice of the radius $\sqrt{n}$ serves as a convenient normalization. The *Hamiltonian* of the model is defined by

$$\mathcal{H}_{G,n}(\sigma) \ := \ - \sum_{\{u,v\} \in E(G)} \langle \sigma_u, \sigma_v \rangle \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. At inverse temperature β, we define the finite Gibbs measure $\mu_{G,n,\beta}$ to be the probability measure on Ω given by

$$d\mu_{G,n,\beta}(\sigma) := \frac{1}{Z^{\text{spin}}_{G,n,\beta}} \exp\left[-\beta \mathcal{H}_{G,n}(\sigma)\right] d\sigma,$$

where $Z^{\text{spin}}_{G,n,\beta}$, the *partition function*, is given by

$$Z^{\text{spin}}_{G,n,\beta} := \int_\Omega \exp\left[-\beta \mathcal{H}_{G,n}(\sigma)\right] d\sigma \tag{1}$$

and $d\sigma$ is the uniform probability measure on Ω (i.e., the product measure of the uniform distributions on $\sqrt{n} \cdot \mathbb{S}^{n-1}$ for each vertex in G).

By taking the weak limit of measures on larger and larger subgraphs of an infinite planar lattice, such as $\mathbb{Z}^2$ or the hexagonal lattice $\mathbb{H}$, an infinite volume measure $\mu_{n,\beta}$ can be defined, and one

Date: December 29, 2014.
Research of H.D.-C. is supported by the Swiss NSF as well as the NCCR SwissMap founded by the Swiss NSF.
Research of R.P. and Y.S. is partially supported by an ISF grant and an IRG grant.
Research of W.S. is partially supported by an ISF grant.

may ask whether a phase transition occurs at some critical inverse temperature. From this point of view, the behavior of the model is very different for different values of n:

- For $n = 1$, the model is simply the Ising model, which is known to undergo a phase transition between an ordered and a disordered phase, as proved by Peierls [24] (refuting Ising's conjecture). The critical inverse temperature has been computed for the square and the hexagonal lattices and it is fair to say that a lot is known about the behavior of the model. We refer the reader to [8, 9, 23] and references therein for an overview of the recent progress on the subject.
- For $n = 2$, the model is the so-called XY model (first introduced in [28]). Since the spin space $\mathbb{S}^1$ is a continuous group, the Mermin–Wagner theorem [21] guarantees that there is no phase transition between ordered and disordered phases. Still, a Kosterlitz–Thouless phase transition occurs as proved in [17, 27, 20, 11]. That is, below some critical inverse temperature, the spin-spin correlations $\mu_{n,\beta}[\langle \sigma_u, \sigma_v \rangle]$ decay exponentially fast in the distance between u and v, while above this critical inverse temperature, they decay only like an inverse power of the distance.
- For $n \geq 3$, it is predicted that no phase transition occurs [25] and that spin-spin correlations decay exponentially fast at every positive temperature. The $n = 3$ case, corresponding to the classical Heisenberg model, is of special interest. Let us mention that this prediction is part of a more general conjecture asserting that planar spin systems with non-Abelian continuous spin space do not exhibit a phase transition. As of today, the $n \geq 3$ case remains wide open. The best known results in this direction can be found in [18], where a $1/n$ expansion is performed as n tends to infinity.

On the hexagonal lattice $\mathbb{H}$, the spin $O(n)$ model can be related to the so-called *loop $O(n)$ model* introduced in [6]. Before providing additional details on the relation, let us define the loop $O(n)$ model. A *loop* is a finite subgraph of $\mathbb{H}$ which is isomorphic to a simple cycle. A *loop configuration* is a spanning subgraph of $\mathbb{H}$ in which every vertex has even degree; see Figure 1. The *non-trivial finite* connected components of a loop configuration are necessarily loops, however, a loop configuration may also contain isolated vertices and infinite simple paths. We shall often identify a loop configuration with its set of edges, disregarding isolated vertices. In this work, a *domain* H is a non-empty finite connected induced subgraph of $\mathbb{H}$ whose complement $\mathbb{H} \setminus V(H)$ induces a connected subgraph of $\mathbb{H}$ (in other words it does not have "holes"). For convenience, all of our results will be stated for domains although the definitions and techniques may sometimes be applied in greater generality. Given a domain H and a loop configuration ξ, we denote by $\mathsf{LoopConf}(H, \xi)$ the collection of all loop configurations ω that agree with ξ on $E(\mathbb{H}) \setminus E(H)$. Finally, for a domain H and a loop configuration ω, we denote by $L_H(\omega)$ the number of loops in ω which *intersect* $E(H)$ and by $o_H(\omega)$ the number of edges of $\omega \cap H$.

Definition 1.1. *Let H be a domain and let ξ be a loop configuration. Let n and x be positive real numbers. The loop $O(n)$ measure on H with edge weight x and boundary conditions ξ is the probability measure $\mathbb{P}^{\xi}_{H,n,x}$ on $\mathsf{LoopConf}(H, \xi)$ defined by*

$$\mathbb{P}^{\xi}_{H,n,x}(\omega) := \frac{x^{o_H(\omega)} n^{L_H(\omega)}}{Z^{\xi}_{H,n,x}}, \quad \omega \in \mathsf{LoopConf}(H, \xi),$$

where $Z^{\xi}_{H,n,x}$ is the unique constant which makes $\mathbb{P}^{\xi}_{H,n,x}$ a probability measure.

We note that the loop $O(n)$ model is defined for any *real* $n > 0$ whereas the spin $O(n)$ model is only defined for positive *integer* n. Let us now briefly discuss the connection between the loop and the spin $O(n)$ models (with integer n) on a domain $H \subset \mathbb{H}$. Rewriting the partition function

$Z^{\mathrm{spin}}_{H,n,\beta}$ given by (1) using the approximation $e^t \approx 1 + t$ gives

$$Z^{\mathrm{spin}}_{H,n,\beta} = \int_\Omega \prod_{\{u,v\}\in E(H)} e^{\beta\langle\sigma_u,\sigma_v\rangle}\, d\sigma \approx \int_\Omega \prod_{\{u,v\}\in E(H)} (1+\beta\langle\sigma_u,\sigma_v\rangle)\, d\sigma$$
$$= \sum_{\omega\subset E(H)} \beta^{o_H(\omega)} \int_\Omega \prod_{\{u,v\}\in E(\omega)} \langle\sigma_u,\sigma_v\rangle\, d\sigma.$$

The integral on the right-hand side equals $n^{L_H(\omega)}$ if $\omega \in \mathsf{LoopConf}(H,\emptyset)$ and 0 otherwise; see Appendix A for the calculation. Here, the normalization of taking spins on the sphere of radius $\sqrt{n}$ is used. Hence, substituting x for β,

$$Z^{\mathrm{spin}}_{H,n,x} \approx \sum_{\omega\in\mathsf{LoopConf}(H,\emptyset)} x^{o_H(\omega)} n^{L_H(\omega)} = Z^{\emptyset}_{H,n,x}.$$

In the same manner, the spin-spin correlation of $u, v \in V(H)$ may be approximated as follows.

$$\mu_{H,n,x}[\langle\sigma_u,\sigma_v\rangle] = \frac{\int_\Omega \langle\sigma_u,\sigma_v\rangle \exp\left[-x\mathcal{H}_{H,n}(\sigma)\right] d\sigma}{Z^{\mathrm{spin}}_{H,n,x}} \approx n\cdot \frac{\sum_{\omega\in\mathsf{LoopConf}(H,\emptyset,u,v)} x^{o_H(\omega)} n^{L'_H(\omega)} J(\omega)}{\sum_{\omega\in\mathsf{LoopConf}(H,\emptyset)} x^{o_H(\omega)} n^{L_H(\omega)}}, \quad (2)$$

where $\mathsf{LoopConf}(H,\emptyset,u,v)$ is the set of spanning subgraphs of H in which the degrees of u and v are odd and the degrees of all other vertices are even. Here, for $\omega \in \mathsf{LoopConf}(H,\emptyset,u,v)$, $o_H(\omega)$ is the number of edges of $\omega\cap H$, $L'_H(\omega)$ is the number of loops in ω which intersect H after removing an arbitrary simple path in ω between u and v, and $J(\omega) := \frac{3n}{n+2}$ if there are three disjoint paths in ω between u and v and $J(\omega) := 1$ otherwise (in which case, there is a unique simple path in ω between u and v); see Appendix A for the calculation.

Unfortunately, the above approximation is not justified for any $x > 0$. Nevertheless, (2) provides a heuristic connection between the spin and the loop $O(n)$ models and suggests that both these models reside in the same universality class. For this reason, it is natural to ask whether the prediction about the absence of phase transition is valid for the loop $O(n)$ model.

Question 1.2. *Does the quantity on the right-hand side of* (2) *decay exponentially fast in the distance between u and v, uniformly in the graph H, whenever $n > 2$ and $x > 0$?*

In this article, we partially answer this question. In Theorem 1.5 below, we show that for all sufficiently large n and any $x > 0$, the quantity on the right-hand side of (2) decays exponentially fast for a large class of graphs H. The theorem is a consequence of a more detailed understanding of the Gibbs measures of the loop $O(n)$ model. We show that for small x the model is in a dilute, disordered phase, where the sampled loop configuration is rather sparse and the probability of seeing long loops surrounding a given vertex decays exponentially in the length (see Figure 2a). For large x, the same exponential decay holds but for a different reason. There, the model is in a dense, ordered phase, which is a perturbation of a periodic ground state. In the ground state all loops have length 6 and a typical perturbation does not make them significantly longer (see Figure 2b).

The $x = \infty$ Model. We shall also consider the limit of the loop $O(n)$ model as the edge weight x tends to infinity. This means restricting the model to 'optimally packed loop configurations', i.e., loop configurations having the maximal possible number of edges.

Definition 1.3. *Let H be a domain and let ξ be a loop configuration. For $n > 0$, the loop $O(n)$ measure on H with edge weight $x = \infty$ and boundary conditions ξ is the probability measure on*

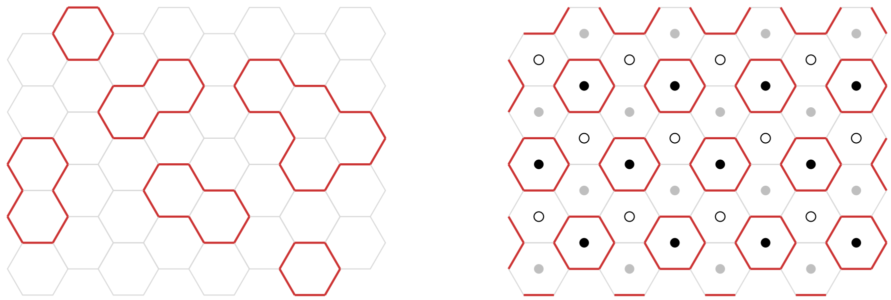


FIGURE 1. On the left, a loop configuration. On the right, a proper 3-coloring of the triangular lattice $\mathbb{T}$ (the dual of the hexagonal lattice $\mathbb{H}$), inducing a partition of $\mathbb{T}$ into three color classes $\mathbb{T}^0$, $\mathbb{T}^1$, and $\mathbb{T}^2$. The 0-phase ground state ω^0_{gnd} is the (fully-packed) loop configuration consisting of trivial loops around each hexagon in $\mathbb{T}^0$.

$\mathsf{LoopConf}(H,\xi)$ *defined by*

$$\mathbb{P}^\xi_{H,n,\infty}(\omega) := \frac{g_\infty(o_H(\omega)) \cdot n^{L_H(\omega)}}{Z^\xi_{H,n,\infty}}, \quad \omega \in \mathsf{LoopConf}(H,\xi),$$

where g_∞ *is defined by*

$$g_\infty(m) := \begin{cases} 0 & \textit{if } m < \max\{o_H(\omega) : \omega \in \mathsf{LoopConf}(H,\xi)\} \\ 1 & \textit{otherwise} \end{cases} \tag{3}$$

and $Z^\xi_{H,n,\infty}$ *is the unique constant making* $\mathbb{P}^\xi_{H,n,\infty}$ *a probability measure.*

We note that if a loop configuration $\omega \in \mathsf{LoopConf}(H,\xi)$ is *fully packed*, i.e., every vertex in $V(H)$ has degree 2, then ω is optimally packed.

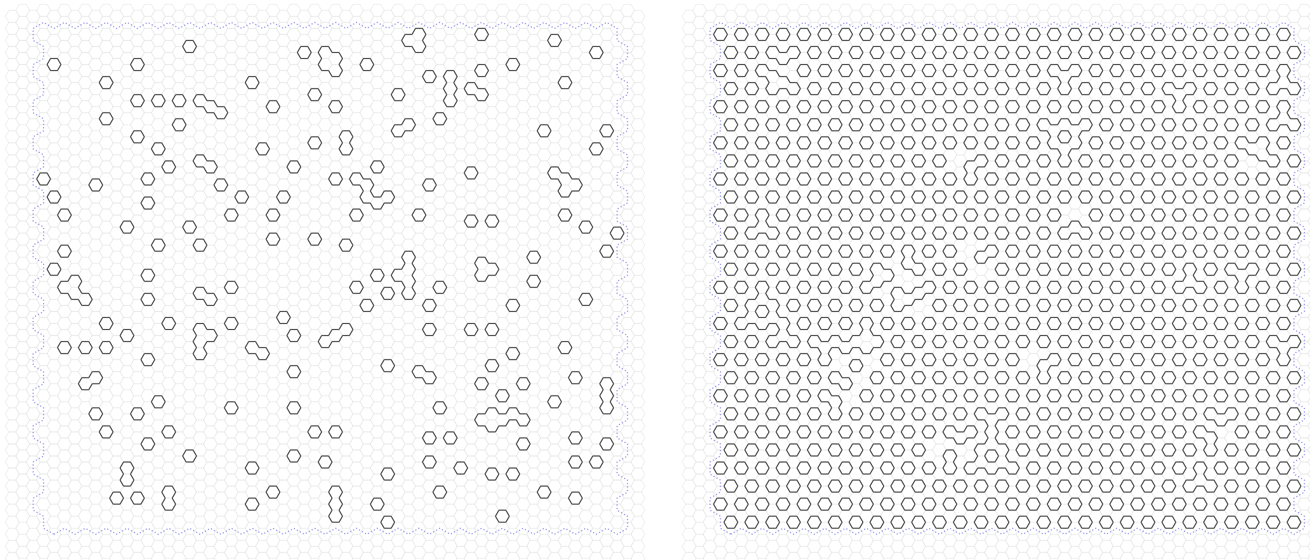
Before concluding this section, let us mention that the loop $O(n)$ model with $n \le 2$ is also of great interest; see Section 4 for a discussion.

1.1. **Results.** In order to state our main results, we need several more definitions (see Figure 1 for their illustration). We consider the triangular lattice $\mathbb{T} := (0,2)\mathbb{Z} + (\sqrt{3},1)\mathbb{Z}$, and view the hexagonal lattice $\mathbb{H}$ as its dual lattice, obtained by placing a vertex at the center of every face (triangle) of $\mathbb{T}$, so that each edge e of $\mathbb{H}$ corresponds to the unique edge e^* of $\mathbb{T}$ which intersects e. Since vertices of $\mathbb{T}$ are identified with faces of $\mathbb{H}$, they will be called *hexagons* instead of vertices. We will also say that a vertex or an edge of $\mathbb{H}$ *borders* a hexagon if it borders the corresponding face of $\mathbb{H}$.

There are exactly 6 proper colorings of $\mathbb{T}$ with the colors $\{0,1,2\}$. For the rest of the paper, we fix an arbitrary proper coloring and let $\mathbb{T}^{\mathsf{c}}$ be the set of hexagons colored by c, $\mathsf{c} \in \{0,1,2\}$. A *trivial loop* is a loop of length exactly 6. Define the c-*phase ground state* $\omega^{\mathsf{c}}_{\text{gnd}}$ to be the (fully-packed) loop configuration consisting of all the trivial loops surrounding hexagons in $\mathbb{T}^{\mathsf{c}}$. We shall say that a domain H is *of type* c, $\mathsf{c} \in \{0,1,2\}$, if every edge $\{u,v\} \in \omega^{\mathsf{c}}_{\text{gnd}}$ satisfies either $u,v \in V(H)$ or $u,v \notin V(H)$. Equivalently, if H is the interior of a circuit contained in $\mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$; see Section 2.1 for the precise definitions. Consequently, H is of type c if and only if

$$\mathsf{LoopConf}(H,\emptyset) = \{\omega \cap E(H) \colon \omega \in \mathsf{LoopConf}(H,\omega^{\mathsf{c}}_{\text{gnd}})\}. \tag{4}$$

Finally, we shall say that a loop *surrounds* a vertex v of $\mathbb{H}$ if any infinite simple path in $\mathbb{H}$ starting at v intersects a vertex of this loop. In particular, if a loop passes through a vertex then it surrounds it as well.



(A) $n = 8$ and $x = 0.5$. Theorem 1.6 shows that the limiting measure is unique for domains with vacant boundary conditions when x is small.

(B) $n = 8$ and $x = 2$. Theorem 1.8 shows that typical configurations are small perturbations of the ground state for large n and x.

FIGURE 2. Two samples of random loop configurations with large n. Configurations are on a 60×45 domain of type 0 and are sampled via Glauber dynamics for 100 million iterations started from the empty configuration.

Theorem 1.4. *There exist $n_0, \alpha > 0$ such that for any $n \geq n_0$ and any $x \in (0, \infty]$ the following holds. For any $\mathsf{c} \in \{0, 1, 2\}$, any domain H of type c, any $u \in V(H)$ and any integer $k > 6$, we have*

$$\mathbb{P}^{\emptyset}_{H,n,x}(\text{there exists a loop of length } k \text{ surrounding } u) \leq n^{-\alpha k}.$$

As follows from Theorem 1.8 below, when n and nx^6 are sufficiently large, it is likely that u is contained in a trivial loop (a loop of length 6). Thus, the assumption that $k > 6$ is necessary. The techniques involved in the proof of Theorem 1.4 also imply the following result, which partially answers Question 1.2.

Theorem 1.5. *There exist $n_0, \alpha > 0$ such that for any $n \geq n_0$ and any $x \in (0, \infty)$ the following holds. For any $\mathsf{c} \in \{0, 1, 2\}$, any domain H of type c, any integer k and any $u, v \in V(H)$ at graph distance k from each other, we have*

$$\frac{\displaystyle\sum_{\omega \in \mathsf{LoopConf}(H,\emptyset,u,v)} x^{o_H(\omega)} n^{L'_H(\omega)} J(\omega)}{\displaystyle\sum_{\omega \in \mathsf{LoopConf}(H,\emptyset)} x^{o_H(\omega)} n^{L_H(\omega)}} \leq n^{-\alpha k} \cdot \begin{cases} x & \text{if } x \geq 1 \\ x^{k/2} & \text{if } x < 1 \end{cases}.$$

Our techniques provide additional information on the Gibbs measures of the loop $O(n)$ model. For small parameter $x > 0$, under vacant boundary conditions, the model is in a dilute, disordered phase, where loops are rare and tend to be short; see Figure 2a. This is relatively simple to show and is proved in Corollary 3.2. A consequence of this fact is the existence of a unique limiting Gibbs measure when exhausting the hexagonal lattice $\mathbb{H}$ via domains with vacant boundary conditions.

Theorem 1.6. *There exists $c > 0$ such that for any $n > 0$ and $0 < x \leq c$ satisfying $nx^6 \leq c$ the following holds. Let H_k be an increasing sequence of domains satisfying $\cup_k H_k = \mathbb{H}$. Then the* *measures $\mathbb{P}^{\emptyset}_{H_k,n,x}$ converge as $k\to\infty$ to an infinite-volume measure $\mathbb{P}_{\mathbb{H},n,x}$ which is supported on loop configurations with no infinite paths.*

It follows that the limiting measure $\mathbb{P}_{\mathbb{H},n,x}$ does not depend on the specific choice of exhausting sequence (H_k) as one may interleave two such sequences to obtain another convergent sequence. Consequently, it also follows that $\mathbb{P}_{\mathbb{H},n,x}$ is invariant under automorphisms of $\mathbb{H}$. Our proofs apply also when one allows H_k to be arbitrary finite subgraphs of $\mathbb{H}$ rather than domains but we do not state this explicitly as our work is mostly concerned with domains. The restriction to vacant boundary conditions is, however, essential for our proofs with the difficulty stemming from the fact that non-vacant boundary conditions may force the existence of long paths in the configuration (see Figure 3b). Still, it may be that there is a unique Gibbs measure in this regime of small x and we provide a discussion of this in Section 4.

