Stability results for random discrete structures

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Abstract

Two years ago, Conlon and Gowers, and Schacht proved general theorems that allow one to transfer a large class of extremal combinatorial results from the deterministic to the probabilistic setting. Even though the two papers solve the same set of long-standing open problems in probabilistic combinatorics, the methods used in them vary significantly and therefore yield results that are not comparable in certain aspects. In particular, the theorem of Schacht yields stronger probability estimates, whereas the one of Conlon and Gowers also implies random versions of some structural statements such as the famous stability theorem of Erdős and Simonovits. In this paper, we bridge the gap between these two transference theorems. Building on the approach of Schacht, we prove a general theorem that allows one to transfer deterministic stability results to the probabilistic setting. We then use this theorem to derive several new results, among them a random version of the Erdős-Simonovits stability theorem for arbitrary graphs, extending the result of Conlon and Gowers, who proved such statement for so-called strictly 2-balanced graphs. The main new idea, a refined approach to multiple exposure when considering subsets of binomial random sets, may be of independent interest.

1   Introduction

One of the most active areas of research within combinatorics has always been the study of various *extremal problems*. In the most classical sense, extremal results in combinatorics give answers to questions of the following general form: For a finite set X, what is the largest subset of X that does not contain subsets of a particular type? Two archetypal examples of such results are the famous theorem of Turán [30], which determines the maximum number of edges in an n-vertex graph that does not contain a complete subgraph on k vertices, and the celebrated theorem of Szemerédi [28], which proves that for every positive δ, every subset A of $\{1, \ldots, n\}$ that satisfies $|A| \geq \delta n$ contains a k-term arithmetic progression, provided that n is sufficiently large (as a function of k and δ).

Extremal results are often accompanied by their structural refinements. Among them, the most notable are various *stability results*, which have the following general form: Suppose that a subset $Y \subseteq X$ does not contain subsets of a particular type and, moreover, the number of elements in Y is maximum possible (or close to maximum possible) among all subsets of X with this property (of not containing subsets of some type). Then Y is very structured. Here, Turán's theorem can again serve as an example as it not only determines the maximum number of edges in an n-vertex graph with no k-vertex complete subgraph, but also shows that (up to isomorphism) the only

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K_k-free n-vertex graph with this many edges is the complete $(k-1)$-partite graph with partite sets of equal or nearly equal size, denoted by $T_{k-1}(n)$ and referred to as the Turán graph. The stability statement of the type we will be considering was only proved much later by Erdős and Simonovits [27]. It states that in fact every K_k-free n-vertex graph whose number of edges is 'close' to the number of edges in $T_{k-1}(n)$ must be very 'close' to the graph $T_{k-1}(n)$, see Theorem 1.2.

A dominant trend in probabilistic combinatorics in the past two decades has been the formulation and study of various 'sparse random' analogues of classical extremal problems such as the aforementioned theorems of Turán and Szemerédi. Usually, these problems are studied in the *binomial random model*. For a finite set X and a real number $p \in [0,1]$, we denote by X_p the *p-random subset of* X, that is, the random subset of X such that each element of X belongs to X_p with probability p, independently of all other elements. A sparse random analogue of the theorem of Szemerédi is the assertion that with probability close to 1, every subset A of $\{1, \dots, n\}_p$ that satisfies $|A| \geq \delta np$ contains a k-term arithmetic progression, provided that p is sufficiently large as a function of n, k, and δ; note that np is the expected size of the random set $\{1, \dots, n\}_p$.

Various problems of this type, in particular the sparse random version of Szemerédi's theorem (Theorem 1.1), have attracted a tremendous amount of attention from many leading researchers. The main goal has been to find the smallest sequences of probabilities (p_n) such that the statements as "every subset A of $\{1, \dots, n\}_{p_n}$ with $|A| \geq \delta np_n$ contains a k-term arithmetic progression" hold *asymptotically almost surely* (a.a.s. for short), that is, with probability tending to 1 as n, the size of the considered structure, tends to infinity. There have been many results in various special cases, but the most important general questions, most notably the random version of Turán's theorem known as the Haxell-Kohayakawa-Łuczak conjecture [16] (or the Kohayakawa-Łuczak-Rödl conjecture [20]) had remained open until very recently, when all those efforts culminated in two breakthrough results of Conlon and Gowers [4] and Schacht [26], which provided a very general and powerful framework to handle problems of this type. The random versions of Szemerédi's and Turán's theorems followed as simple corollaries.