For large parameter $x>0$ and large n, the situation changes dramatically. Here, we obtain that the model is in a dense, ordered phase, where, under the $\omega^{\mathsf{c}}_{\mathrm{gnd}}$ boundary conditions, a typical configuration is a perturbation of that ground state. As a consequence of this structure, the model has at least three different limiting Gibbs measures in this regime of n and x. We state this precisely in the following theorem. To lighten the notation, we write $\mathbb{P}^{\mathsf{c}}_{H,n,x}$ for the loop $O(n)$ measure on H with boundary conditions $\omega^{\mathsf{c}}_{\mathrm{gnd}}$.

Theorem 1.7. *There exists $C>0$ such that for any $n\ge C$ and any $x\in(0,\infty]$ satisfying $nx^6\ge C$ the following holds. Let H_k be an increasing sequence of domains satisfying $\cup_k H_k=\mathbb{H}$. Then, for every $\mathsf{c}\in\{0,1,2\}$, the measures $\mathbb{P}^{\mathsf{c}}_{H_k,n,x}$ converge as $k\to\infty$ to an infinite-volume measure $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$ which is supported on loop configurations with no infinite paths. Furthermore, these three measures are distinct.*

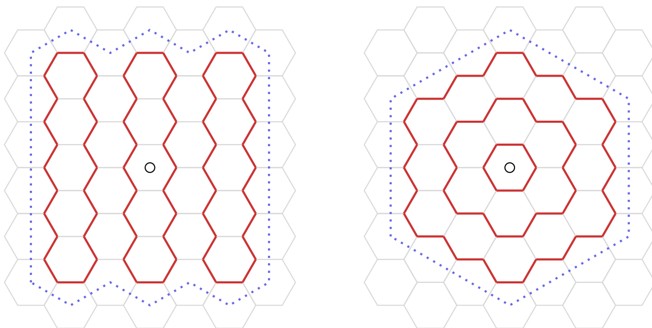
Similarly to before, it follows that, for each $\mathsf{c}\in\{0,1,2\}$, the limiting measure $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$ does not depend on the specific choice of exhausting sequence (H_k) and that $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$ is invariant under automorphisms preserving the set $\mathbb{T}^{\mathsf{c}}$. However, as these measures are distinct for different c, they are not invariant under *all* automorphisms. In particular, if each H_k is of type c, as $\mathbb{P}^{\mathsf{c}}_{H_k,n,x}=\mathbb{P}^{\emptyset}_{H_k,n,x}$ by (4), we have that $\mathbb{P}^{\emptyset}_{H_k,n,x}$ also converges to $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$, in contrast to the behavior obtained in Theorem 1.6 for small x. It would be interesting to determine whether the measures $\{\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}\}_{\mathsf{c}\in\{0,1,2\}}$ are the only *extremal* Gibbs measures. That is, whether every other infinite-volume measure is a convex combination of these three measures.

As mentioned above, in the ordered regime (large x and n), a typical configuration drawn from $\mathbb{P}^{\mathsf{c}}_{H,n,x}$ is a perturbation of the c-phase ground state, $\omega^{\mathsf{c}}_{\mathrm{gnd}}$ (see Figure 2b). This is made precise in the following theorem, which we state for the $\mathsf{c}=0$ phase for concreteness of our definitions. In order to measure how close ω^0_{gnd} and a typical loop configuration are, we introduce the notion of a *breakup*. Fix a domain H and let $\omega\in\mathsf{LoopConf}(H,\omega^0_{\mathrm{gnd}})$ be a loop configuration. Let $A(\omega)$ be the set of vertices of $\mathbb{H}$ belonging to trivial loops surrounding hexagons in $\mathbb{T}^0$ and let $B(\omega)$ be the unique infinite connected component of $A(\omega)$. For $u\in\mathbb{H}$, define the breakup $\mathcal{C}(\omega,u)$ of u to be the connected component of $\mathbb{H}\setminus B(\omega)$ containing u, setting $\mathcal{C}(\omega,u)=\emptyset$ if $u\in B(\omega)$. We also define $\partial\mathcal{C}(\omega,u)$ to be the internal vertex boundary of $\mathcal{C}(\omega,u)$, i.e., the set of vertices of $\mathcal{C}(\omega,u)$ adjacent to a vertex not in $\mathcal{C}(\omega,u)$ (thus in $B(\omega)$).

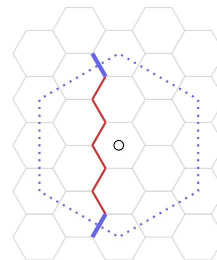
Theorem 1.8. *There exists $c>0$ such that for any $x\in(0,\infty]$, any $n>0$, any domain H, any $u\in V(\mathbb{H})$ and any positive integer k, we have*

$$\mathbb{P}^0_{H,n,x}(|\partial\mathcal{C}(\omega,u)|\ge k)\le(cn\cdot\min\{x^6,1\})^{-k/15}.$$

One should note that the above theorem contains the implicit assumption that $n\ge C$ and $nx^6\ge C$, as otherwise the statement is trivial.



(A) Domains for which there exists a single fully-packed loop configuration (with vacant boundary conditions). Using such domains, one may obtain many weak limits of the probability measures $\mathbb{P}^{\emptyset}_{H,n,\infty}$.



(B) A domain with boundary conditions inducing a unique loop configuration with minimal number of edges. Such domains give rise to a Gibbs measure for $x = 0$ which contains an infinite interface passing near 0.

FIGURE 3. Constructing multiple Gibbs measures when $x = 0$ or $x = \infty$ through suitable domains and boundary conditions.

In this work, we mainly study the loop $O(n)$ model with either vacant or ground state boundary conditions. To obtain a complete picture regarding the possible Gibbs measures, one must also study the model for general boundary conditions. As mentioned above, understanding the Gibbs measures in each regime of n and x, and in particular, determining the number of extremal Gibbs measures, is an interesting problem. Theorem 1.6 and Theorem 1.7 bring us closer to this goal, providing a partial answer in the regimes $nx^6 \leq c$ and $nx^6 \geq C$, for large n. In this regard, one may ask what happens in the intermediate regime, i.e., when $c < nx^6 < C$ and n is large. For instance, one may ask whether or not there is a single transition curve, perhaps of the form $nx^6 = c'$. If indeed this is the case, it would be interesting to investigate the number of extremal Gibbs measures on this curve, determining whether there is a unique such Gibbs measure (as Theorem 1.6 suggests for $nx^6 \leq c$), 3 such measures (as Theorem 1.7 suggests for $nx^6 \geq C$), 4 such measures, or perhaps a different quantity.

Remark. For $x = 0$ and $x = \infty$, many other Gibbs measures can be constructed. For instance, for positive integers a and b, let $H_{a,b}$ be the “rectangle” of width $2a + 1$ and height b (measured in hexagons) with the origin at the center, as in Figure 3a (on the left). It is not hard to check that the configuration depicted in the figure is the unique fully-packed loop configuration (with vacant boundary conditions) inside $H_{a,b}$. Thus, the probability measure $\mathbb{P}^{\emptyset}_{H_{a,b},n,\infty}$ is supported on a single configuration. The measures $\mathbb{P}^{\emptyset}_{H_{a,b},n,\infty}$ converge (as $a, b \to \infty$) to the constant configuration of infinite vertical paths covering the entire lattice. By considering different domains, one may construct many more examples of this nature (once again, see Figure 3a). One may also look at the limiting model as x tends to 0, which corresponds to requiring the configuration to have the minimal number of edges. For the vacant boundary conditions, the limiting Gibbs measure is a Dirac measure on the empty configuration. Using alternative boundary conditions, one may construct several distinct Gibbs measures (see, e.g., Figure 3b).

1.2. **Overview of the proof.** Our proofs make use of the following simple lemma.

Lemma 1.9. *Let $p, q > 0$ and let E and F be two events in a discrete probability space. If there exists a map $\mathsf{T}\colon E \to F$ such that $\mathbb{P}(\mathsf{T}(e)) \geq p \cdot \mathbb{P}(e)$ for every $e \in E$, and $|\mathsf{T}^{-1}(f)| \leq q$ for every $f \in F$, then*

$$\mathbb{P}(E) \leq \frac{q}{p} \cdot \mathbb{P}(F).$$

Proof. We have

$$p \cdot \mathbb{P}(E) \leq \sum_{e \in E} \mathbb{P}(\mathsf{T}(e)) = \sum_{e \in E} \sum_{f \in F} \mathbb{P}(f) \mathbf{1}_{\{\mathsf{T}(e)=f\}} = \sum_{f \in F} |\mathsf{T}^{-1}(f)| \cdot \mathbb{P}(f) \leq q \cdot \mathbb{P}(F). \qquad \square$$

The results for small x are obtained via a fairly standard, and short, Peierls argument, by applying the above lemma to the map which removes loops. For details, we refer the reader to Section 3.1. The main novelty of this work lies in the study of the loop $O(n)$ model for large x.

In the large x regime the idea is to apply the above lemma to a suitably defined 'repair map'. This map takes a configuration ω sampled with 0-phase ground state boundary conditions (or vacant boundary conditions in a domain of type 0) and having a large breakup and returns a 'repaired' configuration in which the breakup is significantly reduced. The map operates by identifying regions in which the configuration resembles one of the three ground states. Regions resembling the ω^1_{gnd} state are 'shifted down' by one hexagon to resemble ω^0_{gnd} and similarly regions resembling ω^2_{gnd} are 'shifted up' by one hexagon to resemble ω^0_{gnd}. Regions resembling the ω^0_{gnd} state are left untouched. Regions which do not resemble any of the ground states are completely replaced by trivial loops from the ω^0_{gnd} state. We show that this yields a new loop configuration, compatible with the boundary conditions, and having much higher probability. To finish using Lemma 1.9, we further show that the number of preimages of a given loop configuration is exponentially smaller than the probability gain. This yields the main lemma of our paper, Lemma 2.11, from which our results for large x are later deduced. The repair map is illustrated in Figure 6 and is formally defined in Section 2.3 following the definitions of 'flowers', 'gardens' and 'clusters' which we require to make precise the notion of resembling a ground state.

1.3. **Graph notation.** Throughout this paper, given a graph G, we shall denote its vertex and edge sets by $V(G)$ and $E(G)$, respectively. If $x, y \in V(G)$ are such that $\{x, y\} \in E(G)$, we say that x and y are *adjacent* (or *neighbors*) in G and we drop the dependence on G if it is clear from the context. For a vertex x and an edge e such that $x \in e$, we say that e is *incident* to x and that x is an *endpoint* of e. For $A \subset V(G)$, we define its *(vertex) boundary* ∂A by

$$\partial A := \big\{x \in A \colon \{x, y\} \in E(G) \text{ for some } y \notin A\big\}.$$

The following is a standard lemma which gives a bound on the number of connected induced subgraphs of a graph.

Lemma 1.10 ([2, Chapter 45]). *Let G be a graph with maximum degree $d \geq 3$. The number of connected subsets of $V(G)$ containing a given vertex and k other vertices is at most $(e(d-1))^k$.*

1.4. **Organization of the article.** The rest of the article is structured as follows. Section 2 introduces the repair map and proves the main lemma, Lemma 2.11. In Section 3 we derive our theorems. The statements regarding large x are deduced from the main lemma whereas the parts pertaining to small x, being simpler, are obtained directly. In Section 4 we discuss several directions for future research.

2. Flowers, gardens and the repair map

This section is devoted to the formulation and proof of the main lemma, Lemma 2.11. We start by stating a few definitions in Section 2.1. In particular, we introduce the notions of a circuit, c-*flower*, c-*garden* and c-*cluster*, and gather some easy general facts about these objects. The main lemma is stated in Section 2.2 and the remaining sections are devoted to its proof. Section 2.3 introduces the repair map, which will play the role of T in Lemma 1.9. Section 2.4 compares the probability of a configuration and its image under the repair map (which corresponds to estimating p in Lemma 1.9). Section 2.5 gathers the last ingredients (mainly an estimate for the number of possible preimages under the repair map, which corresponds to bounding q in Lemma 1.9) to conclude the proof of Lemma 2.11.

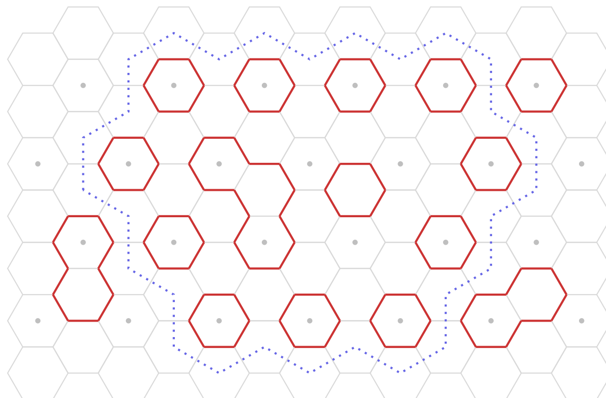


FIGURE 4. A garden. The dashed line denotes a vacant circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$, where $\mathsf{c} \in \{0, 1, 2\}$. The edges inside σ, along with the edges crossing σ, then comprise a c-garden of ω, since every hexagon in $\mathbb{T}^{\mathsf{c}} \cap \partial \mathrm{Int}^{\mathrm{hex}}(\sigma)$ is surrounded by a trivial loop.

2.1. **Definitions and gardening.** A *circuit* is a simple closed path in $\mathbb{T}$, which may be viewed as a sequence of hexagons $\gamma = (\gamma_0, \dots, \gamma_m)$, $m \geq 3$, satisfying the following two properties:

- $\gamma_m = \gamma_0$ and $\gamma_i \neq \gamma_j$ for every $0 \leq i < j < m$,
- γ_i and γ_{i+1} are neighbors (in $\mathbb{T}$) for every $0 \leq i < m$.

Define γ^* to be the set of edges $\{\gamma_i, \gamma_{i+1}\}^* \in E(\mathbb{H})$ for $0 \leq i < m$.

We proceed with three standard geometric facts regarding circuits and domains. For completeness, these facts are proved in Appendix B.

Fact 2.1. *If γ is a circuit then the removal of γ^* splits $\mathbb{H}$ into exactly two connected components, one of which is infinite, denoted by $\mathrm{Ext}(\gamma)$, and one of which is finite, denoted by $\mathrm{Int}(\gamma)$. Moreover, each of these are induced subgraphs of $\mathbb{H}$.*

Let γ be a circuit. We denote the vertex sets and edge sets of $\mathrm{Int}(\gamma), \mathrm{Ext}(\gamma)$ by $\mathrm{Int}^{\mathrm{V}}(\gamma), \mathrm{Ext}^{\mathrm{V}}(\gamma)$ and $\mathrm{Int}^{\mathrm{E}}(\gamma), \mathrm{Ext}^{\mathrm{E}}(\gamma)$, respectively. Note that $\{\mathrm{Int}^{\mathrm{V}}(\gamma), \mathrm{Ext}^{\mathrm{V}}(\gamma)\}$ is a partition of $V(\mathbb{H})$ and that $\{\mathrm{Int}^{\mathrm{E}}(\gamma), \mathrm{Ext}^{\mathrm{E}}(\gamma), \gamma^*\}$ is a partition of $E(\mathbb{H})$. We also define $\mathrm{Int}^{\mathrm{hex}}(\gamma)$ to be the set of faces of $\mathrm{Int}(\gamma)$, i.e., the set of hexagons $z \in \mathbb{T}$ having all their six bordering vertices in $\mathrm{Int}(\gamma)$. Since $\mathrm{Int}(\gamma)$ is induced, this is equivalent to having all six bordering edges in $\mathrm{Int}(\gamma)$.

Note that, by Fact 2.1, $\mathrm{Int}(\gamma)$ is a domain. The converse is also true.

Fact 2.2. *Circuits are in one-to-one correspondence with domains via $\gamma \leftrightarrow \mathrm{Int}(\gamma)$.*

Hence, every domain H may be written as $H = \mathrm{Int}(\gamma)$ for some circuit γ. Recalling the definition from Section 1.1 of a domain of type $\mathsf{c} \in \{0, 1, 2\}$, one should also note that H is of type c if and only if $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$.

Fact 2.3. *Let σ and σ' be two circuits such that $\sigma^* \cap (\sigma')^* \neq \emptyset$ or $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\sigma') \neq \emptyset$. Then there exists a circuit $\gamma \subset \sigma \cup \sigma'$ such that $\gamma^* \subset \sigma^* \cup (\sigma')^*$ and $\mathrm{Int}(\sigma) \cup \mathrm{Int}(\sigma') \subset \mathrm{Int}(\gamma)$.*

Definition 2.4 (c-flower, c-garden, vacant circuit; see Figure 4)**.** *Let $\mathsf{c} \in \{0, 1, 2\}$ and let ω be a loop configuration. A hexagon $x \in \mathbb{T}^{\mathsf{c}}$ is a* c-flower of ω *if it is surrounded by a trivial loop in ω. A subset $E \subset E(\mathbb{H})$ is a* c-garden of ω *if there exists a circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ such that $E = \mathrm{Int}^{\mathrm{E}}(\sigma) \cup \sigma^*$ and every $x \in \mathbb{T}^{\mathsf{c}} \cap \partial \mathrm{Int}^{\mathrm{hex}}(\sigma)$ is a c-flower of ω. In this case, we denote $\sigma(E) := \sigma$. A circuit σ is* vacant *in ω if $\omega \cap \sigma^* = \emptyset$.*

We say that $E \subset E(\mathbb{H})$ is a garden of ω if it is a c-garden of ω for some $\mathsf{c} \in \{0, 1, 2\}$. We stress the fact that a garden is a *subset of the edges of* $\mathbb{H}$. We continue with several simple properties of circuits, gardens and loop configurations which will be used throughout the paper.

Lemma 2.5. *Let ω and ω' be two loop configurations.*

(a) *If σ is a vacant circuit in ω then $\omega \cap \mathrm{Int}^{\mathrm{E}}(\sigma)$ and $\omega \cap \mathrm{Ext}^{\mathrm{E}}(\sigma)$ are loop configurations.*
(b) *If E is a garden of ω then $\sigma(E)$ is a vacant circuit in ω.*
(c) *If E is a garden of ω then $\omega \cap E$ and $\omega \setminus E$ are loop configurations.*
(d) *If ω and ω' are disjoint then $\omega \cup \omega'$ is a loop configuration.*
(e) *If ω' is contained in ω then $\omega \setminus \omega'$ is a loop configuration.*

Proof. To see (a), let σ be a vacant circuit in ω. Since any path between $\mathrm{Int}(\sigma)$ and $\mathrm{Ext}(\sigma)$ intersects σ^*, and since $\omega \cap \sigma^* = \emptyset$, every loop of ω is contained in either $\mathrm{Int}(\sigma)$ or $\mathrm{Ext}(\sigma)$, and thus, (a) follows.

We now show (b). Let E be a c-garden of ω, $\mathsf{c} \in \{0,1,2\}$, and let $\sigma := \sigma(E)$. One of the endpoints of every edge $e \in \sigma^*$ must border a hexagon in $\mathbb{T}^{\mathsf{c}} \cap \partial\mathrm{Int}^{\mathrm{hex}}(\sigma)$. By the definition of a c-garden, this hexagon is a c-flower, and hence, e cannot belong to ω. Thus, σ is vacant in ω.

In light of (a) and (b), (c) is immediate.

To establish (d), it suffices to show that no vertex has degree 3 in $\omega' \cup \omega$. Indeed, if a vertex has degree 3 then one of the edges incident to it must be contained in both ω and ω', which is a contradiction.

Finally, the last statement is straightforward. □

Lemma 2.6. *Let $\mathsf{c} \in \{0,1,2\}$, let $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ be a circuit, let $z \in \mathbb{T}^{\mathsf{c}}$ be a hexagon and let $V(z)$ denote the six vertices in $\mathbb{H}$ bordering z. Then*

$$z \in \mathrm{Int}^{\mathrm{hex}}(\sigma) \quad \iff \quad V(z) \cap \mathrm{Int}^{\mathrm{V}}(\sigma) \neq \emptyset.$$

Proof. Recall that, by definition, $z \in \mathrm{Int}^{\mathrm{hex}}(\sigma)$ if and only if $V(z) \subset \mathrm{Int}^{\mathrm{V}}(\sigma)$. Thus, it suffices to check that if $v \in V(z) \cap \mathrm{Int}^{\mathrm{V}}(\sigma)$ and $u \in V(z)$ is adjacent to v then $u \in \mathrm{Int}^{\mathrm{V}}(\sigma)$. Indeed this is the case, as otherwise, $\{u,v\} \in \sigma^*$ and $z \in \sigma$, which contradicts the assumption that $\sigma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$. □

We proceed to discuss disjointness and containment properties of c-gardens.