Following [4], let us say that a set A of integers is (δ, k)-*Szemerédi* if every subset of A of cardinality at least $\delta|A|$ contains a k-term arithmetic progression. Also, let us abbreviate $\{1, \dots, n\}$ by $[n]$. The methods of Conlon and Gowers, and Schacht imply that in the p-random subset of $[n]$, the property of being (δ, k)-Szemerédi has a threshold at $n^{-1/(k-1)}$.

Theorem 1.1 ([4, 26])**.** *For every positive δ and every integer k with $k \geq 3$, there exist positive constants c and C such that*

$$\lim_{n \to \infty} P([n]_{p_n} \text{ is } (\delta, k)\text{-Szemerédi}) = \begin{cases} 1, & \text{if } p_n \geq Cn^{-\frac{1}{k-1}}, \\ 0, & \text{if } p_n \leq cn^{-\frac{1}{k-1}}. \end{cases}$$

Given two graphs G and H, let $\mathrm{ex}(G, H)$ denote the maximum number of edges in a subgraph of G that is H-*free*, that is, does not contain H as a subgraph, i.e.,

$$\mathrm{ex}(G, H) = \max\{e(G') \colon G' \subseteq G \text{ and } G' \not\supseteq H\}.$$

The aforementioned theorem of Turán determines $\mathrm{ex}(K_n, K_k)$ for all k and n. It was later generalised by Erdős and Stone [8], and Erdős and Simonovits [6], who proved that for an arbitrary graph H with at least one edge,

$$\mathrm{ex}(K_n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \binom{n}{2}, \tag{1}$$

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where $\chi(H)$ is the chromatic number of H. On the other hand, it is easy to see that for every graph G,

$$\text{ex}(G, H) \geq \left(1 - \frac{1}{\chi(H) - 1}\right) e(G). \tag{2}$$

Erdős and Simonovits [27] proved the following structural refinement of (1), known since under the name of the Erdős-Simonovits stability theorem:

Theorem 1.2. *For every positive δ and every graph H with at least one edge, there exists a positive ε such that every n-vertex H-free graph with at least $\text{ex}(K_n, H) - \varepsilon n^2$ edges can be made $(\chi(H) - 1)$-partite by removing from it at most δn^2 edges.*

Let $G(n, p)$ denote the binomial random graph on the vertex set $[n]$ with edge probability p and note that in our notation, $G(n, p) = E(K_n)_p$. A notion that is intrinsic to the study of subgraphs of random graphs is that of 2-*density*. Let H be a graph with at least 3 vertices. We define the 2-density of H, denoted by $m_2(H)$, by

$$m_2(H) = \max\left\{\frac{e(K) - 1}{v(K) - 2} : K \subseteq H \text{ with } v(K) \geq 3\right\},$$

where $v(K)$ and $e(K)$ denote the number of vertices and the number of edges of K, respectively. A fairly straightforward computation (see, for example, [26]) shows that for every H, if $p_n \ll n^{-1/m_2(H)}$, then a.a.s. the number of copies of some $H' \subseteq H$ in $G(n, p_n)$ is much smaller than $\binom{n}{2}p_n$, the expected number of edges in $G(n, p_n)$, and therefore $\text{ex}(G(n, p_n), H) = (1 + o(1))\binom{n}{2}p_n$, which is very far from (2). Haxell, Kohayakawa, and Łuczak [16] conjectured that once $p_n \geq C_H n^{-1/m_2(H)}$, then the trivial estimate (2) becomes essentially best possible and hence a natural generalisation of (1) holds in $G(n, p_n)$ asymptotically almost surely. Their conjecture was confirmed by Conlon and Gowers, and Schacht.

Theorem 1.3 ([4, 26]). *For every graph H with maximum degree at least two and every positive ε, there exist positive constants c and C such that*

$$\lim_{n \to \infty} P\left(\text{ex}(G(n, p_n), H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \varepsilon\right)\binom{n}{2}p_n\right) = \begin{cases} 1, & \text{if } p_n \geq C n^{-1/m_2(H)}, \\ 0, & \text{if } p_n \leq c n^{-1/m_2(H)}. \end{cases}$$

Theorem 1.3 showed a certain advantage of the approach of Schacht [26] over the methods of Conlon and Gowers [4], which allow one to prove the above statement only in the case when H is *strictly* 2-*balanced*[1]; a graph H is strictly 2-balanced if it has the largest 2-density among all of its subgraphs or, in other words, if every proper subgraph $H' \subsetneq H$ satisfies $m_2(H') < m_2(H)$. Moreover, Schacht's approach yielded an asymptotically best possible estimate on the rate of convergence in the above limit, showing that the 'error probability' is $\exp(-\Omega(n^2 p_n))$, whereas the other approach yields only an $n^{-\omega(1)}$ bound[2]. On the other hand, Conlon and Gowers were able to prove the following sparse random analogue of the Erdős-Simonovits stability theorem (Theorem 1.2), which did not follow from Schacht's general theorem.

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[1]Actually, the methods of [4] allow one to prove the 1-statement in Theorem 1.3 and Theorem 1.4 also in the case when H is an arbitrary graph, under the somewhat stronger assumption that $p_n \geq n^{-1/m_2(H)}(\log n)^C$ for some constant C.

[2]We remark that obtaining the correct estimates on the error probability is essential in Schacht's approach as it facilitates induction, cf. the proofs of [26, Lemma 3.4] and Lemma 3.6

Theorem 1.4 ([4]). *For every strictly 2-balanced graph H and every positive δ, there exist positive constants C and ε such that if $p_n \geq Cn^{-1/m_2(H)}$, then a.a.s. every H-free subgraph of $G(n, p_n)$ with at least $\left(1 - \frac{1}{\chi(H)-1} - \varepsilon\right)\binom{n}{2}p_n$ edges may be made $(\chi(H) - 1)$-partite by removing from it at most $\delta n^2 p_n$ edges.*

In view of these disparities, it is natural to ask whether some synthesis of the methods of [4] and [26] can bridge the gap between their results; that is, whether one can prove a transference theorem that is applicable in all the settings in which Schacht's result can be applied, gives exponentially decaying bounds on the 'error probability', and yet implies structural statements. In this paper, we give an affirmative answer to these questions. We show how the approach of Schacht can be adapted to yield structural results of the form of Theorem 1.4 in the cases where the methods of Conlon and Gowers are not applicable. We prove a version of the general transference theorem from [26] tailored for stability statements. As corollaries of this general theorem, we then derive several new results. In particular, we remove the assumption that H is strictly 2-balanced from the statement of Theorem 1.4, where we also improve the implicit probability estimate from $n^{-\omega(1)}$ to $\exp(-\Omega(n^2 p_n))$, which is asymptotically best possible. Finally, we remark that our approach removes the somewhat artificial condition $p_n \ll 1$ present in the general transference theorems of both Conlon and Gowers [4], and Schacht [26] (the case $p_n = \Omega(1)$ in Theorems 1.3 and 1.4 did not follow directly from the respective transference theorem and required additional arguments). We postpone the formulation of our main result, Theorem 3.4, to Section 3 and first discuss several of its most important corollaries.

1.1 New results

In this section, we give a brief overview of the applications of our main result, Theorem 3.4. The proofs of these statements are given in Section 5.

1.1.1 Graphs

Our first result generalizes and strengthens Theorem 1.4 by removing the assumption that H is strictly 2-balanced and improving the probability estimate implicit in the "asymptotically almost surely" statement. Theorem 1.5, conjectured by Kohayakawa, Łuczak, and Rödl [20], is an essentially best possible random analogue of the stability theorem of Erdős and Simonovits (Theorem 1.2).

Theorem 1.5. *For every graph H with maximum degree at least two and every positive δ, there exist positive constants C and ε such that if $p_n \geq Cn^{-1/m_2(H)}$, then with probability at least $1 - \exp(-\Omega(n^2 p_n))$, every H-free subgraph of $G(n, p_n)$ with at least $\left(1 - \frac{1}{\chi(H)-1} - \varepsilon\right)\binom{n}{2}p_n$ edges may be made $(\chi(H) - 1)$-partite by removing from it at most $\delta n^2 p_n$ edges.*

1.1.2 Hypergraphs

Given two ℓ-uniform hypergraphs G and H, as in the graph case ($\ell = 2$), we define $\text{ex}(G, H)$ to be the maximum number of edges in an H-free subhypergraph of G. Unlike the graph case, if $\ell \geq 3$, then even the asymptotic behaviour of the function $\text{ex}(K_n^\ell, H)$ is not known apart from some very specific hypergraphs H. Still, for an arbitrary H, it makes sense to define the *Turán density* of H,