Lemma 2.7. *Let ω be a loop configuration and let E_1 and E_2 be two c-gardens of ω for some $\mathsf{c} \in \{0,1,2\}$. If there exists a vertex which is the endpoint of an edge in E_1 and an edge in E_2, then $E_1 \cup E_2$ is contained in a c-garden of ω.*

Proof. Denote $\sigma_1 := \sigma(E_1)$ and $\sigma_2 := \sigma(E_2)$. Let us first show that necessarily $\mathrm{Int}(\sigma_1) \cap \mathrm{Int}(\sigma_2) \neq \emptyset$ or $\sigma_1^* \cap \sigma_2^* \neq \emptyset$. To this end, let $v, u, w \in V(\mathbb{H})$ be such that $\{v,u\} \in E_1$ and $\{v,w\} \in E_2$. If $v \in \mathrm{Int}^{\mathrm{V}}(\sigma_1) \cap \mathrm{Int}^{\mathrm{V}}(\sigma_2)$ then we are done. Otherwise, suppose without loss of generality that $v \in \mathrm{Ext}^{\mathrm{V}}(\sigma_1)$ so that $u \in \mathrm{Int}^{\mathrm{V}}(\sigma_1)$. If also $v \in \mathrm{Ext}^{\mathrm{V}}(\sigma_2)$ then necessarily $w = u$ and $w \in \mathrm{Int}^{\mathrm{V}}(\sigma_2)$ as $\sigma_1, \sigma_2 \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$. If instead $v \in \mathrm{Int}^{\mathrm{V}}(\sigma_2)$ then either $u \in \mathrm{Int}^{\mathrm{V}}(\sigma_2)$ or $\{v,u\} \in \sigma_1^* \cap \sigma_2^*$.

By Fact 2.3, there exists a circuit $\gamma \subset \sigma_1 \cup \sigma_2$ such that $\mathrm{Int}(\sigma_1) \cup \mathrm{Int}(\sigma_2) \subset \mathrm{Int}(\gamma)$. In particular, $E_1 \cup E_2 \subset E$, where $E := \mathrm{Int}(\gamma) \cup \gamma^*$. It remains to show that E is a c-garden of ω. Since, by Lemma 2.6, $\mathbb{T}^{\mathsf{c}} \cap \partial\mathrm{Int}^{\mathrm{hex}}(\gamma) \subset \partial\mathrm{Int}^{\mathrm{hex}}(\sigma_1) \cup \partial\mathrm{Int}^{\mathrm{hex}}(\sigma_2)$, this follows from the assumption that E_1 and E_2 are c-gardens of ω. □

Lemma 2.8. *Let ω be a loop configuration, let E_0 be a c_0-garden of ω and let E_1 be a c_1-garden of ω with $\mathsf{c}_0, \mathsf{c}_1 \in \{0,1,2\}$ distinct. Then, either $E_0 \subset E_1$, $E_1 \subset E_0$ or $E_0 \cap E_1 = \emptyset$.*

Proof. Assume without loss of generality that $\mathsf{c}_0 = 0$, $\mathsf{c}_1 = 1$ and that $E_0 \cap E_1 \neq \emptyset$. Denote $\sigma_0 := \sigma(E_0) \subset \mathbb{T} \setminus \mathbb{T}^0$ and $\sigma_1 := \sigma(E_1) \subset \mathbb{T} \setminus \mathbb{T}^1$. Consider an infinite path in $\mathbb{H}$ beginning with some edge of $E_0 \cap E_1$ and let $e \in E(\mathbb{H})$ be the first edge on this path that is not in $\mathrm{Int}^{\mathrm{E}}(\sigma_0) \cap \mathrm{Int}^{\mathrm{E}}(\sigma_1)$ (maybe the first edge itself). We may assume without loss of generality that $e \notin \mathrm{Int}^{\mathrm{E}}(\sigma_0)$. Thus, $e \in \sigma_0^*$, and, therefore, e is bordered by a hexagon $z \in \mathbb{T}^1$ and a hexagon in $\mathbb{T}^2$. Since e is also in E_1, z belongs to $\mathrm{Int}^{\mathrm{hex}}(\sigma_1)$. Now, begin at z and move along the circuit σ_0 until reaching a hexagon $y \in \mathrm{Int}^{\mathrm{hex}}(\sigma_1)$ whose successor y' is not in $\mathrm{Int}^{\mathrm{hex}}(\sigma_1)$. If such a hexagon does not exist

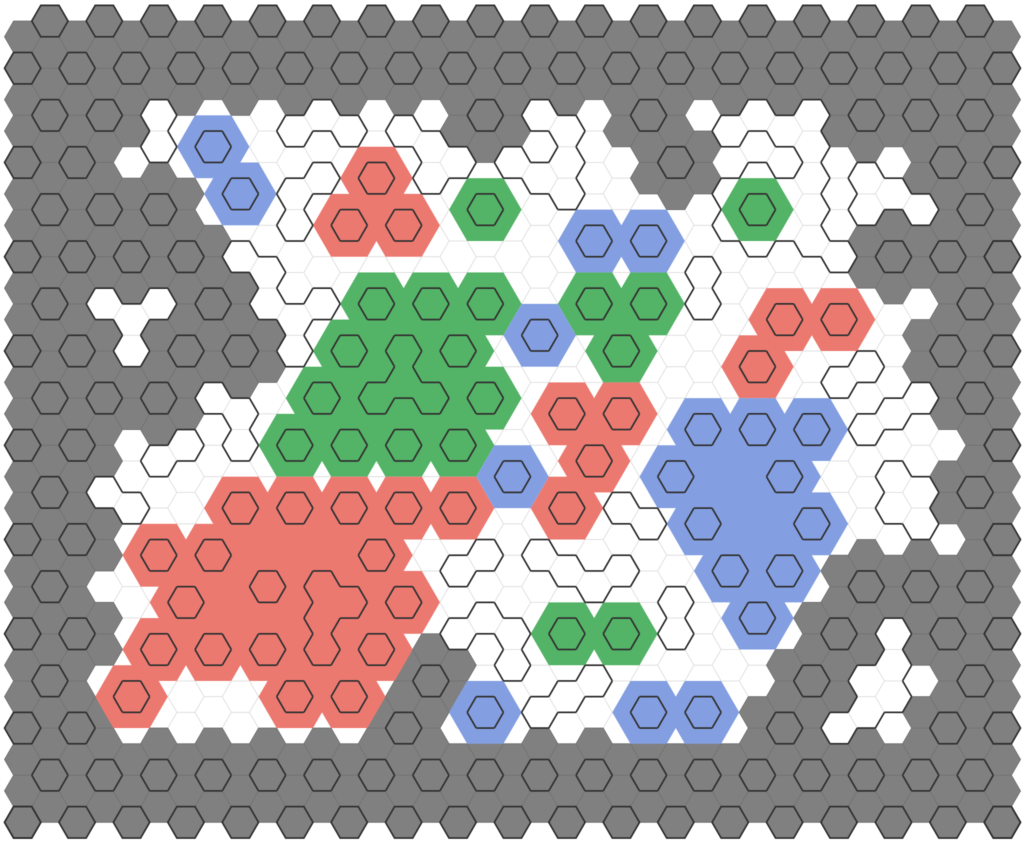


FIGURE 5. A loop configuration $\omega \in \mathsf{LoopConf}(H, \omega^0_{\mathrm{gnd}})$. The 0-clusters are denoted in green, the 1-clusters in red and the 2-clusters in blue; all taken with respect to the circuit surrounding the large unshaded domain.

then $\sigma_0 \subset \mathrm{Int}^{\mathrm{hex}}(\sigma_1)$, and hence, $E_0 \subset E_1$. On the other hand, if such a hexagon does exist, then y' must be in $\sigma_0 \cap \sigma_1 \subset \mathbb{T}^2$, so that y must be in $\mathbb{T}^1$. Since y is also in $\partial\mathrm{Int}^{\mathrm{hex}}(\sigma_1)$, it must be a 1-flower of ω. But since y is also on σ_0, it must be adjacent to a 0-flower of ω, which is a contradiction. This latter case is therefore impossible, and the lemma is proved. ☐

Definition 2.9 (c-cluster, c-cluster inside γ)**.** *Let $\mathsf{c} \in \{0,1,2\}$ and let ω be a loop configuration. A subset $C \subset E(\mathbb{H})$ is a* c-cluster *of ω if it is a c-garden of ω and it is not contained in any other garden of ω. Let γ be a vacant circuit in ω and note that $\omega \cap \mathrm{Int}^{\mathrm{E}}(\gamma)$ is a loop configuration by Lemma 2.5a. A subset $C \subset E(\mathbb{H})$ is a* c-cluster *of ω* inside γ *if it is a c-cluster of $\omega \cap \mathrm{Int}^{\mathrm{E}}(\gamma)$.*

We say that $C \subset E(\mathbb{H})$ is a cluster (inside γ) if it is a c-cluster (inside γ) for some $\mathsf{c} \in \{0,1,2\}$. Once again, note that a cluster (inside γ) is a *subset of edges of* $\mathbb{H}$. Evidently, a cluster of ω inside γ is also a garden of ω, but it is *not* necessarily a cluster of ω. The notion of c-cluster inside γ will be important in the definition of the repair map in Section 2.3. Note that, by Lemma 2.7 and Lemma 2.8,

$$\text{any two distinct clusters of } \omega \text{ (inside } \gamma) \text{ are edge disjoint,} \tag{5}$$

and, moreover, for any $\mathsf{c} \in \{0,1,2\}$,

$$\text{the union of any two distinct } \mathsf{c}\text{-clusters of } \omega \text{ (inside } \gamma) \text{ is a disconnected set of edges,} \tag{6}$$

where a set of edges E is said to be connected if the graph whose vertex set is the set of endpoints of edges in E and whose edge set is E is connected.

2.2. **Statement of the main lemma.** We are now in a position to state the main lemma. It is convenient to unite the discussion of the cases of $x \in (0, \infty)$ and $x = \infty$ so that we may handle both simultaneously. In fact, our arguments are rather robust and extend to a larger class of measures on $\mathsf{LoopConf}(H, \xi)$ defined as follows.

Definition 2.10. *Let H be a domain, let ξ be a loop configuration and let $n > 0$. Let $g\colon \{0, 1, 2, ...\} \to [0, \infty)$ be a function for which there exists some $\omega \in \mathsf{LoopConf}(H, \xi)$ with $g(o_H(\omega)) > 0$. Define the probability measure $\mathbb{P}^{\xi}_{H,n,g}$ on $\mathsf{LoopConf}(H, \xi)$ by*

$$\mathbb{P}^{\xi}_{H,n,g}(\omega) := \frac{g(o_H(\omega)) \cdot n^{L_H(\omega)}}{Z^{\xi}_{H,n,g}}, \quad \omega \in \mathsf{LoopConf}(H, \xi),$$

where $Z^{\xi}_{H,n,g}$ is the unique constant making $\mathbb{P}^{\xi}_{H,n,g}$ a probability measure.

Note that one recovers the standard loop $O(n)$ model with edge weight x described in Section 1 as a special case of the above definition by letting $g = g_x$ if $x \in (0, \infty)$ and $g = g_\infty$ if $x = \infty$, where $g_x(m) := x^m$ and g_∞ is the function defined in (3). In later occurrences of $\mathbb{P}^{\xi}_{H,n,g}$ it will be implicitly assumed that g is such that this measure is well defined (i.e., that there exists some $\omega \in \mathsf{LoopConf}(H, \xi)$ with $g(o_H(\omega)) > 0$). This is certainly the case in the only examples we rely upon, when $g = g_x$ for some $x \in (0, \infty]$.

A function $g\colon \{0, 1, 2, ...\} \to [0, \infty)$ is said to have *ϵ-bounded decay* ($\epsilon > 0$) if

$$g(m + 1) \geq \epsilon \cdot g(m) \quad \text{for all } m \geq 0. \tag{7}$$

The main lemma will be stated for functions g with x-bounded decay for $0 < x \leq 1$. Observing that for $x \in (0, \infty]$, the function g_x has $\min\{x, 1\}$-bounded decay, we will later be able to apply the main lemma for any x.

For a loop configuration ω and a vacant circuit γ in ω, denote by $V(\omega, \gamma)$ the set of vertices $v \in \mathrm{Int}^{\mathrm{V}}(\gamma)$ such that the three edges of $\mathbb{H}$ incident to v are not all contained in the same cluster of ω inside γ.

Lemma 2.11. *There exists an absolute constant $c > 0$ such that for any $n > 0$, any $0 < x \leq 1$, any g with x-bounded decay, any circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ and any positive integer k, we have*

$$\mathbb{P}^{\emptyset}_{\mathrm{Int}(\gamma),n,g}\big(\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega, \gamma) \text{ and } |V(\omega, \gamma)| \geq k\big) \leq (cnx^6)^{-k/15}.$$

By symmetry, we may take γ to be a circuit in $\mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ for any $\mathsf{c} \in \{0, 1, 2\}$. Moreover, by the domain Markov property, we may allow any domain H and any boundary conditions ξ, as long as γ is vacant. Therefore, we obtain the following corollary.

Corollary 2.12. *There exists an absolute constant $c > 0$ such that for any $n > 0$, any $0 < x \leq 1$, any g with x-bounded decay, any domain H, any loop configuration ξ, any $\mathsf{c} \in \{0, 1, 2\}$, any circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ for which $\mathrm{Int}(\gamma) \subset H$ and $\mathbb{P}^{\xi}_{H,n,g}(\gamma \text{ vacant}) > 0$ and any positive integer k, we have*

$$\mathbb{P}^{\xi}_{H,n,g}\big(\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega, \gamma) \text{ and } |V(\omega, \gamma)| \geq k \;\big|\; \gamma \text{ vacant}\big) \leq (cnx^6)^{-k/15}.$$

In particular,

$$\mathbb{P}^{\xi}_{H,n,g}\big(\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega, \gamma) \;\big|\; \gamma \text{ vacant}\big) \leq (cnx^6)^{-|\partial \mathrm{Int}^{\mathrm{V}}(\gamma)|/15}.$$

One should note that Lemma 2.11 and Corollary 2.12 contain the implicit assumption that $n \geq nx^6 \geq C$, as otherwise the statement is trivial.

2.3. **Definition of the repair map.** For the remainder of this section, we fix a circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ and set $H := \mathrm{Int}(\gamma)$. Consider a loop configuration ω such that γ is vacant in ω. The idea of the repair map is to modify ω as follows:

- Edges in 1-clusters inside γ are shifted down "into the 0-phase".
- Edges in 2-clusters inside γ are shifted up "into the 0-phase".
- Edges in 0-clusters inside γ are left untouched.
- Finally, the remaining edges which are not inside clusters, but are in the interior of γ (these edges will be called *bad*), are overwritten to "match" the 0-phase ground state, ω^0_{gnd}.

See Figure 6 for an illustration of this map.

In order to formalize this idea, we need a few definitions. A *shift* is a graph automorphism of $\mathbb{T}$ which maps every hexagon to one of its neighbors. We henceforth fix a shift $\uparrow$ which maps $\mathbb{T}^0$ to $\mathbb{T}^1$ (and hence, maps $\mathbb{T}^1$ to $\mathbb{T}^2$ and $\mathbb{T}^2$ to $\mathbb{T}^0$), and denote its inverse by $\downarrow$. A shift naturally induces mappings on the set of vertices and the set of edges of $\mathbb{H}$. We shall use the same symbols, $\uparrow$ and $\downarrow$, to denote these mappings. Recall from Section 1.1 that $\mathbb{T}$ has a coordinate system given by $(0,2)\mathbb{Z} + (\sqrt{3},1)\mathbb{Z}$ and that $(\mathbb{T}^0, \mathbb{T}^1, \mathbb{T}^2)$ are the color classes of an arbitrary proper 3-coloring of $\mathbb{T}$. In our figures we make the choice that $(0,0) \in \mathbb{T}^0$ and $(0,2) \in \mathbb{T}^1$ so that $\uparrow$ is the map $(a,b) \mapsto (a, b+2)$.

For a loop configuration $\omega \in \mathsf{LoopConf}(H,\emptyset)$ and $\mathsf{c} \in \{0,1,2\}$, let $E^{\mathsf{c}}(\omega) \subset E(\mathbb{H})$ be the union of all c-clusters of ω. Note that, since $H = \mathrm{Int}(\gamma)$, for $\omega \in \mathsf{LoopConf}(H,\emptyset)$, the notions of a c-cluster and a c-cluster inside γ coincide. For a $\omega \in \mathsf{LoopConf}(H,\emptyset)$, define also

$$E^{\mathrm{bad}}(\omega) := (\mathrm{Int}^{\mathrm{E}}(\gamma) \cup \gamma^*) \setminus \big(E^0(\omega) \cup E^1(\omega)^{\downarrow} \cup E^2(\omega)^{\uparrow}\big) \tag{8}$$

and

$$\overline{E}(\omega) := (\mathrm{Int}^{\mathrm{E}}(\gamma) \cup \gamma^*) \setminus \big(E^0(\omega) \cup E^1(\omega) \cup E^2(\omega)\big). \tag{9}$$

Note that, by (5), $\{E^0(\omega), E^1(\omega), E^2(\omega), \overline{E}(\omega)\}$ is a partition of $\mathrm{Int}^{\mathrm{E}}(\gamma) \cup \gamma^*$. Thus, Lemma 2.5 implies that

$$\omega \cap E^0(\omega),\ \omega \cap E^1(\omega),\ \omega \cap E^2(\omega) \text{ and } \omega \cap \overline{E}(\omega) \text{ are pairwise disjoint loop configurations.} \tag{10}$$

See Figure 6 and Figure 5 for an illustration of these notions. Finally, we define the *repair map*

$$R_\gamma \colon \mathsf{LoopConf}(H,\emptyset) \to \mathsf{LoopConf}(H,\emptyset)$$

by

$$R_\gamma(\omega) := \big(\omega \cap E^0(\omega)\big) \cup \big(\omega \cap E^1(\omega)\big)^{\downarrow} \cup \big(\omega \cap E^2(\omega)\big)^{\uparrow} \cup \big(\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}(\omega)\big).$$

The fact that the mapping is well-defined, i.e., that $R_\gamma(\omega)$ is indeed in $\mathsf{LoopConf}(H,\emptyset)$, is not completely straightforward. This follows from the following proposition, together with the simple property in Lemma 2.5d.

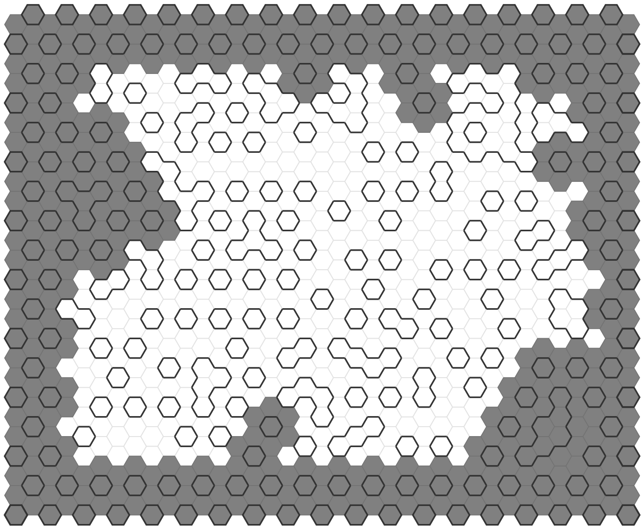
Proposition 2.13. *Let $\omega \in \mathsf{LoopConf}(H,\emptyset)$. Then $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}(\omega)$, $\omega \cap E^0(\omega)$ and $(\omega \cap E^1(\omega))^{\downarrow} \cup (\omega \cap E^2(\omega))^{\uparrow}$ are pairwise disjoint loop configurations in $\mathsf{LoopConf}(H,\emptyset)$.*

We require the following simple geometric lemma.

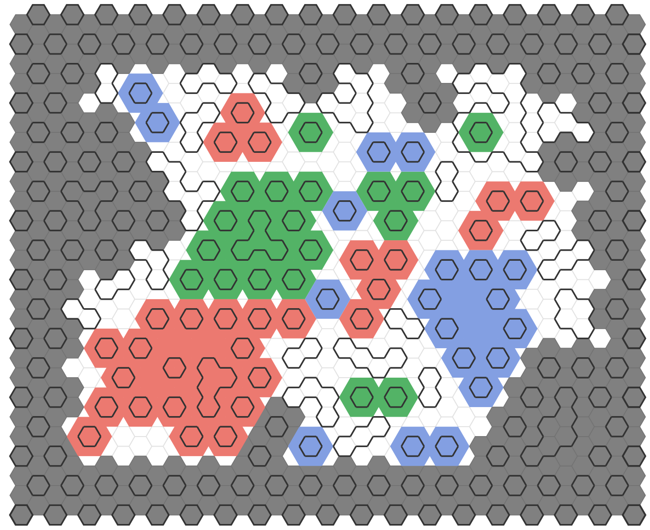
Lemma 2.14. *Let $\sigma \subset \mathbb{T} \setminus \mathbb{T}^0$ and $\sigma' \subset \mathbb{T} \setminus \mathbb{T}^1$ be circuits.*

(a) If $\mathrm{Int}(\sigma') \subset \mathrm{Int}(\sigma)$ then $\mathrm{Int}(\sigma')^{\downarrow} \subset \mathrm{Int}(\sigma)$.
(b) If $\mathrm{Int}(\sigma') \subset \mathrm{Ext}(\sigma)$ then $\mathrm{Int}(\sigma')^{\downarrow} \subset \mathrm{Ext}(\sigma)$.
(c) If $\mathrm{Int}(\sigma') \cap \mathrm{Int}(\sigma) = \emptyset$ then $\mathrm{Int}(\sigma')^{\downarrow} \cap \mathrm{Int}(\sigma) = \emptyset$.

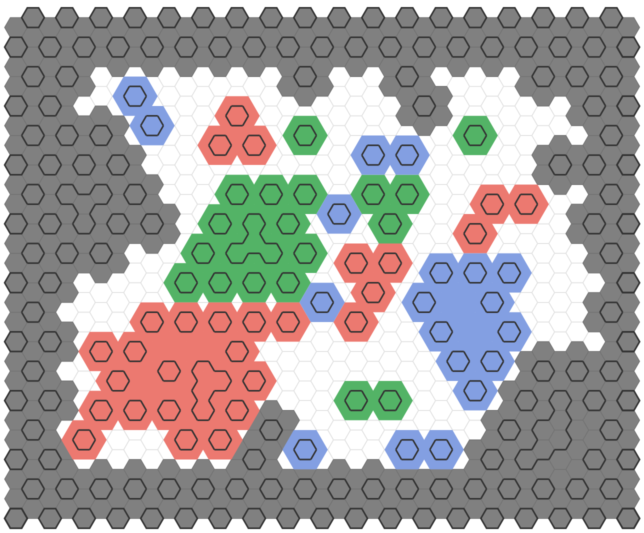
Proof. We first prove (a). The assumption that $\mathrm{Int}(\sigma') \subset \mathrm{Int}(\sigma)$ implies that $\mathrm{Int}^{\mathrm{hex}}(\sigma') \subset \mathrm{Int}^{\mathrm{hex}}(\sigma)$. By Lemma 2.6, any vertex v in $\mathrm{Int}(\sigma)$ borders a hexagon in $\mathrm{Int}^{\mathrm{hex}}(\sigma)$. Thus, it suffices to show that $\mathrm{Int}^{\mathrm{hex}}(\sigma')^{\downarrow} \subset \mathrm{Int}^{\mathrm{hex}}(\sigma)$. Assume towards a contradiction that there exists a hexagon $z \in \mathrm{Int}^{\mathrm{hex}}(\sigma')$ such that $z^{\downarrow} \notin \mathrm{Int}^{\mathrm{hex}}(\gamma)$. In such case, $z^{\downarrow}$ must be in $\sigma \cap \sigma' \subset \mathbb{T}^2$, and consequently, $z \in \mathbb{T}^0$.



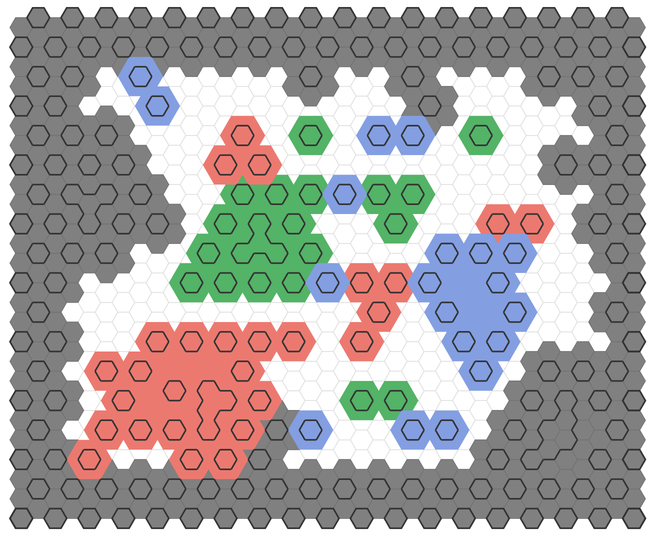
(A) The breakup is found by exploring 0-flowers from the boundary.



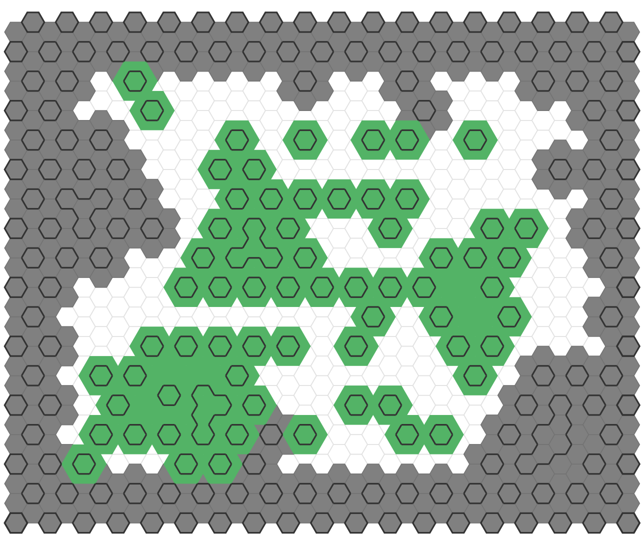
(B) The clusters are found within the breakup.



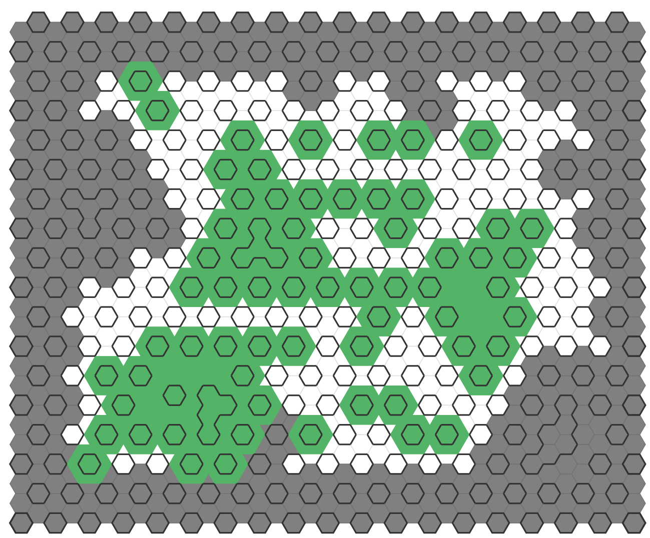
(C) Bad edges are discarded.



(D) The clusters are shifted into the 0-phase.



(E) The empty area outside the shifted clusters is now compatible with the 0-phase ground state.



(F) Trivial loops are packed in the empty area outside the shifted clusters.

FIGURE 6. An illustration of finding the breakup and applying the repair map in it. The initial loop configuration is modified step-by-step, resulting in a loop configuration with many more loops and at least as many edges. Formal definitions are in Section 2.3.

Therefore, as $z \in \mathrm{Int}^{\mathrm{hex}}(\sigma')$ and $\sigma' \subset \mathbb{T} \setminus \mathbb{T}^1$, Lemma 2.6 implies that the three neighbors of z in $\mathbb{T}^1$ belong to $\mathrm{Int}^{\mathrm{hex}}(\sigma') \subset \mathrm{Int}^{\mathrm{hex}}(\sigma)$. This implies that $z^\downarrow$ has three neighbors in $\mathbb{T}^0 \cap \mathrm{Int}^{\mathrm{hex}}(\sigma)$. In particular, the six vertices bordering $z^\downarrow$ belong to $\mathrm{Int}(\sigma)$, implying that $z^\downarrow \in \mathrm{Int}^{\mathrm{hex}}(\sigma)$, which is a contradiction.

The proof of (b) is very similar to that of (a) and so we omit it.

Finally, (c) follows from (b), as $\mathrm{Int}(\sigma') \cap \mathrm{Int}(\sigma) = \emptyset$ implies that $\mathrm{Int}(\sigma') \subset \mathrm{Ext}(\sigma)$, since $\mathrm{Int}(\sigma')$ is an induced subgraph. □

Proof of Proposition 2.13. For the sake of brevity, throughout the proof, we drop ω from the notation of the above sets and write E^{bad}, E^0, E^1 and E^2.

Step 1: $\omega \cap E^0$, $(\omega \cap E^1)^\downarrow \cup (\omega \cap E^2)^\uparrow$ and $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}$ are contained in $\mathrm{Int}(\gamma)$.

Since γ is vacant in both ω and ω^0_{gnd}, it follows that $\omega \cap E^0$ and $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}$ are contained in $\mathrm{Int}(\gamma)$. It remains to show that $(\omega \cap E^1)^\downarrow$ and $(\omega \cap E^2)^\uparrow$ are contained in $\mathrm{Int}(\gamma)$. We show this only for $(\omega \cap E^1)^\downarrow$, as the other case is symmetric. Let E be a 1-garden of ω. We must show that $(\omega \cap E)^\downarrow \subset \mathrm{Int}(\gamma)$. Since $(\omega \cap E)^\downarrow \subset \mathrm{Int}(\sigma(E))^\downarrow$, this follows from Lemma 2.14.

Step 2: $\omega \cap E^0$, $(\omega \cap E^1)^\downarrow \cup (\omega \cap E^2)^\uparrow$ and $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}$ are pairwise disjoint.

By definition, E^{bad} (and therefore $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}$) is disjoint from the other two sets. It remains to show that $\omega \cap E^0$ is disjoint from $(\omega \cap E^1)^\downarrow$ and $(\omega \cap E^2)^\uparrow$. We show this only for $\omega \cap E^0$ and $(\omega \cap E^1)^\downarrow$, as the other case is symmetric. Let E and E' be 0- and 1-gardens of ω, respectively. We must show that $(\omega \cap E) \cap (\omega \cap E')^\downarrow = \emptyset$. By Lemma 2.5b, $(\omega \cap E) \cap (\omega \cap E')^\downarrow \subset \mathrm{Int}(\sigma(E)) \cap \mathrm{Int}(\sigma(E'))^\downarrow$, which is empty by (5) and Lemma 2.14.

Step 3: $\omega \cap E^0$, $(\omega \cap E^1)^\downarrow \cup (\omega \cap E^2)^\uparrow$ and $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}$ are loop configurations.

We first show that $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}}$ is a loop configuration. Observe that $E^0 \cup (E^1)^\downarrow \cup (E^2)^\uparrow$ is the union of $\mathrm{Int}^{\mathrm{E}}(\sigma) \cup \sigma^*$ for a collection of circuits $\sigma \subset \mathbb{T} \setminus \mathbb{T}^0$. Since every circuit $\sigma \subset \mathbb{T} \setminus \mathbb{T}^0$ is vacant in ω^0_{gnd}, Lemma 2.5 implies that $\omega^0_{\mathrm{gnd}} \cap (E^0 \cup (E^1)^\downarrow \cup (E^2)^\uparrow)$ is a loop configuration, and thus, also that $\omega^0_{\mathrm{gnd}} \cap E^{\mathrm{bad}} = (\omega^0_{\mathrm{gnd}} \setminus (E^0 \cup (E^1)^\downarrow \cup (E^2)^\uparrow)) \setminus \mathrm{Ext}(\gamma)$ is a loop configuration.

Since $\omega \cap E^0$ is a loop configuration, by (10), it only remains to check that $(\omega \cap E^1)^\downarrow \cup (\omega \cap E^2)^\uparrow$ is a loop configuration. In light of (10) and Lemma 2.5(d,e), it suffices to show that $(\omega \cap E^1)^\downarrow \cap (\omega \cap E^2)^\uparrow$ is a loop configuration. For convenience, we prove this separately in the next lemma. □

For a hexagon $z \in \mathbb{T}$, we denote by $E(z)$ the six edges bordering z. We call a hexagon $z \in \mathbb{T}$ *double-clustered* for ω if $E(z^\uparrow) \in E^1(\omega)$ and $E(z^\downarrow) \in E^2(\omega)$. Denote by $\mathrm{dbl}(\omega)$ the subset of all hexagons in $\mathrm{Int}^{\mathrm{hex}}(\gamma)$ that are double-clustered for ω.

Lemma 2.15. *Let* $\omega \in \mathsf{LoopConf}(H, \emptyset)$. *Then* $\mathrm{dbl}(\omega) \subset \mathbb{T}^0$ *and* $(\omega \cap E^1(\omega))^\downarrow \cap (\omega \cap E^2(\omega))^\uparrow$ *consists solely of the trivial loops surrounding the hexagons in* $\mathrm{dbl}(\omega)$. *That is,*

$$(\omega \cap E^1(\omega))^\downarrow \cap (\omega \cap E^2(\omega))^\uparrow = \bigcup_{z \in \mathrm{dbl}(\omega)} E(z).$$

Proof. Let $z \in \mathrm{dbl}(\omega)$. Then $z^\uparrow \in \mathrm{Int}^{\mathrm{hex}}(\sigma(E_1))$ and $z^\downarrow \in \mathrm{Int}^{\mathrm{hex}}(\sigma(E_2))$, where E_1 and E_2 are 1- and 2-clusters of ω, respectively. If follows from Lemma 2.6 and (5) that $z \in \mathbb{T}^0$ and that $z \notin \mathrm{Int}^{\mathrm{hex}}(\sigma(E_1)) \cup \mathrm{Int}^{\mathrm{hex}}(\sigma(E_2))$. Thus, $z^\uparrow$ is a 1-flower of ω and $z^\downarrow$ is a 2-flower of ω. In particular, $E(z) \subset (\omega \cap E^1(\omega))^\downarrow \cap (\omega \cap E^2(\omega))^\uparrow$.

For the opposite containment, let $e \in (\omega \cap E^1(\omega))^\downarrow \cap (\omega \cap E^2(\omega))^\uparrow$. Then $e^\uparrow \in \mathrm{Int}^{\mathrm{E}}(\sigma(E_1)) \cup \sigma(E_1)^*$ and $e^\downarrow \in \mathrm{Int}^{\mathrm{E}}(\sigma(E_2)) \cup \sigma(E_2)^*$, where E_1 and E_2 are 1- and 2-clusters of ω, respectively. Since, by Lemma 2.5b, $\sigma(E_1)$ and $\sigma(E_2)$ are vacant in ω, we have $e^\uparrow \in \mathrm{Int}^{\mathrm{E}}(\sigma(E_1))$ and $e^\downarrow \in$

$\mathrm{Int}^{\mathrm{E}}(\sigma(E_2))$. In particular, both endpoints of $e^{\uparrow}$ belong to $\mathrm{Int}^{\mathrm{V}}(\sigma(E_1))$ and both endpoints of $e^{\downarrow}$ belong to $\mathrm{Int}^{\mathrm{V}}(\sigma(E_2))$. Therefore, by Lemma 2.6, e must border a hexagon z in $\mathbb{T}^0$, and $V(z^{\uparrow}) \subset E_1$ and $V(z^{\downarrow}) \subset E_2$, where $V(z)$ denotes the six vertices bordering z. Thus, $z \in \mathrm{dbl}(\omega)$. $\square$

2.4. **Comparing the probabilities of $R_\gamma(\omega)$ and ω.** As in Section 2.3, we henceforth fix a circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ and denote $H := \mathrm{Int}(\gamma)$. Our goal now is to compare the probabilities of $R_\gamma(\omega)$ and ω.

Proposition 2.16. *Let $n > 0$ and $0 < x \le 1$ satisfy $nx^6 \ge 1$. Then, for any g with x-bounded decay and any $\omega \in \mathsf{LoopConf}(H, \emptyset)$, we have*

$$\mathbb{P}^{\emptyset}_{H,n,g}(R_\gamma(\omega)) \ge (nx^6)^{|V(\omega,\gamma)|/15} \cdot \mathbb{P}^{\emptyset}_{H,n,g}(\omega).$$

The proof of Proposition 2.16 is based on showing that applying the repair map can only increase the number of loops and edges and estimating carefully the amounts by which they increase.

We begin with two preliminary lemmas. Denote by $V^{\mathrm{bad}}(\omega)$ the subset of $\mathrm{Int}^{\mathrm{V}}(\gamma)$ composed of endpoints of edges in $E^{\mathrm{bad}}(\omega)$. Recall the definition of $V(\omega, \gamma)$ from Section 2.2.

Lemma 2.17. *For any $\omega \in \mathsf{LoopConf}(H, \emptyset)$, we have*

$$|V^{\mathrm{bad}}(\omega)| = |V(\omega,\gamma)| + 6 \cdot |\,\mathrm{dbl}(\omega)|.$$

Proof. As before, set $E^{\mathsf{c}} := E^{\mathsf{c}}(\omega)$ for $\mathsf{c} \in \{0,1,2\}$. Let $U := \mathrm{Int}^{\mathrm{V}}(\gamma) \setminus V(\omega,\gamma)$ be the set of vertices whose three incident edges are all contained in exactly one of the sets E^0, E^1 or E^2. Let $U' := \mathrm{Int}^{\mathrm{V}}(\gamma) \setminus V^{\mathrm{bad}}(\omega)$ be the set of vertices whose three incident edges are all contained in exactly one of the sets E^0, $(E^1)^{\downarrow}$ or $(E^2)^{\uparrow}$. The lemma will follow if we show that $|U| - |U'| = 6 \cdot |\,\mathrm{dbl}(\omega)|$.

For $E \subset E(\mathbb{H})$, denote by $\mathrm{Int}(E)$ the set of vertices whose three incident edges belong to E. Note that, for a c-garden E of ω, we have $\mathrm{Int}(E) = \mathrm{Int}^{\mathrm{V}}(\sigma(E))$. Thus, it follows from Lemma 2.14 that if E and E' are 0- and 1-gardens of ω, respectively, then $\mathrm{Int}(E) \cap \mathrm{Int}(E')^{\downarrow} = \emptyset$. On the other hand, $\mathrm{Int}(E^0)$ is the union of the interiors of the 0-clusters of ω in γ which comprise E^0, because, by (6), the union of every two of them is disconnected. Since the analogous statement is also true for $\mathrm{Int}(E^1)$, we conclude that $\mathrm{Int}(E^0) \cap \mathrm{Int}(E^1)^{\downarrow} = \emptyset$. By symmetry, we also have $\mathrm{Int}(E^0) \cap \mathrm{Int}(E^2)^{\uparrow} = \emptyset$.

By the inclusion-exclusion principle, recalling from (5) that the sets E^0, E^1 and E^2 are pairwise disjoint, we obtain

$$\begin{aligned}
|U'| &= |\mathrm{Int}(E^0) \cup \mathrm{Int}(E^1)^{\downarrow} \cup \mathrm{Int}(E^2)^{\uparrow}| \\
&= |\mathrm{Int}(E^0)| + |\mathrm{Int}(E^1)^{\downarrow}| + |\mathrm{Int}(E^2)^{\uparrow}| - |\mathrm{Int}(E^1)^{\downarrow} \cap \mathrm{Int}(E^2)^{\uparrow}| \\
&= |\mathrm{Int}(E^0)| + |\mathrm{Int}(E^1)| + |\mathrm{Int}(E^2)| - |\mathrm{Int}(E^1)^{\downarrow} \cap \mathrm{Int}(E^2)^{\uparrow}| \\
&= |\mathrm{Int}(E^0) \cup \mathrm{Int}(E^1) \cup \mathrm{Int}(E^2)| - |\mathrm{Int}(E^1)^{\downarrow} \cap \mathrm{Int}(E^2)^{\uparrow}| \\
&= |U| - |\mathrm{Int}(E^1)^{\downarrow} \cap \mathrm{Int}(E^2)^{\uparrow}|.
\end{aligned}$$

Finally, observe that, by Lemma 2.6, $\mathrm{Int}(E^1)^{\downarrow} \cap \mathrm{Int}(E^2)^{\uparrow}$ is precisely the set of vertices that border the hexagons in $\mathrm{dbl}(\omega)$ and that each such vertex is incident to a unique double-clustered hexagon (since $\mathrm{dbl}(\omega) \subset \mathbb{T}^0$, by Lemma 2.15). Consequently,

$$|\mathrm{Int}(E^1)^{\downarrow} \cap \mathrm{Int}(E^2)^{\uparrow}| = 6 \cdot |\,\mathrm{dbl}(\omega)|. \qquad \square$$

For our next lemma, we require the following definition. A *functional on loops* is a map ϕ that assigns a real number to each loop in $\mathbb{H}$. We say that ϕ is $\uparrow$*-invariant* if $\phi(L^{\uparrow}) = \phi(L)$ for every loop L and $\phi(L) = \phi(L')$ for any two trivial loops L and L'. Given such a functional, we extend ϕ to finite loop configurations ω by summing over all the loops, i.e., by setting

$$\phi(\omega) := \sum_{L \in \mathrm{loops}(\omega)} \phi(L),$$

where loops(ω) is the set of loops in ω.

Recall the definition of $\overline{E}(\omega)$ from (9) and let $\mathsf{TrivLoop} \subset \mathbb{H}$ denote a trivial loop.

Lemma 2.18. *For any $\omega \in \mathsf{LoopConf}(H, \emptyset)$ and any $\uparrow$-invariant functional ϕ on loops, we have*

$$\phi\big(R_\gamma(\omega)\big) - \phi(\omega) = \phi(\mathsf{TrivLoop}) \cdot |V(\omega,\gamma)|/6 - \phi(\omega \cap \overline{E}(\omega)).$$

Proof. As before, set $E^{\mathsf{c}} := E^{\mathsf{c}}(\omega)$ for $\mathsf{c} \in \{0,1,2\}$ and $E^{\text{bad}} := E^{\text{bad}}(\omega)$. Recall from Proposition 2.13 that each loop of $R_\gamma(\omega)$ belongs to one of the following pairwise disjoint loop configurations: $\omega \cap E^0$, $\omega^0_{\text{gnd}} \cap E^{\text{bad}}$, or $(\omega \cap E^1)^{\downarrow} \cup (\omega \cap E^2)^{\uparrow}$. Thus, the definition of a functional implies that

$$\phi(R_\gamma(\omega)) = \phi(\omega \cap E^0) + \phi(\omega^0_{\text{gnd}} \cap E^{\text{bad}}) + \phi\big((\omega \cap E^1)^{\downarrow} \cup (\omega \cap E^2)^{\uparrow}\big). \tag{11}$$

We claim that $\omega^0_{\text{gnd}} \cap E^{\text{bad}}$ consists of $|V^{\text{bad}}(\omega)|/6$ trivial loops. As $\omega^0_{\text{gnd}} \cap E^{\text{bad}}$ is a loop configuration and ω^0_{gnd} is a fully-packed loop configuration containing only trivial loops, it suffices to show that each vertex in $V^{\text{bad}}(\omega)$ is incident to at least two edges in E^{bad}. We may write

$$E^{\text{bad}} = (\text{Int}^{\text{E}}(\gamma) \cup \gamma^*) \setminus \bigcup_i (\text{Int}^{\text{E}}(\sigma_i) \cup \sigma_i^*) = \bigcap_i \text{Ext}^{\text{E}}(\sigma_i) \setminus \text{Ext}^{\text{E}}(\gamma)$$

for some circuits $\sigma_i \subset \mathbb{T} \setminus \mathbb{T}^0$. Let $v \in V^{\text{bad}}(\omega)$ and let z be the hexagon in $\mathbb{T}^0$ which v borders. By Lemma 2.6, the six edges bordering z must belong to $\text{Int}^{\text{E}}(\gamma)$ and to $\text{Ext}^{\text{E}}(\sigma_i)$ for each i. Hence, they belong to E^{bad}, and, in particular, two edges incident to v belong to E^{bad}, as required.

Thus, the $\uparrow$-invariance of ϕ implies

$$\phi(\omega^0_{\text{gnd}} \cap E^{\text{bad}}) = \phi(\mathsf{TrivLoop}) \cdot |V^{\text{bad}}(\omega)|/6. \tag{12}$$

Recalling Lemma 2.15, the inclusion-exclusion principle and the $\uparrow$-invariance of ϕ imply that

$$\begin{aligned}\phi\big((\omega \cap E^1)^{\downarrow} \cup (\omega \cap E^2)^{\uparrow}\big) &= \phi((\omega \cap E^1)^{\downarrow}) + \phi((\omega \cap E^2)^{\uparrow}) - \phi\big((\omega \cap E^1)^{\downarrow} \cap (\omega \cap E^2)^{\uparrow}\big) \\ &= \phi(\omega \cap E^1) + \phi(\omega \cap E^2) - \phi(\mathsf{TrivLoop}) \cdot |\operatorname{dbl}(\omega)|.\end{aligned} \tag{13}$$

Using Lemma 2.17 and identities (11), (12) and (13), we obtain

$$\phi(R_\gamma(\omega)) = \phi(\omega \cap E^0) + \phi(\omega \cap E^1) + \phi(\omega \cap E^2) + \phi(\mathsf{TrivLoop}) \cdot |V(\omega,\gamma)|/6.$$

Finally, by (10),

$$\phi(\omega) = \phi(\omega \cap E^0) + \phi(\omega \cap E^1) + \phi(\omega \cap E^2) + \phi(\omega \cap \overline{E}(\omega)),$$

and the lemma follows by subtracting the last two displayed equations. ☐

Proof of Proposition 2.16. Fix a loop configuration $\omega \in \mathsf{LoopConf}(H, \emptyset)$ and denote

$$\begin{aligned}\Delta L &:= L_H(R_\gamma(\omega)) - L_H(\omega), \\ \Delta o &:= o_H(R_\gamma(\omega)) - o_H(\omega).\end{aligned}$$

With these definitions, we have

$$\frac{\mathbb{P}^{\emptyset}_{H,n,g}(R_\gamma(\omega))}{\mathbb{P}^{\emptyset}_{H,n,g}(\omega)} = \frac{g(o_H(R_\gamma(\omega))) \cdot n^{L_H(R_\gamma(\omega))}}{g(o_H(\omega)) \cdot n^{L_H(\omega)}} = \frac{g(o_H(\omega) + \Delta o)}{g(o_H(\omega))} \cdot n^{\Delta L}.$$

We will show first that

$$\mathbb{P}^{\emptyset}_{H,n,g}(R_\gamma(\omega)) \geq (nx^6)^{\Delta L} \cdot \mathbb{P}^{\emptyset}_{H,n,g}(\omega). \tag{14}$$

Since $0 < x \leq 1$ and g has x-bounded decay, see (7), in order to obtain (14), it suffices to prove that $0 \leq \Delta o \leq 6\Delta L$. Lemma 2.18 applied to the $\uparrow$-invariant functionals ϕ_1 and ϕ_2 defined by

$$\phi_1(L) := 1 \quad \text{and} \quad \phi_2(L) := |E(L)| \quad \text{for every loop } L$$

implies (respectively) that

$$\Delta L = |V(\omega,\gamma)|/6 - |\operatorname{loops}(\omega \cap \overline{E}(\omega))|, \tag{15}$$

$$\Delta o = |V(\omega,\gamma)| - |\omega \cap \overline{E}(\omega)|. \tag{16}$$

Since every loop contains at least six edges, we have

$$6 \cdot \Delta L - \Delta o = |\omega \cap \overline{E}(\omega)| - 6 \cdot |\operatorname{loops}(\omega \cap \overline{E}(\omega))| \geq 0.$$

Furthermore, the fact that $\omega \cap \overline{E}(\omega)$ is a loop configuration implies that

$$|\omega \cap \overline{E}(\omega)| = \sum_{\substack{L \in \operatorname{loops}(\omega) \\ E(L) \subset \overline{E}(\omega)}} |E(L)| = \sum_{\substack{L \in \operatorname{loops}(\omega) \\ E(L) \subset \overline{E}(\omega)}} |V(L)| \leq |V(\omega,\gamma)|, \tag{17}$$

where the last inequality follows from the simple observation that each vertex in a loop contained in $\overline{E}(\omega)$ is incident to at most one edge of a cluster. Now, (16) and (17) imply that $\Delta o \geq 0$, which completes the proof of (14).

As we have assumed that $nx^6 \geq 1$, it remains to show that $\Delta L \geq |V(\omega,\gamma)|/15$. Since every trivial loop of ω is contained in a cluster, there are no trivial loops of ω in $\overline{E}(\omega)$. Any non-trivial loop contains at least 10 edges, and hence, by (17),

$$|\operatorname{loops}(\omega \cap \overline{E}(\omega))| \leq |\omega \cap \overline{E}(\omega)|/10 \leq |V(\omega,\gamma)|/10.$$

Substituting this estimate into (15) yields

$$\Delta L = |V(\omega,\gamma)|/6 - |\operatorname{loops}(\omega \cap \overline{E}(\omega)| \geq |V(\omega,\gamma)|/15,$$

thus concluding the proof. □

2.5. **Proof of the main lemma.** In this section, we establish several properties of the repair map R_γ and prove Lemma 2.11. Let us start with two technical lemmas regarding the connectedness of $V(\omega,\gamma)$. Let $\mathbb{H}^\times$ be the graph obtained from $\mathbb{H}$ by adding an edge between each pair of opposite vertices of every hexagon, so that $\mathbb{H}^\times$ is a 6-regular non-planar graph.

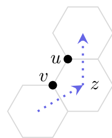
Lemma 2.19. *Let $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ be a circuit with $\mathsf{c} \in \{0,1,2\}$. Then $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ and $\partial\mathrm{Ext}^{\mathrm{V}}(\gamma)$ are connected in $\mathbb{H}^\times$.*

Proof. We only prove that $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ is connected, as the proof of the second statement is very similar. Orient γ with a positive orientation, so that when we move along it, $\mathrm{Int}^{\mathrm{V}}(\gamma)$ is always on the left. Let z be a hexagon on γ. Consider its predecessor z_0 and its successor z_1 in the cyclic ordering of γ. Since z_0 and z_1 belong to the same color class of $\mathbb{T}$, there are only two possible constellations, up to translation and rotation, of the segment (z_0, z, z_1): relative to the step from z_0 to z, the step from z to z_1 may be either a left turn or a right turn (see Figure 7).

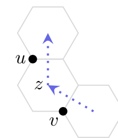
If (z_0, z, z_1) is a left turn then precisely two vertices bordering z lie in $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ and these two vertices are adjacent in $\mathbb{H}$. If (z_0, z, z_1) is a right turn then again precisely two vertices bordering z are in $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ and these two vertices are opposite corners of the hexagon z, and thus adjacent in $\mathbb{H}^\times$. In either case, exactly two vertices bordering z are in $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ and they are adjacent in $\mathbb{H}^\times$. Since every vertex in $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ borders exactly two (consecutive) hexagons in γ, we conclude that $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ is connected in $\mathbb{H}^\times$. □

Lemma 2.20. *Let ω be a loop configuration and let $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ be a vacant circuit for ω. If $\partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma)$ then $V(\omega,\gamma)$ is connected in $\mathbb{H}^\times$.*

Proof. First, by Lemma 2.19, $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ is connected in $\mathbb{H}^\times$. Therefore, it suffices to show that any connected component of $V(\omega,\gamma)$ in $\mathbb{H}^\times$ intersects $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$. In order to prove this, we shall show that for any vertex $v \in V(\omega,\gamma) \setminus \partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ there exists a vertex $v' \in V(\omega,\gamma)$ further to the right than v (i.e., with larger first coordinate in the coordinate system on $\mathbb{T}$ defined in Section 1.1),

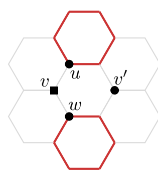


(A) A left turn. Precisely two vertices u and v bordering the hexagon z belong to the interior γ. In particular, they must belong to $\partial\mathrm{Int}(\gamma)$. These vertices lie on one of the edges bordering z, and thus, are adjacent in $\mathbb{H}$.

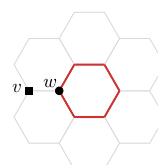


(B) A right turn. Four vertices bordering the hexagon z belong to the interior of γ, precisely two of which, u and v, belong to $\partial\mathrm{Int}(\gamma)$. These vertices lie on opposite corners of z, and thus, are adjacent in $\mathbb{H}^\times$.

FIGURE 7. A circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$, denoted by the dotted line, is oriented with a positive orientation so that the interior is to its left and the exterior is to its right. Every segment (z_0, z, z_1) of γ then constitutes either a left turn or a right turn.



(A) v is the left-most vertex of some hexagon.



(B) v is the right-most vertex of some hexagon.

FIGURE 8. Left-most and right-most vertices.

which is connected to v by an $\mathbb{H}^\times$-path in $V(\omega,\gamma)$. Since $V(\omega,\gamma) \subset \mathrm{Int}^{\mathrm{V}}(\gamma)$, this will imply that the connected component of $V(\omega,\gamma)$ in $\mathbb{H}^\times$ containing v intersects $\partial\mathrm{Int}^{\mathrm{V}}(\gamma)$, as required.

Observe that every vertex in $\mathbb{H}$ is either the left-most or the right-most vertex of some hexagon. Assume first that $v \in V(\omega,\gamma) \setminus \partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ is the left-most vertex of a hexagon (see Figure 8a). Assume that the vertices u and w to the top-right and to the bottom-right of v are not in $V(\omega,\gamma)$. Then there exist two clusters E and E' of ω such that $u \in \mathrm{Int}^{\mathrm{V}}(\sigma(E))$ and $w \in \mathrm{Int}^{\mathrm{V}}(\sigma(E'))$. If z and z' are the unique hexagons containing u and w but not v, respectively, then, by Lemma 2.6, $z \in \mathrm{Int}^{\mathrm{hex}}(\sigma(E))$ and $z' \in \mathrm{Int}^{\mathrm{hex}}(\sigma(E'))$. Thus, z and z' are surrounded by trivial loops. In this case, the vertex directly to the right of v cannot be an internal vertex of a cluster, and hence, it belongs to $V(\omega,\gamma)$. Since it is adjacent to v in $\mathbb{H}^\times$, we found v' as required.

Assume now that $v \in V(\omega,\gamma) \setminus \partial\mathrm{Int}^{\mathrm{V}}(\gamma)$ is the right-most vertex of a hexagon (see Figure 8b). If the vertex w to the right of v is not in $V(\omega,\gamma)$ then there exists $\mathsf{c} \in \{0,1,2\}$ and a c-cluster E of ω such that $w \in \mathrm{Int}^{\mathrm{V}}(\sigma)$. Clearly, $v \in \partial\mathrm{Ext}^{\mathrm{V}}(\sigma) \subset V(\omega,\gamma)$. Moreover, since w is in $\mathrm{Int}^{\mathrm{V}}(\sigma)$, $\partial\mathrm{Ext}^{\mathrm{V}}(\sigma)$ contains a vertex which is further to the right than v. The claim now follows by noticing that, by Lemma 2.19, $\partial\mathrm{Ext}^{\mathrm{V}}(\sigma)$ is connected in $\mathbb{H}^\times$. $\square$

Proof of Lemma 2.11. Let $\gamma \subset \mathbb{T}\setminus\mathbb{T}^0$ be a circuit and denote $H := \mathrm{Int}(\gamma)$. Let $n > 0$ and $0 < x \le 1$. We may assume throughout the proof that nx^6 is sufficiently large, as otherwise the statement is trivial. We shall show that for any $V \subset \mathrm{Int}^{\mathrm{V}}(\gamma)$,

$$\mathbb{P}^{\emptyset}_{H,n,g}(V(\omega,\gamma) = V) \le (2\sqrt{2})^{|V|} \cdot (nx^6)^{-|V|/15}. \tag{18}$$

In light of Lemma 2.20 and Lemma 1.10, Lemma 2.11 will then follow from (18) by summing over all sets V with $\partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V \subset \mathrm{Int}^{\mathrm{V}}(\gamma)$ such that V is connected in $\mathbb{H}^\times$ and has cardinality at least k.

In order to prove (18), we shall apply Lemma 1.9 to the (restricted) repair map

$$R_\gamma \colon \{\omega \in \mathsf{LoopConf}(H,\emptyset) \ : \ V(\omega,\gamma) = V\} \to \mathsf{LoopConf}(H,\emptyset).$$

By Proposition 2.16, we may take $p = (nx^6)^{|V|/15}$. It remains to estimate, for each V, the maximum number of preimages under R_γ of a given loop configuration.

Let ω be such that $V(\omega, \gamma) = V$ and let $E(V)$ be the set of edges with both endpoints in V. We claim that the set $\omega \setminus E(V)$ is determined by $R_\gamma(\omega)$. Indeed, for every $e \in E(\mathbb{H}) \setminus E(V)$, the following is true:

- If $e \notin \mathrm{Int}^{\mathrm{E}}(\gamma)$ then $e \notin \omega$ (since $\omega \in \mathsf{LoopConf}(H, \emptyset)$).
- If one of the two endpoints of e is in V (note that the second endpoint cannot be in V since $e \notin E(V)$), then $e \in \sigma(E)^*$ for some cluster E of ω. In particular, $e \notin \omega$, since $\sigma(E)$ is vacant in ω, by Lemma 2.5b.
- If both endpoints of e are not in V, then e belongs to a c-cluster E of ω for some $\mathsf{c} \in \{0, 1, 2\}$. In this case, $\omega \cap E$ equals either $R_\gamma(\omega) \cap E$, $R_\gamma(\omega)^\uparrow \cap E$ or $R_\gamma(\omega)^\downarrow \cap E$, depending on whether $\mathsf{c} = 0$, $\mathsf{c} = 1$ or $\mathsf{c} = 2$, respectively. Hence, it suffices to determine the value c from V. To this end, consider a path from an endpoint of e to V, and let $\{u, v\}$ be the first edge on this path such that $v \in V$ and $u \notin V$. Then, since $\{u, v\} \in \sigma(E)^*$ and $\sigma(E) \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$, we see that c is the unique element in $\{0, 1, 2\}$ such that $y, z \notin \mathbb{T}^{\mathsf{c}}$, where $\{y, z\}^* = \{u, v\}$.

In conclusion, since given $V(\omega, \gamma) = V$, $R_\gamma(\omega)$ uniquely determines $\omega \setminus E(V)$, the number of preimages of a given loop configuration is at most the number of subsets of $E(V)$. Since there are at most $3|V|/2$ edges with both endpoints in V, there are at most $2^{3|V|/2}$ subsets of $E(V)$. Thus, Lemma 1.9 implies (18). □

3. Proofs of main theorems

Throughout this section, we continue to use the notation introduced in Section 2.1. The proofs of the main theorems mostly rely on the main lemma, Lemma 2.11, and its corollary, Corollary 2.12. We apply these for the functions $g = g_x$ for $x \in (0, \infty]$, where, as before, $g_x(m) = x^m$ for $x \in (0, \infty)$ and g_∞ is the function defined in (3), so that the probability measures $\mathbb{P}^\xi_{H,n,g_x}$ and $\mathbb{P}^\xi_{H,n,x}$ coincide. As the lemma and corollary require that g has x-bounded decay for some $0 < x \le 1$, we note that g_x has $\min\{x, 1\}$-bounded decay for all $x \in (0, \infty]$.

3.1. Exponential decay of loop lengths.

As mentioned in the introduction, the results for small x follow via a Peierls argument. The following lemma gives an upper bound on the probability that a given collection of loops appears in a random loop configuration.

Lemma 3.1. *Let H be a domain and let ξ be a loop configuration. Then, for any $n > 0$, any $x > 0$ and any $A \in \mathsf{LoopConf}(H, \emptyset)$, we have*

$$\mathbb{P}^\xi_{H,n,x}(A \subset \omega) \le n^{L_H(A)} x^{o_H(A)}.$$

Proof. Consider the map

$$\mathsf{T} \colon \{\omega \in \mathsf{LoopConf}(H, \xi) : A \subset \omega\} \to \mathsf{LoopConf}(H, \xi)$$

defined by

$$\mathsf{T}(\omega) := \omega \setminus A.$$

Clearly, T is well-defined (see Lemma 2.5e) and injective. Moreover, since $L_H(\mathsf{T}(\omega)) = L_H(\omega) - L_H(A)$ and $o_H(\mathsf{T}(\omega)) = o_H(\omega) - o_H(A)$, we have

$$\mathbb{P}^\emptyset_{H,n,x}(\mathsf{T}(\omega)) = \mathbb{P}^\emptyset_{H,n,x}(\omega) \cdot n^{-L_H(A)} x^{-o_H(A)}.$$

Hence, the statement follows from Lemma 1.9. □

Corollary 3.2. *For any $n > 0$, any $x > 0$, any domain H, any vertex $u \in V(\mathbb{H})$ and any positive integer k, we have*

$$\mathbb{P}^\emptyset_{H,n,x}(\textit{there exists a loop of length } k \textit{ surrounding } u) \le kn(2x)^k.$$

Proof. Denote by a_k the number of simple paths of length k in $\mathbb{H}$ starting at a given vertex. Clearly, $a_k \leq 3 \cdot 2^{k-1}$. It is then easy to see that the number of loops of length k surrounding u is at most $ka_{k-1} \leq k2^k$. Thus, the result follows by the union bound and Lemma 3.1. □

We also require the following lemma. We say that a circuit γ *surrounds* a subgraph $A \subset \mathbb{H}$ if $A \subset \mathrm{Int}(\gamma)$ and that γ is *inside* A if $\mathrm{Int}(\gamma) \subset A$. We say that a circuit γ *contains* a circuit σ if $\mathrm{Int}(\sigma) \subset \mathrm{Int}(\gamma)$.

Lemma 3.3. *Let $\mathsf{c}' \in \{0,1,2\}$, let H be a domain of type c' and let $\omega \in \mathsf{LoopConf}(H, \emptyset)$. Let $U \subset V(H)$ be a connected subset and assume that no vertex in U belongs to a trivial loop in ω. Then there exists a $\mathsf{c} \in \{0,1,2\}$ and a circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ inside H such that γ is vacant in ω and $U \cup \partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega, \gamma)$.*

Proof. Let us first extend the notion of a breakup. For $\mathsf{c} \in \{0,1,2\}$ and a circuit $\gamma \in \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ which is vacant in ω, let $A(\omega,\gamma)$ be the set of vertices of $\mathbb{H}$ belonging to trivial loops in ω surrounding hexagons in $\mathbb{T}^{\mathsf{c}}$ (i.e., the vertices bordering c-flowers in ω), and let $B(\omega,\gamma)$ be the unique infinite connected component of $\mathrm{Ext}^{\mathrm{V}}(\gamma) \cup A(\omega,\gamma)$. For $u \in \mathbb{H}$, define $\mathcal{C}(\omega,\gamma,u)$ to be the connected component of $\mathbb{H} \setminus B(\omega,\gamma)$ containing u, setting $\mathcal{C}(\omega,\gamma,u) = \emptyset$ if $u \in B(\omega,\gamma)$. By definition, when $\mathcal{C}(\omega,\gamma,u)$ is non-empty, the subgraph of $\mathbb{H}$ induced by $\mathcal{C}(\omega,\gamma,u)$ is a domain. Assume that $\mathcal{C}(\omega,\gamma,u)$ is non-empty and let $\Gamma(\omega,\gamma,u)$ be the circuit satisfying $\mathcal{C}(\omega,\gamma,u) = \mathrm{Int}^{\mathrm{V}}(\Gamma(\omega,\gamma,u))$.

We claim that $\Gamma(\omega,\gamma,u)$ is vacant in ω, is contained in $\mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ and satisfies $\partial\mathcal{C}(\omega,\gamma,u) \subset V(\omega, \Gamma(\omega,\gamma,u))$. To see this, denote $\sigma := \Gamma(\omega,\gamma,u)$ and let $e = \{v,w\} \in \sigma^*$ be an edge with $v \in \mathcal{C}(\omega,\gamma,u)$ and $w \notin \mathcal{C}(\omega,\gamma,u)$. In particular, $v \notin B(\omega,\gamma)$ and $w \in B(\omega,\gamma)$. If $w \notin A(\omega,\gamma)$ then $e \in \gamma^*$ so that $e \notin \omega$ and $e \in \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$. Otherwise, $w \in A(\omega,\gamma)$ and so it borders a c-flower z of ω. Since $v \notin A(\omega,\gamma)$, v does not border z, and thus, $e \notin \omega$ and $e \in \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$. It remains to check that $v \in V(\omega,\sigma)$. Indeed, since $v \notin A(\omega,\gamma)$, v does not border a c-flower of ω, and hence, cannot belong to any cluster of ω inside σ (as σ is vacant in ω). Thus $v \in V(\omega,\sigma)$.

We are now ready to find the required circuit γ. Fix a vertex $u \in U$. Begin with the circuit γ corresponding to H, i.e., $H = \mathrm{Int}(\gamma)$. Now, replace γ with the circuit $\Gamma(\omega,\gamma,u)$. If $u \in V(\omega,\gamma)$ then, since U is connected and no vertex in U belongs to a trivial loop in ω, $U \subset V(\omega,\gamma)$ and we are finished. Otherwise, all three edges incident to u are contained in some cluster E of ω inside γ. Note that $\mathrm{Int}(\sigma(E))$ is strictly contained in $\mathrm{Int}(\gamma)$, since $\mathrm{Int}^{\mathrm{V}}(\sigma(E)) \cap V(\omega,\gamma) = \emptyset$ and $\partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma)$. Thus, replacing γ with $\sigma(E)$ and iterating the above process, we obtain the required circuit. □

Proof of Theorem 1.4. Suppose that n_0 is a sufficiently large constant, let $n \geq n_0$ and let $x \in (0,\infty]$ be arbitrary. Let $\mathsf{c}' \in \{0,1,2\}$, let H be a domain of type c' and let $u \in V(H)$. We shall estimate the probability that, in a random loop configuration drawn from $\mathbb{P}^{\emptyset}_{H,n,x}$, the vertex u is surrounded by a non-trivial loop of length k. We consider two cases, depending on the relative values of n and x.

Suppose first that $nx^6 < n^{1/50}$. Since $n \geq n_0$, we may assume that $2x \leq n^{-4/25}$ and that $kn^{-k/120} \leq 1$ for all $k > 0$. By Corollary 3.2, for every $k \geq 7$,

$$\begin{aligned}\mathbb{P}^{\emptyset}_{H,n,x}(\text{there exists a loop of length } k \text{ surrounding } u) &\leq kn(2x)^k \leq kn^{1-4k/25}\\ &\leq kn^{-k/60} \leq n^{-k/120}.\end{aligned}$$

We now assume that $nx^6 \geq n^{1/50}$. Since $n \geq n_0$, we may assume that $n \cdot \min\{x^6, 1\}$ is sufficiently large for our arguments to hold. Let $L \subset H$ be a non-trivial loop of length k surrounding u. Note that, if $\omega \in \mathsf{LoopConf}(H,\emptyset)$ has $L \subset \omega$ then, by Lemma 3.3, for some $\mathsf{c} \in \{0,1,2\}$, there exists a vacant circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ in ω such that $V(L) \cup \partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma)$. By Corollary 2.12, for every fixed circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$,

$$\mathbb{P}^{\emptyset}_{H,n,x}(\gamma \text{ vacant and } V(L) \cup \partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma)) \leq (cn \cdot \min\{x^6, 1\})^{-|V(L)\cup\partial\mathrm{Int}^{\mathrm{V}}(\gamma)|/15}.$$

Thus, denoting by $\mathcal{G}(u)$ the set of circuits γ contained in $\mathbb{T} \setminus \mathbb{T}^{\mathsf{c}}$ for some $\mathsf{c} \in \{0,1,2\}$ and having $u \in \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(H)$, we obtain

$$\begin{aligned}\mathbb{P}^{\emptyset}_{H,n,x}(L \subset \omega) &\leq \sum_{\gamma \in \mathcal{G}(u)} (cn \cdot \min\{x^6, 1\})^{-|V(L) \cup \partial \mathrm{Int}^{\mathrm{V}}(\gamma)|/15} \\ &\leq \sum_{\ell=1}^{\infty} D^{\ell} (cn \cdot \min\{x^6,1\})^{-\max\{\ell,k\}/15} \\ &\leq (c'n \cdot \min\{x^6,1\})^{-k/15},\end{aligned}$$

where we used the facts that the length of a circuit γ such that $|\partial \mathrm{Int}^{\mathrm{V}}(\gamma)| = \ell$ is at most 3ℓ, that the number of circuits of length at most 3ℓ surrounding u is bounded by D^{ℓ} for some sufficiently large constant D, and in the last inequality we used the assumption that $n \cdot \min\{x^6, 1\}$ is sufficiently large. Since the number of loops of length k surrounding a given vertex is smaller than $k2^k$, our assumptions that $nx^6 \geq n^{1/50}$ and $n \geq n_0$ yield

$$\mathbb{P}^{\emptyset}_{H,n,x}(\text{there exists a loop of length } k \text{ surrounding } u) \leq k2^k (c'n^{1/50})^{-k/15} \leq n^{-k/800}.$$

Proof of Theorem 1.5. The proof is very similar to that of Theorem 1.4. The main difference is the following replacement of Lemma 2.11. Recall that every $\lambda \in \mathsf{LoopConf}(H, \emptyset, u, v)$ contains a simple path between u and v. For every $\lambda \in \mathsf{LoopConf}(H, \emptyset, u, v)$, let $p(\lambda, u)$ be the connected component of u (equivalently, of v) in λ and denote $\omega_\lambda := \lambda \setminus p(\lambda, u)$, so that ω_λ is a loop configuration in $\mathsf{LoopConf}(H, \emptyset)$. For a circuit γ for which $\mathrm{Int}(\gamma) \subset H$, let $\mathcal{E}(H, u, v, \gamma, k)$ be the set of configurations $\lambda \in \mathsf{LoopConf}(H, \emptyset, u, v)$ such that

- γ is vacant in ω_λ;
- $V(p(\lambda, u)) \subset V(\omega_\lambda, \gamma)$;
- $\partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega_\lambda, \gamma)$;
- $|V(\omega_\lambda, \gamma)| \geq k$.

For $\omega \in \mathsf{LoopConf}(H, \emptyset)$ and $\lambda \in \mathsf{LoopConf}(H, \emptyset, u, v)$, denote

$$\begin{aligned}\phi_{H,n,x}(\omega) &:= x^{o_H(\omega)} n^{L_H(\omega)}, \\ \phi_{H,n,x}(\lambda) &:= x^{o_H(\lambda)} n^{L'_H(\lambda)} J(\lambda).\end{aligned}$$

Lemma 3.4. *There exist absolute constants $C, c > 0$ such that for any $n \geq C$ and $x \in (0, \infty)$ satisfying $nx^6 \geq C$ the following holds. For any circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$, any vertices $u, v \in V(\mathbb{H})$ and any positive integer k, we have*

$$\sum_{\lambda \in \mathcal{E}(\mathrm{Int}(\gamma), u, v, \gamma, k)} \phi_{H,n,x}(\lambda) \leq \min\{x, x^{d(u,v)}\} \cdot (cn \cdot \min\{x^6, 1\})^{-k/15} \sum_{\omega \in \mathsf{LoopConf}(H, \emptyset)} \phi_{H,n,x}(\omega).$$

Proof. Fix a circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ and denote $H := \mathrm{Int}(\gamma)$. Recall the repair map R_γ defined in Section 2.3 and Proposition 2.16 which compared the probabilities of $R_\gamma(\omega)$ and ω. Let us first show that one may strengthen the conclusion of Proposition 2.16 when $g = g_x$ to obtain

$$\mathbb{P}^{\emptyset}_{H,n,x}(R_\gamma(\omega)) \geq \max\{x, 1\}^{|V'(\omega,\gamma)|} (n \cdot \min\{x^6, 1\})^{|V(\omega,\gamma)|/15} \cdot \mathbb{P}^{\emptyset}_{H,n,x}(\omega),$$

for any $\omega \in \mathsf{LoopConf}(H, \emptyset)$, where $V'(\omega, \gamma)$ denotes the vertices in $V(\omega, \gamma)$ which are isolated in ω. Indeed, for $0 < x \leq 1$, this is precisely the conclusion of the original proposition. For $x > 1$, one repeats the proof with the following modifications. In (17), modifying the last step, we may obtain the stronger inequality

$$|\omega \cap \overline{E}(\omega)| \leq |V(\omega, \gamma) \setminus V'(\omega, \gamma)|.$$

This then shows that $\Delta o \geq |V'(\omega, \gamma)|$. Hence, we conclude that

$$\mathbb{P}^{\emptyset}_{H,n,x}(R_\gamma(\omega)) = x^{\Delta o} n^{\Delta L} \cdot \mathbb{P}^{\emptyset}_{H,n,x}(\omega) \geq x^{|V'(\omega,\gamma)|} n^{|V(\omega,\gamma)|/15} \cdot \mathbb{P}^{\emptyset}_{H,n,x}(\omega),$$

as required.

Now, fix a subset $V \subset \mathrm{Int}^{\mathrm{V}}(\gamma)$ and consider the map

$$S\colon \{\lambda \in \mathcal{E}(H,u,v,\gamma,k) \;:\; V(\omega_\lambda,\gamma)=V\} \to \mathsf{LoopConf}(H,\omega^0_{\mathrm{gnd}})$$

defined by $S(\lambda) := R_\gamma(\omega_\lambda)$. Then, as $\phi_{H,n,x}(\lambda) = x^{|E(p(\lambda,u))|}J(\lambda)\cdot\phi_{H,n,x}(\omega_\lambda)$, we have just shown that

$$\frac{\phi_{H,n,x}(S(\lambda))}{\phi_{H,n,x}(\lambda)} \geq \frac{1}{J(\lambda)} x^{-|E(p(\lambda,u,v))|} \max\{x,1\}^{|V'(\omega_\lambda,\gamma)|}(n\cdot\min\{x^6,1\})^{|V(\omega_\lambda,\gamma)|/15}.$$

Thus, since $J(\lambda)\leq 3$ and $d(u,v) \leq |E(p(\lambda,u))| \leq |V'(\omega_\lambda,\gamma)|+1$, we obtain

$$\frac{\phi_{H,n,x}(S(\lambda))}{\phi_{H,n,x}(\lambda)} \geq \tfrac{1}{3}\max\{x^{-1},x^{-|d(u,v)|}\}\cdot(n\min\{x^6,1\})^{|V(\omega_\lambda,\gamma)|/15}.$$

Now, as was shown in the proof of Lemma 2.11, $S(\lambda)$ uniquely determines $\omega_\lambda \setminus E(V)$. Noting that $\lambda \setminus E(V) = \omega_\lambda \setminus E(V)$, since $V(p(\lambda,u)) \subset V(\omega_\lambda,\gamma)$, we see that the number of preimages of ω under S is bounded by the number of subsets of $E(V)$, i.e., $|S^{-1}(\omega)| \leq 2^{3|V|/2}$. Therefore, a minor variant of Lemma 1.9 (a variant in which the mapping is between different discrete measure spaces, not necessarily probability spaces, which is proved with the same argument) implies that

$$\sum_{\substack{\lambda\in\mathcal{E}(H,u,v,\gamma,k)\\ V(\omega_\lambda)=V}} \phi_{H,n,x}(\lambda) \leq 3\min\{x,x^{d(u,v)}\}\cdot(cn\cdot\min\{x^6,1\})^{-k/15}\sum_{\omega\in\mathsf{LoopConf}(H,\emptyset)}\phi_{H,n,x}(\omega).$$

The above is the analogue of (18). Finally, as in the proof of Lemma 2.11, the lemma follows by summing over all sets V with $\partial\mathrm{Int}^{\mathrm{V}}(\gamma) \subset V \subset \mathrm{Int}^{\mathrm{V}}(\gamma)$ such that V is connected in $\mathbb{H}^\times$ and has cardinality at least k. □

Once this lemma is available to us, the proof then goes along the same lines as Section 3.1. For small x, one easily adapts Lemma 3.1 and its corollary, Corollary 3.2, to show that the ratio decays exponentially fast in the distance between u and v (simply use the map $\lambda\mapsto\omega_\lambda$). For large x, one uses Lemma 3.3 to find a vacant circuit $\gamma\subset\mathbb{T}\setminus\mathbb{T}^{\mathsf{c}}$ in ω_λ (for some $\mathsf{c}\in\{0,1,2\}$) such that $V(p(\lambda,u))\cup\partial\mathrm{Int}^{\mathrm{V}}(\gamma)\subset V(\omega_\lambda,\gamma)$. The rest of the proof is then the same.

3.2. **Small perturbation of ground state.**

Proof of Theorem 1.8. By definition, the subgraph of $\mathbb{H}$ induced by $\mathcal{C}(\omega,u)$ is a domain when it is non-empty. Let $\Gamma(\omega,u)$ be the circuit satisfying $\mathcal{C}(\omega,u)=\mathrm{Int}^{\mathrm{V}}(\Gamma(\omega,u))$. It follows that $\Gamma(\omega,u)$ is vacant and contained in $\mathbb{T}\setminus\mathbb{T}^0$. To see this, note that the edge boundary of $B(\omega)$ consists only of edges $\{v,w\}$ such that v is on the boundary of a 0-flower y and w is the unique neighbor of v not lying on the boundary of y; in particular, $\{v,w\}$ borders a hexagon from $\mathbb{T}^1$ and a hexagon from $\mathbb{T}^2$ and $\{v,w\}\notin\omega$. Furthermore, $\partial\mathcal{C}(\omega,u)\subset V(\omega,\Gamma(\omega,u))$. Indeed, if $v\in\partial\mathrm{Int}^{\mathrm{V}}(\Gamma(\omega,u))$ then, by the definition of $B(\omega)$, v does not belong to a trivial loop surrounding a hexagon in $\mathbb{T}^0$. It follows that v does not belong to any cluster of ω inside $\Gamma(\omega,u)$ as $\Gamma(\omega,u)$ is vacant in ω. Thus $v\in V(\omega,\Gamma(\omega,u))$.

Finally, denoting by $\mathcal{G}_k(u)$ the set of circuits $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ having $u \in \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(H)$ and $|\partial \mathrm{Int}^{\mathrm{V}}(\gamma)| \geq k$, Corollary 2.12 implies that

$$\begin{aligned}
\mathbb{P}^0_{H,n,x}(|\partial \mathcal{C}(\omega,u)| \geq k) &\leq \sum_{\gamma \in \mathcal{G}_k(u)} \mathbb{P}^0_{H,n,x}(\Gamma(\omega,u) = \gamma) \\
&\leq \sum_{\gamma \in \mathcal{G}_k(u)} \mathbb{P}^0_{H,n,x}(\gamma \text{ vacant and } \partial \mathrm{Int}^{\mathrm{V}}(\gamma) \subset V(\omega,\gamma)) \\
&\leq \sum_{\gamma \in \mathcal{G}_k(u)} (cn \cdot \min\{x^6,1\})^{-|\partial \mathrm{Int}^{\mathrm{V}}(\gamma)|/15} \\
&\leq \sum_{\ell \geq k} D^\ell (cn \cdot \min\{x^6,1\})^{-\ell/15} \leq (c' n \cdot \min\{x^6,1\})^{-k/15},
\end{aligned}$$

where c', D, C' are positive constants. In the final inequality, we used the facts that the length of a circuit γ such that $|\partial \mathrm{Int}^{\mathrm{V}}(\gamma)| = \ell$ is at most 3ℓ, and that the number of circuits of length at most 3ℓ surrounding u is bounded by D^ℓ for some sufficiently large constant D.

3.3. **Limiting Gibbs measures.** Before proving the last two theorems, we require the following lemma.

Lemma 3.5. *Let H and H' be two domains, let $A \subset H \cap H'$ be a non-empty subgraph and let ξ and ξ' be loop configurations. Let $n > 0$, let $x \in (0,\infty]$ and let $\omega \sim \mathbb{P}^{\xi}_{H,n,x}$ and $\omega' \sim \mathbb{P}^{\xi'}_{H',n,x}$ be independent. Denote by Ω the event that there exists a circuit surrounding A and inside $H \cap H'$ which is vacant in both ω and ω'. Assume that Ω has positive probability. Then, conditioned on Ω, the marginal distributions of ω and ω' on A are equal.*

Proof. In this proof, a doubly-vacant circuit is a circuit which is vacant in both ω and ω'. Let $\mathcal{G}$ denote the collection of circuits surrounding A and inside $H \cap H'$. Let $\sigma \in \mathcal{G}$ and $\sigma' \in \mathcal{G}$ be doubly-vacant circuits. Then, since both circuits surround A, $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\sigma') \neq \emptyset$. By Fact 2.3, there exists a circuit $\gamma \subset \sigma \cup \sigma'$ having $\gamma^* \subset \sigma^* \cup (\sigma')^*$ which contains both σ and σ'. Clearly, γ is doubly-vacant, surrounds A and is inside $H \cap H'$, and hence, $\gamma \in \mathcal{G}$. Thus, we have a notion of the "outermost" doubly-vacant circuit in $\mathcal{G}$. On Ω, define Γ to be this circuit. Then, we claim that, for any circuit $\gamma \in \mathcal{G}$ for which the event $\Omega \cap \{\Gamma = \gamma\}$ has positive probability, conditioned on $\Omega \cap \{\Gamma = \gamma\}$, the marginal distribution of (ω, ω') on A^2 is the same as the marginal distribution of two independent loop configurations sampled from $\mathbb{P}^{\emptyset}_{\mathrm{Int}(\gamma),n,x}$. Indeed, since the event $\Omega \cap \{\Gamma = \gamma\}$ is determined by $\omega \setminus \mathrm{Int}(\gamma)$ and $\omega' \setminus \mathrm{Int}(\gamma)$, this follows from the domain Markov property. □

Proof of Theorem 1.6. We start with a lemma.

Lemma 3.6. *Let $n > 0$ and $x > 0$. For any two domains H and H', any vertex $u \in V(\mathbb{H})$ and any positive integer k, we have*

$$\mathbb{P}(\textit{the connected component of } u \textit{ in } \omega \cup \omega' \textit{ has exactly } k \textit{ edges}) \leq (18e \max\{n^{1/6},1\}x)^k,$$

where $\omega \sim \mathbb{P}^{\emptyset}_{H,n,x}$ and $\omega' \sim \mathbb{P}^{\emptyset}_{H',n,x}$ are independent.

Proof. We may assume that $\max\{n^{1/6},1\}x \leq 1$, since the statement is trivial otherwise. Let $\mathcal{C}_k$ be the set of connected subgraphs of $\mathbb{H}$ that have exactly k edges, at most k vertices and contain u. For $C \in \mathcal{C}_k$, call a pair of loop configurations (A, A') *compatible with* C if $E(A) \cup E(A') = E(C)$.

Let C be the connected component of u in $\omega \cup \omega'$. Then

$$\begin{aligned}\mathbb{P}(|\mathsf{C}| = k) &\leq \sum_{C \in \mathcal{C}_k} \sum_{(A,A') \text{ compatible with } C} \mathbb{P}(A \subset \omega,\ A' \subset \omega') \\ &\leq \sum_{C \in \mathcal{C}_k} \sum_{(A,A') \text{ compatible with } C} (n^{1/6}x)^{o_H(A)+o_{H'}(A')} \\ &\leq (18e)^k (\max\{n^{1/6}, 1\}x)^k.\end{aligned}$$

The second inequality follows from Lemma 3.1 and the fact that ω and ω' are independent, any loop consists of at least six edges and $n \geq 1$. The last inequality follows from the following three facts:

- $o_H(A) + o_{H'}(A') \geq |E(C)| = k$ and $\max\{n^{1/6}, 1\}x \leq 1$;
- since each edge in C must be in either A, A' or in both, the number of possible pairs of loop configurations (A, A') compatible with C is bounded by 3^k;
- choosing first a connected set of k vertices and then a subgraph (not necessarily connected) on these vertices, Lemma 1.10 gives that $|\mathcal{C}_k|$ is bounded by $3^k(2e)^k$. □

Let us conclude the proof of Theorem 1.6. Let H and H' be two domains and let $A \subset B \subset H \cap H'$ be two subdomains. Let $\omega \sim \mathbb{P}^{\emptyset}_{H,n,x}$ and $\omega' \sim \mathbb{P}^{\emptyset}_{H',n,x}$ be independent. Let $\mathcal{E}$ be the event that the union of the connected components of the vertices of A in the graph $\omega \cup \omega'$ intersects $\mathbb{H} \setminus B$. Lemma 3.6 implies that

$$\mathbb{P}(\mathcal{E}) \leq \sum_{v \in V(A)} \sum_{k=d(\{v\}, \mathbb{H}\setminus B)}^{\infty} (18e \max\{n^{1/6}, 1\}x)^k \leq |V(A)| \sum_{k=d(A, \mathbb{H}\setminus B)}^{\infty} (18e \max\{n^{1/6}, 1\}x)^k, \quad (19)$$

where $d(E, F)$ is the minimum of the graph distances between a vertex in E and a vertex in F.

Let us now show that, on the complement of $\mathcal{E}$, there exists a circuit γ surrounding A and inside $H \cap H'$ which is vacant in both ω and ω'. We first define the notion of the *outer circuit* of a non-empty finite connected subset U of $V(\mathbb{H})$. Let U' be the unique infinite connected component of $V(\mathbb{H}) \setminus U$ and let $U'' := V(\mathbb{H}) \setminus U'$. Evidently, U'' is a domain containing U. The outer circuit σ of U is then the circuit corresponding to the subgraph of $\mathbb{H}$ induced by U'', i.e., $U'' = \mathrm{Int}^{\mathrm{V}}(\sigma)$, which exists by Fact 2.2. Note also that $\partial \mathrm{Int}^{\mathrm{V}}(\sigma) \subset \partial U$ and that if U is contained in some domain then U'' is also contained in the same domain.

Let D be the union of the connected components of vertices of A in $\omega \cup \omega'$. Let γ be the outer circuit of $V(A) \cup \mathsf{D}$, and note that, on the complement of $\mathcal{E}$, γ is inside B. Let us show that γ is vacant in both ω and ω'. To this end, let $e = (u, v) \in \gamma^*$ be an edge with $u \in V(A) \cup \mathsf{D}$ and $v \notin V(A) \cup \mathsf{D}$. Assume first that $u \in \mathsf{D}$. Clearly $e \notin \omega \cup \omega'$, as otherwise, v would also belong to D. Assume now that $u \in V(A) \setminus \mathsf{D}$. Then, by definition of D, u is not contained in a loop of neither ω nor ω'. In particular, e does not belong to neither ω nor ω'. Thus, γ is vacant in both ω and ω'.

Thus, by Lemma 3.5, the total variation between the measures $\mathbb{P}^{\emptyset}_{H,n,x}(\cdot_{|A})$ and $\mathbb{P}^{\emptyset}_{H',n,x}(\cdot_{|A})$ is at most $\mathbb{P}(\mathcal{E})$. In light of (19), by taking B large enough, we may make $\mathbb{P}(\mathcal{E})$ arbitrarily small. This implies the convergence of the measures $\mathbb{P}^{\emptyset}_{H_k,n,x}(\cdot_{|A})$ towards a limit. Since this holds for any domain A, we have established the convergence of $\mathbb{P}^{\emptyset}_{H_k,n,x}$ as $k \to \infty$ towards an infinite-volume measure $\mathbb{P}^{\emptyset}_{\mathbb{H},n,x}$.

The fact that the limiting measure is supported on loop configurations with no infinite paths is an immediate consequence of Corollary 3.2. Indeed, the corollary shows that in the measure $\mathbb{P}^{\emptyset}_{H_k,n,x}$ the probability that a given vertex is contained in a loop of length m tends to zero with m, uniformly in k.

Proof of Theorem 1.7. Let us first show that if $n \cdot \min\{x^6, 1\}$ is sufficiently large then the limiting measures $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$, $\mathsf{c} \in \{0,1,2\}$, are distinct (assuming they exist). By Theorem 1.8, if $n \cdot \min\{x^6, 1\}$ is sufficiently large then, for any $z \in \mathbb{T}^0$,

$$\mathbb{P}^0_{\mathbb{H},n,x}(z \text{ is surrounded by a trivial loop}) > 1/2.$$

Since $\mathbb{P}^1_{\mathbb{H},n,x}$ and $\mathbb{P}^2_{\mathbb{H},n,x}$ are the measures induced by applying the shifts $\downarrow$ and $\uparrow$, respectively, to $\mathbb{P}^0_{\mathbb{H},n,x}$, the same statement holds for any $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$ with $z \in \mathbb{T}^{\mathsf{c}}$. Thus, since adjacent hexagons cannot both be surrounded by trivial loops simultaneously, we conclude that the measures $\{\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}\}_{\mathsf{c}\in\{0,1,2\}}$ are distinct.

It remains to show that, for any $\mathsf{c} \in \{0,1,2\}$, $\mathbb{P}^{\mathsf{c}}_{H_k,n,x}$ converges as $k \to \infty$ to an infinite-volume measure $\mathbb{P}^{\mathsf{c}}_{\mathbb{H},n,x}$ which is supported on loop configurations with no infinite paths. Without loss of generality, we may assume that $\mathsf{c} = 0$. The proof bears similarity with the proof of Theorem 1.6.

We start with a lemma. Recall the definition of $B(\omega)$ and $\mathcal{C}(\omega, u)$ from Section 1.1. For a domain H and a loop configuration $\omega \in \mathsf{LoopConf}(H, \omega^0_{\text{gnd}})$, denote $\mathcal{C}(\omega) := V(\mathbb{H}) \setminus B(\omega) = \cup_{u \in V(\mathbb{H})} \mathcal{C}(\omega, u)$. Note that, by definition, every two breakups $\mathcal{C}(\omega, u)$ and $\mathcal{C}(\omega, v)$, where $u, v \in V(\mathbb{H})$, are either equal or their union is disconnected in $\mathbb{H}^\times$ (as the definition implies that if a vertex belongs to $\mathcal{C}(\omega)$ then all vertices bordering the same hexagon in $\mathbb{T}^0$ also belong to $\mathcal{C}(\omega)$). Thus, every connected component of $\mathcal{C}(\omega)$ is a breakup of some vertex, and every $\mathbb{H}^\times$-connected component of $\partial\mathcal{C}(\omega)$ is the boundary of a breakup of some vertex, i.e., equals $\partial\mathcal{C}(\omega, u)$ for some $u \in V(\mathbb{H})$ (recall that this set is $\mathbb{H}^\times$-connected, by Lemma 2.19).

Lemma 3.7. *There exists an absolute constant $c > 0$ such that for any $n > 0$ and $x \in (0, \infty]$ the following holds. For any two domains H and H', any vertex $u \in V(\mathbb{H})$ and any positive integer k,*

$$\mathbb{P}(\textit{the } \mathbb{H}^\times\textit{-connected component of } u \textit{ in } \partial\mathcal{C}(\omega) \cup \partial\mathcal{C}(\omega') \textit{ has cardinality } k) \le (cn \cdot \min\{x^6, 1\})^{-k/15},$$

where $\omega \sim \mathbb{P}^0_{H,n,x}$ and $\omega' \sim \mathbb{P}^0_{H',n,x}$ are independent.

Proof. Let $\mathcal{C}_k$ be the set of $\mathbb{H}^\times$-connected subsets of $V(\mathbb{H})$ of cardinality k containing u. For $C \in \mathcal{C}_k$, call a pair (A, A') of subsets of $V(\mathbb{H})$ *compatible with* C if $A \cup A' = C$. We write $A \prec \mathcal{C}(\omega)$ if A is the union of some $\mathbb{H}^\times$-connected components of $\partial\mathcal{C}(\omega)$, or equivalently, if every $\mathbb{H}^\times$-connected component of A is equal to $\partial\mathcal{C}(\omega, v)$ for some $v \in V(\mathbb{H})$. Now, we claim that for each fixed A we have

$$\mathbb{P}^0_{H,n,x}(A \prec \mathcal{C}(\omega)) \le (cn \cdot \min\{x^6, 1\})^{-|A|/15}. \tag{20}$$

To see this, note that for the probability to be positive, A needs to be a union of $\partial\mathrm{Int}^{\mathrm{V}}(\gamma_i)$ for a collection of circuits $\gamma_i \subset \mathbb{T} \setminus \mathbb{T}^0$ with disjoint interiors. Next, by conditioning on all of the γ_i being vacant, we may apply the domain Markov property and Theorem 1.8 to obtain the estimate (20). Similarly, for each fixed A' we have that

$$\mathbb{P}^0_{H',n,x}(A' \prec \mathcal{C}(\omega')) \le (cn \cdot \min\{x^6, 1\})^{-|A'|/15}.$$

We may assume that $cn \cdot \min\{x^6, 1\} \ge 1$, since the statement is trivial otherwise. Let C be the $\mathbb{H}^\times$-connected component of u in $\partial\mathcal{C}(\omega) \cup \partial\mathcal{C}(\omega')$. Then

$$\begin{aligned}
\mathbb{P}(|\mathsf{C}| = k) &\le \sum_{C \in \mathcal{C}_k} \sum_{(A,A') \text{ compatible with } C} \mathbb{P}(A \prec \mathcal{C}(\omega),\ A' \prec \mathcal{C}(\omega')) \\
&\le \sum_{C \in \mathcal{C}_k} \sum_{(A,A') \text{ compatible with } C} (cn \cdot \min\{x^6, 1\})^{-(|A|+|A'|)/15} \\
&\le (15e)^k (cn \cdot \min\{x^6, 1\})^{-k/15}.
\end{aligned}$$

In the second inequality we used the fact that ω and ω' are independent. The last inequality follows from the following three facts:

- $|A| + |A'| \geq |C| = k$ and $cn \cdot \min\{x^6, 1\} \geq 1$;
- since each vertex in C is either in A, in A' or in both, the number of possible pairs (A, A') compatible with C is bounded by 3^k;
- since $\mathbb{H}^\times$ is 6-regular, Lemma 1.10 implies that $|\mathcal{C}_k|$ is bounded by $(5e)^k$. □

Let us conclude the proof of Theorem 1.7. Let H and H' be two domains and let $A \subset B \subset H \cap H'$ be two domains of type 0. Let $\omega \sim \mathbb{P}^0_{H,n,x}$ and $\omega' \sim \mathbb{P}^0_{H',n,x}$ be independent. Let $\mathcal{E}$ be the event that the union of $\mathbb{H}^\times$-connected components of vertices in A in $\partial\mathcal{C}(\omega) \cup \partial\mathcal{C}(\omega')$ intersects $V(\mathbb{H}) \setminus B$. Lemma 3.7 implies that

$$\mathbb{P}(\mathcal{E}) \leq \sum_{u \in V(A)} \sum_{k=d(\{u\},\mathbb{H}\setminus B)}^{\infty} (cn \cdot \min\{x^6, 1\})^{-k/15} \leq |V(A)| \sum_{k=d(A,\mathbb{H}\setminus B)}^{\infty} (cn \cdot \min\{x^6, 1\})^{-k/15},$$

where $d(E, F)$ is the minimum of the graph distances between a vertex in E and a vertex in F. Let $\mathcal{E}'$ be the event that A is contained in either $\mathcal{C}(\omega)$ or $\mathcal{C}(\omega')$, i.e., that A is contained entirely in one breakup (of either ω or ω'). Denote by $\rho(m)$ the smallest possible size of ∂U for a subset $U \subset V(\mathbb{H})$ of size m. Then Theorem 1.8 implies that

$$\mathbb{P}(\mathcal{E}') \leq (cn \cdot \min\{x^6, 1\})^{-\rho(|V(A)|)/15}.$$

Let us now show that, on the complement of $\mathcal{E} \cup \mathcal{E}'$, there exists a circuit $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ surrounding A and inside $H \cap H'$ which is vacant in both ω and ω'. We require the following simple geometric claim. For brevity, in the rest of the proof we identify a domain with its set of vertices.

$$\begin{aligned}&\text{If } S, T \text{ are two domains of type 0 with } S \not\subset T \text{ and } T \not\subset S \text{ such that } S \cup T \text{ is connected}\\ &\text{then } \partial S \cup \partial T \text{ is } \mathbb{H}^\times\text{-connected. If, in addition, } S \cap T \neq \emptyset \text{ then also } \partial S \cap \partial T \neq \emptyset\end{aligned} \tag{21}$$

To see this, note first that ∂S and ∂T are $\mathbb{H}^\times$-connected by Fact 2.2 and Lemma 2.19. If $S \cap T = \emptyset$ then the assumption that $S \cup T$ is connected implies that a vertex of ∂S is adjacent to a vertex of ∂T yielding that $\partial S \cup \partial T$ is $\mathbb{H}^\times$-connected. Assume that $S \cap T \neq \emptyset$. By considering a path in T from $T \setminus S$ to $T \cap S$ it follows that $\partial S \cap T \neq \emptyset$. Similarly, considering a path in T^c from $S \setminus T$ to $(S \cup T)^c$ shows that $\partial S \setminus T \neq \emptyset$. We conclude that $\partial S \cap \partial T \neq \emptyset$, yielding the claim.

Recall the notion of the outer circuit of a non-empty finite connected subset U of $V(\mathbb{H})$ from the proof of Theorem 1.6. Let D be the union of the connected components of the vertices of A in $\mathcal{C}(\omega) \cup \mathcal{C}(\omega')$. Let γ be the outer circuit of $A \cup \mathsf{D}$. It follows that $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ and that γ is vacant in both ω and ω'. Indeed, $\gamma \subset \mathbb{T} \setminus \mathbb{T}^0$ since A is a domain of type 0 and, by the definition of breakup, each circuit $\Gamma(\omega, u)$ corresponding to the breakup $\mathcal{C}(\omega, u)$ is in $\mathbb{T} \setminus \mathbb{T}^0$. Thus, no edge of γ^* can belong to $\omega \cup \omega'$ since otherwise both its endpoints would belong to a breakup.

We claim that, on the complement of $\mathcal{E} \cup \mathcal{E}'$, γ is inside B. By the definition of γ and since B is a domain, it suffices to show that $A \cup \mathsf{D} \subset B$. As $A \subset B$, we need only verify that $\mathsf{D} \subset B$. On the complement of $\mathcal{E}'$, we may write $A \cup \mathsf{D}$ as the union of domains D_i of type 0 such that no one contains another, $D_0 = A$ and each D_i, $i \neq 0$, is a breakup of either ω or ω'. Let D' be the union of the $\mathbb{H}^\times$-connected components of the vertices of A in $\partial\mathcal{C}(\omega) \cup \partial\mathcal{C}(\omega')$. On the complement of $\mathcal{E}$, $\partial A \cup \mathsf{D}'$ is contained in B. By (21), $\cup_i \partial D_i$ is $\mathbb{H}^\times$-connected and if $D_i \cap A \neq \emptyset$ then $\partial D_i \cap \partial A \neq \emptyset$. Thus $\cup_i \partial D_i \subset \partial A \cup \mathsf{D}'$. We conclude that $\partial \mathsf{D} \subset \cup_i \partial D_i \subset B$, whence $\mathsf{D} \subset B$ as we wanted to show.

Thus, by Lemma 3.5, the total variation between the measures $\mathbb{P}^0_{H,n,x}(\cdot_{|A})$ and $\mathbb{P}^0_{H',n,x}(\cdot_{|A})$ is at most $\mathbb{P}(\mathcal{E} \cup \mathcal{E}')$. In particular, fixing a subgraph $A' \subset A$, the same holds for the measures $\mathbb{P}^0_{H,n,x}(\cdot_{|A'})$ and $\mathbb{P}^0_{H',n,x}(\cdot_{|A'})$. Since $\rho(m)$ clearly tends to infinity as m tends to infinity, by first taking A large enough and then taking B large enough, we may make $\mathbb{P}(\mathcal{E} \cup \mathcal{E}')$ arbitrarily small. This implies the convergence of the measures $\mathbb{P}^0_{H_k,n,x}(\cdot_{|A'})$ towards a limit. Since this holds for any finite subgraph A' of $\mathbb{H}$, we have established the convergence of $\mathbb{P}^0_{H_k,n,x}$ as $k \to \infty$ towards an infinite-volume measure $\mathbb{P}^0_{\mathbb{H},n,x}$.

The fact that the limiting measure is supported on loop configurations with no infinite paths is a consequence of Theorem 1.8. Indeed, if a given vertex u is contained in a loop of length $m > 6$ then the breakup $\mathcal{C}(\omega, u)$ must be of size at least m, and hence, $\partial\mathcal{C}(\omega, u)$ is necessarily of size at least $\rho(m)$, which tends to infinity with m. Thus, in the measure $\mathbb{P}^0_{H_k,n,x}$, the probability that u is contained in a loop of length $m > 6$ tends to zero with m, uniformly in k.

4. Discussion and open questions

In this work, we investigate the structure of loop configurations in the loop $O(n)$ model with large parameter n. We show that the chance of having a loop of length k surrounding a given vertex decays exponentially in k. In addition, we show, under appropriate boundary conditions, that if nx^6 is small, the model is in a dilute, disordered phase whereas if nx^6 is large, configurations typically resemble one of the three ground states. In this section we briefly discuss several open directions.

Spin O(n). As described in the introduction, the loop $O(n)$ model can be viewed as an approximation of the spin $O(n)$ model, with the length of loops related to the spin-spin correlation function. Thus, our results prove an analogue of the well-known conjecture that spin-spin correlations decay exponentially in the spin $O(n)$ model with $n \geq 3$ at any positive temperature. Proving the conjecture itself remains a tantalizing challenge.

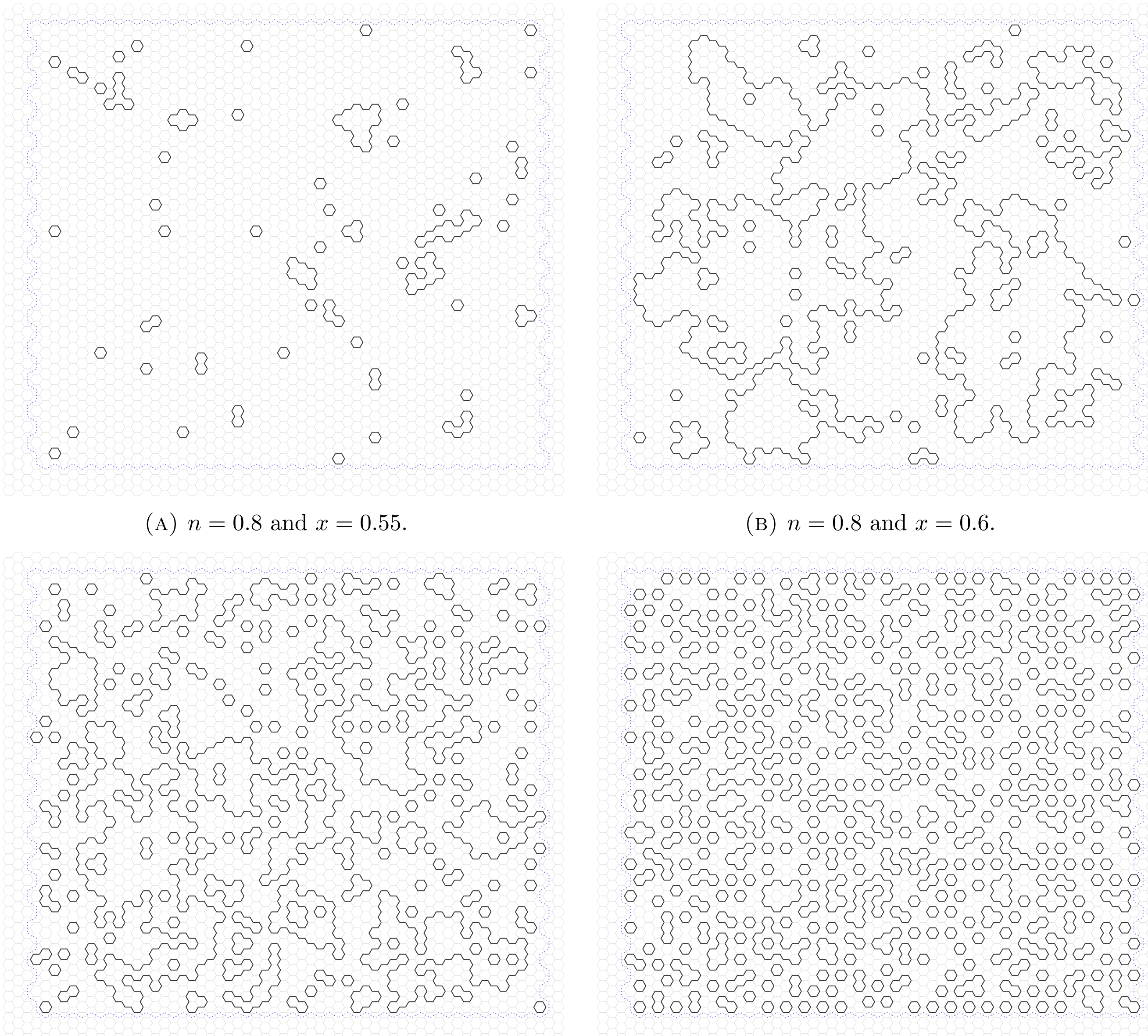
Small n**.** Studying the loop $O(n)$ model for small values of n is of great interest. It is predicted that the model displays critical behavior only when $n \leq 2$. There, it is expected to undergo a Kosterlitz–Thouless phase transition at $x_c = 1/\sqrt{2+\sqrt{2-n}}$, see [22], and exhibit conformal invariance when $x \geq x_c$. Mathematical results on this are currently restricted to the cases $n = 1$ and $n = 0$, which correspond to the *Ising model* and the *self-avoiding walk*, respectively. For these two cases, the critical values have been identified rigorously in [16] and [10], respectively. In the $n = 1$ case, the model has been proved [3, 4] to be conformally invariant at $x_c = 1/\sqrt{3}$. For $n = 1$ and $x = \infty$ the height function of the model may be viewed as a uniformly chosen lozenge tiling of a domain in the plane. This viewpoint leads to a determinantal process, the *dimer model*, which has been analyzed in great detail (see, e.g., [14] for an introduction). Conformal invariance has also been proved for the *double dimer model* which is closely related to the case $n = 2$ and $x = \infty$ (see [15]).

Our results are limited to the case $n \geq n_0$ and understanding the various behaviors for small values of n remains a beautiful mathematical challenge. To give a taste of the different possibilities, we provide some simulation results in Figure 9.

The Gibbs measures. Our results shed light on the Gibbs measures of the loop $O(n)$ model when $n \geq n_0$ and either $nx^6 \leq c$ or $nx^6 \geq C$. The structure for $n \geq n_0$ and $c \leq nx^6 \leq C$ remains unclear; see Figure 9d and Figure 2. Is there a single $x_c(n)$ at which the model transitions from the dilute, disordered phase to the dense, ordered phase? What happens when $x = x_c(n)$?

When $n \geq n_0$ and $nx^6 \geq C$ we prove that the model has at least three different Gibbs measures, distinguished by a choice of a sublattice of the triangular lattice. Are these the only extremal Gibbs measures in this regime (i.e., is every other measure a convex combination of these three measures)? This would be in the spirit of the Aizenman–Higuchi theorem [1, 13] which proves that the only extremal Gibbs measures for the 2D Ising model are the two pure states. This theorem was recently extended to the q-state Potts model in [5].

For small values of $\max\{n, 1\}x^6$ we prove the existence of a limiting Gibbs measure when exhausting space via an increasing sequence of domains with vacant boundary conditions. Is this Gibbs measure unique for each choice of n and x in this regime? Intuitively, the difficulty in proving this lies in dealing with domains with boundary conditions which force an interface (i.e., part of a loop) through the domain (similarly to the situation in Figure 3b). If this interface passes



(A) $n = 0.8$ and $x = 0.55$.

(B) $n = 0.8$ and $x = 0.6$.

(C) $n = 2$ and $x = 1/\sqrt{2} \approx 0.707$.

(D) $n = 8$ and $x = 1$.

FIGURE 9. A few samples of random loop configurations. Configurations are on a 60×45 domain of type 0 and are sampled via Glauber dynamics for 100 million iterations started from the empty configuration. The conjectured phase transition point for $n = 0.8$ is $x_c = 1/\sqrt{2 + \sqrt{2 - 0.8}} \approx 0.568$ and for $n = 2$ is $x_c = 1/\sqrt{2} \approx 0.707$. Theorem 1.4 shows that long loops are exponentially unlikely for large n.

near the origin with non-negligible probability, one would obtain a limiting Gibbs measure having an infinite path with positive probability. However, one expects interfaces to follow diffusive scaling, similarly to random walk paths, and as such should have negligible probability to pass close to the origin when the domain is large. Making such an intuition rigorous is quite non-trivial and was recently carried out successfully in [5] for planar Potts models. Adapting the ideas in [5] to the loop $O(n)$ model poses quite a challenge as these rely on specific properties of the Potts model. Roughly, the strategy in [5] proceeds by showing that when starting from a large domain H

with arbitrary boundary conditions, only a uniformly bounded number of interfaces will reach the boundary of a smaller subdomain H'. Then it is shown that these bounded number of interfaces follow diffusive scaling as in the intuition above. The first part, bounding the number of interfaces between the boundary of H and H', may possibly be carried out for the loop $O(n)$ model by using Lemma 1.9; configurations with many long interfaces may be 'rewired', erasing most of these interfaces and replacing them with short connections along the boundary of H, yielding configurations with much higher probability. The second part, however, showing the diffusive scaling, remains a major obstacle.

Appendix A. Integrals

In this section, we present a detailed derivation of the formulas approximating the partition function and the spin-spin correlations in the spin $O(n)$ model on a finite subgraph H of the hexagonal lattice. Let $u, v \in V(H)$ be distinct vertices and let H^+ be the (possibly multi-)graph obtained by adding an edge $e_{u,v}$ between u and v to H. In the introductory section, the derivation was reduced to computing integrals of the form

$$I(\omega) := \int_\Omega \prod_{\{w,w'\}\in E(\omega)} \langle \sigma_w, \sigma_{w'} \rangle \, d\sigma,$$

where $\Omega = (\sqrt{n} \cdot \mathbb{S}^{n-1})^{V(H)}$, ω is an arbitrary subgraph of H^+, and $d\sigma$ is the product of $|V(H)|$ uniform probability measures on $\sqrt{n} \cdot \mathbb{S}^{n-1}$. Note first that, by symmetry, making the substitution $\sigma_w \leftarrow -\sigma_w$ for some $w \in V(H)$ does not change the value of this integral and consequently $I(\omega) = 0$ unless every vertex has even degree in ω. In other words, if $\omega \subset H$ then $I(\omega) = 0$ unless ω is a loop configuration, i.e., $\omega \in \mathsf{LoopConf}(H, \emptyset)$, and $I(\omega + e_{u,v}) = 0$ unless the degrees of u and v in ω are odd and the degrees of all other vertices are even, i.e., $\omega \in \mathsf{LoopConf}(H, \emptyset, u, v)$.

We shall repeatedly make use of the following identity. For every $x, y \in \mathbb{R}^n$,

$$\int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle x, z \rangle \langle z, y \rangle \, dz = \langle x, y \rangle, \tag{22}$$

where dz is the uniform probability measure on $\sqrt{n} \cdot \mathbb{S}^{n-1}$. Note that both sides of (22) are bilinear functions of x and y and therefore it is enough to verify that (22) holds when x and y are two vectors from the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. By symmetry, for each i,

$$\int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle e_i, z \rangle \langle z, e_i \rangle \, dz = \frac{1}{n} \sum_{i=1}^{n} \int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle z, e_i \rangle^2 \, dz = \frac{1}{n} \int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \|z\|^2 dz = 1,$$

If $i \neq j$, substituting $(z_1, \dots, z_n) \leftarrow (z_1, \dots, z_{i-1}, -z_i, z_{i+1}, \dots, z_n)$ yields

$$\int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle e_i, z \rangle \langle z, e_j \rangle \, dz = -\int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle e_i, z \rangle \langle z, e_j \rangle \, dz = 0.$$

Suppose first that $\omega \in \mathsf{LoopConf}(H, \emptyset)$. Since the loops of ω are vertex-disjoint, $I(\omega) = \prod_{L\subset\omega} I(L)$, where L ranges over all loops of ω. Suppose now that L is a loop through vertices $v_0, \dots, v_\ell$, where $v_\ell = v_0$. Invoking (22) repeatedly yields

$$I(L) = \int_\Omega \langle \sigma_{v_0}, \sigma_{v_1} \rangle \cdots \langle \sigma_{v_{\ell}-1}, \sigma_{v_\ell} \rangle \, d\sigma = \int_\Omega \langle \sigma_{v_0}, \sigma_{v_0} \rangle \, d\sigma = n,$$

giving $I(\omega) = n^{L_H(\omega)}$.

Suppose now that $\omega \in \mathsf{LoopConf}(H, \emptyset, u, v)$, let C be the connected component of u (and v) in ω, and note that C must contain a simple path P connecting u and v. Since we have already proved that $I(L) = n$ for every loop L, in order to compute $I(\omega + e_{u,v})$, it is enough to compute $I(C + e_{u,v})$. A simple case analysis shows that C is either (i) the path P, (ii) the path P and a loop intersecting P in one of its endpoints, (iii) the path P and two vertex-disjoint loops, each intersecting P in one of its endpoints, or (iv) the path P and two other simple paths connecting u and v, each pair of paths sharing only the vertices u and v. Since the edge $e_{u,v}$ closes P into a loop, invoking (22) repeatedly to 'contract' loops yields that $I(C + e_{u,v})$ equals n in case (i), n^2 in case (ii), and n^3 in case (iii). In case (iv), since C is not a collection of edge-disjoint loops, invoking (22) repeatedly only gives

$$I(C + e_{u,v}) = \iint_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle x, y\rangle^4 \, dxdy,$$

which is somewhat more difficult to compute. Using symmetry and the fact that the projection of the Lebesgue measure on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ onto the first coordinate gives the measure on $[-1, 1]$ with density $(1 - t^2)^{\frac{n-3}{2}}$ up to a normalization constant, we obtain

$$\begin{aligned} I(C + e_{u,v}) &= \int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle x, \sqrt{n}e_1\rangle^4 \, dx = n^4 \int_{\sqrt{n}\cdot\mathbb{S}^{n-1}} \langle x/\sqrt{n}, e_1\rangle^4 \, dx \\ &= n^4 \cdot \frac{\int_{-1}^{1} t^4 (1 - t^2)^{\frac{n-3}{2}} \, dt}{\int_{-1}^{1} (1 - t^2)^{\frac{n-3}{2}} \, dt} = \frac{3n^3}{n + 2}, \end{aligned}$$

where one may obtain the final identity using integration by parts.

Appendix B. Circuits and domains

Here we prove some facts about circuits and domains.

Proof of Fact 2.1. Let γ be a circuit and denote by $\mathbb{H}_\gamma$ the subgraph of $\mathbb{H}$ obtained by removing from $\mathbb{H}$ all edges in γ^*. Let $\mathrm{Ext}(\gamma)$ be the set of vertices that are the endpoint of some infinite simple path in $\mathbb{H}_\gamma$.

First, we claim that $\mathrm{Ext}(\gamma)$ is a connected component of $\mathbb{H}_\gamma$. To see this, note first that by definition, $\mathrm{Ext}(\gamma)$ is a union of connected components of $\mathbb{H}_\gamma$. Furthermore, since γ^* is finite, there exists an R and a vertex $u \in V(\mathbb{H})$ such that the complement of the ball of radius R (in the graph distance determined by $\mathbb{H}$) centered at u induces the same connected graph $\mathbb{H}_R$ in both $\mathbb{H}$ and $\mathbb{H}_\gamma$. Finally, every infinite simple path in $\mathbb{H}$ intersects $\mathbb{H}_R$ and therefore $\mathrm{Ext}(\gamma)$ consists of a single connected component.

Second, we claim that the set of endpoints of the edges in γ^* intersects at most two connected components of $\mathbb{H}_\gamma$, one of which is $\mathrm{Ext}(\gamma)$. To see this, suppose that $\gamma = (\gamma_0, \dots, \gamma_m)$ as in the definition in Section 2.1. In order to prove the first part of our claim, it suffices to show that for each $i \in \{1, \dots, m - 1\}$, there are two connected sets of vertices each of which intersects both $\{\gamma_{i-1}, \gamma_i\}^*$ and $\{\gamma_i, \gamma_{i+1}\}^*$. To see this, note that $\{\gamma_{i-1}, \gamma_i\}^*$ and $\{\gamma_i, \gamma_{i+1}\}^*$ are the only two out of six edges surrounding the hexagon γ_i that belong to γ^*. Consequently, the removal of γ^* partitions the six vertices surrounding γ_i into two connected sets, each of which intersects both $\{\gamma_{i-1}, \gamma_i\}^*$ and $\{\gamma_i, \gamma_{i+1}\}^*$. For the second part of the claim, consider an arbitrary infinite simple path in $\mathbb{H}$ which uses an edge from γ^*. Let $\{v, w\}$ be the last edge of γ^* on this path and observe that either v or w belongs to $\mathrm{Ext}(\gamma)$. Therefore, $\mathrm{Ext}(\gamma)$ is one of the two connected components that contains an endpoint of an edge of γ^*.

Third, we claim that $\mathrm{Ext}(\gamma) \neq V(\mathbb{H})$. If this were not the case, then in particular there would be a $\{v, w\} \in \gamma^*$ such that both v and w belong to the same connected component of $\mathbb{H}_\gamma$. Consequently, there would be a simple path P in $\mathbb{H}_\gamma$ that connects v and w. The edge $\{v, w\}$ and P would then form a cycle in $\mathbb{H}$ that contains exactly one edge of γ^*. This is impossible since the basic 6-cycles surrounding the hexagons of $\mathbb{T}$ generate the cycle space of $\mathbb{H}$ and each of these basic cycles intersects γ^* in either 0 or 2 edges.

Fourth, we claim that $V(\mathbb{H}) \setminus \mathrm{Ext}(\gamma)$ is connected, that is, every two $v, w \notin \mathrm{Ext}(\gamma)$ are in the same connected component of $\mathbb{H}_\gamma$. To see this, consider two infinite simple paths P_v and P_w in $\mathbb{H}$ that start at v and w, respectively. Since $v, w \notin \mathrm{Ext}(\gamma)$, both P_v and P_w contain an edge from γ^*.

Let $\{v_1, v_2\} \in \gamma^*$ and $\{w_1, w_2\} \in \gamma^*$ be the first such edges on P_v and P_w, respectively. Clearly v, v_1 and w, w_1 lie in the same connected components, other than $\mathrm{Ext}(\gamma)$. By our second claim, v_1 and w_1 must belong to the same connected component. Hence, v and w also belong to the same connected component, which we shall from now on denote by $\mathrm{Int}(\gamma)$.

Finally, we show that both $\mathrm{Ext}(\gamma)$ and $\mathrm{Int}(\gamma)$ are induced subgraphs of $\mathbb{H}$ and that $\mathrm{Int}(\gamma)$ is finite. The first assertion follows from the fact that the two endpoints of each edge of γ^* belong to different connected components, which we have already established above. If the second assertion were false, then $\mathrm{Int}(\gamma)$ would be an infinite connected graph and hence it would contain an infinite path, contradicting the fact that $\mathrm{Int}(\gamma) \cap \mathrm{Ext}(\gamma) = \emptyset$. □

Proof of Fact 2.2. Let H be a domain. Let A denote the vertex set of H and let E be the set of edges of $\mathbb{H}$ with exactly one endpoint in A. It suffices to show that $E = \gamma^*$ for some circuit γ.

We first claim that for each hexagon $x \in \mathbb{T}$, either zero or two out of the six edges surrounding x belong to E. To see this, note first that exactly one of the two endpoints of each edge of E belongs to A and hence the number of edges surrounding x that are in E is even. If it were more than two, there would be four vertices v_1, v_2, v_3, v_4 bordering x such that (v_1, v_2, v_3, v_4) is their clockwise ordering with respect to x, $v_1, v_3 \in A$ and $v_2, v_4 \notin A$. As both A and $V(\mathbb{H}) \setminus A$ are connected, there is a path in A from v_1 to v_3 and a path in $V(\mathbb{H}) \setminus A$ from v_2 to v_4. These two paths are clearly vertex disjoint, but as $\mathbb{H}$ is planar, they must intersect, a contradiction.

Let T be the auxiliary graph with vertex set $\mathbb{T}$ whose edges are all pairs $\{x, y\}$ such that $\{x, y\}^* \in E$. It follows from the above claim that the degree of every vertex of T is either 0 or 2. In particular, every nontrivial connected component of T is a circuit. Let γ be one of these circuits. By Fact 2.1, γ^* splits $\mathbb{H}$ into exactly two connected components. As $\gamma^* \subseteq E$ and $\mathrm{Int}(\gamma)$ is finite and non-empty, it must be that $V(\mathbb{H}) \setminus A \subseteq \mathrm{Ext}(\gamma)$ and $A \subseteq \mathrm{Int}(\gamma)$. Consequently, $A = \mathrm{Int}(\gamma)$. □

Proof of Fact 2.3. Denote $A := \mathrm{Int}^{\mathrm{V}}(\sigma)$, $A' := \mathrm{Int}^{\mathrm{V}}(\sigma')$ and $B := A \cup A'$. Let us first show that B is connected. If $A \cap A' \neq \emptyset$ then this is immediate. Otherwise, by assumption, there exists an edge $\{v, u\} \in \sigma^* \cap (\sigma')^*$. Assume without loss of generality that $v \in A$ and $u \notin A$. Then $u \in A'$ and $v \notin A'$, and thus, B is connected.

Let C be the unique infinite connected component of $V(\mathbb{H}) \setminus B$ and let $H := V(\mathbb{H}) \setminus C$. It is straightforward to check that H is finite, $B \subset H$ and $\partial H \subset \partial B$. Since B is connected, this implies that H is connected. Thus, as $V(\mathbb{H}) \setminus H = C$ is connected, the subgraph of $\mathbb{H}$ induced by H is a domain.

By Fact 2.2, there exists a circuit γ such that $H = \mathrm{Int}^{\mathrm{V}}(\gamma)$. It remains to check that $\gamma^* \subset \sigma^* \cup (\sigma')^*$ (the fact that $\gamma \subset \sigma \cup \sigma'$ follows from this). Let $\{v, u\} \in \gamma^*$ be such that $v \in H$ and $u \notin H$. In particular, $v \in B$ and $u \notin B$. Thus, either $v \in A$ so that $\{v, u\} \in \sigma^*$, or $v \in A'$ so that $\{v, u\} \in (\sigma')^*$. □

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