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denoted $\pi(H)$, by

$$\pi(H) = \lim_{n \to \infty} \frac{\mathrm{ex}(K_n^{(\ell)}, H)}{\binom{n}{\ell}},$$

as a standard averaging argument shows that the above limit always exists and that $\pi(H) < 1$ for every H. Moreover, it is not very hard to see that for every hypergraph G,

$$\mathrm{ex}(G, H) \geq \pi(H)e(G). \tag{3}$$

Let $G^{(\ell)}(n, p)$ denote the binomial random ℓ-uniform hypergraph on the vertex set $[n]$ with edge probability p. Let H be an ℓ-uniform hypergraph with at least $\ell + 1$ vertices. As in the graph case, we define the ℓ-density of H, denoted by $m_\ell(H)$, by

$$m_\ell(H) = \max \left\{ \frac{e(K) - 1}{v(K) - \ell} : K \subseteq H \text{ with } v(K) \geq \ell + 1 \right\}.$$

As in the case $\ell = 2$, one can see that if $p_n \ll n^{-1/m_\ell(H)}$, then $\mathrm{ex}(G^\ell(n, p), H) = (1 + o(1))\binom{n}{\ell}p$, which is very far from (3). A natural generalisation of the conjecture of Haxell, Kohayakawa, and Łuczak [16] would state that once $p_n \geq C_H n^{-1/m_k(H)}$, then the trivial estimate (3) is essentially best possible. Such statement was proved by Conlon and Gowers, and Schacht.

Theorem 1.6 ([4, 26]). *For every ℓ-uniform hypergraph H with maximum degree at least two and every positive ε, there exist positive constants c and C such that*

$$\lim_{n \to \infty} P\left(\mathrm{ex}(G^\ell(n, p_n), H) \leq (\pi(H) + \varepsilon)\binom{n}{\ell}p_n \right) = \begin{cases} 1, & \text{if } p_n \geq Cn^{-1/m_\ell(H)}, \\ 0, & \text{if } p_n \leq cn^{-1/m_\ell(H)}. \end{cases}$$

Similarly as in Theorem 1.3, the methods of Conlon and Gowers allow one to prove the above statement only in the case when H is *strictly ℓ-balanced*[3], i.e., if it has the largest ℓ-density among all of its subhypergraphs or, in other words, if every proper subhypergraph $H' \subsetneq H$ satisfies $m_\ell(H') < m_\ell(H)$.

The techniques of Conlon and Gowers can also be used to transfer stability theorems for (strictly) ℓ-balanced ℓ-uniform hypergraphs into the sparse random setting. Unfortunately, unlike in the graph case, where we have the very general theorem of Erdős and Simonovits (Theorem 1.2), there is only a handful of stability results known for ℓ-uniform hypergraphs with $\ell \geq 3$. To the best of our knowledge, the only hypergraphs for which an 'Erdős-Simonovits-type' stability result is known are: the Fano plane (the 3-uniform hypergraph with 7 vertices and 7 edges defined by the points and lines of the finite projective plane of order 2), proved independently by Füredi and Simonovits [12] and Keevash and Sudakov [18]; the 3-book of 2 pages (the 3-uniform hypergraph on the vertex set $\{1, \ldots, 5\}$ with edge set $\{123, 124, 345\}$), proved by Keevash and Mubayi [17]; F^3 (the 3-uniform hypergraph on the vertex set $\{1, \ldots, 5\}$ with edge set $\{123, 124, 125, 345\}$), proved by Füredi, Pikhurko, and Simonovits [10]; the 4-book of 3 pages (the 4-uniform hypergraph on the vertex set $\{1, \ldots, 7\}$ with edge set $\{1234, 1235, 1236, 4567\}$), proved by Füredi, Pikhurko, and Simonovits [11]; F^4 (the 4-uniform hypergraph on the vertex set $\{1, \ldots, 7\}$ with edge set $\{1234, 1235, 1236, 1237, 4567\}$), proved by Füredi, Mubayi, and Pikhurko [9]; the generalised triangle (the 4-uniform hypergraph on the vertex set $\{1, \ldots, 7\}$ with edge set $\{1234, 1235, 4567\}$), proved

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[3]Or when H is an arbitrary ℓ-uniform hypergraph, under the somewhat stronger assumption that $p_n \geq n^{1-1/m_\ell(H)}(\log n)^C$ in the 1-statement.

by Pikhurko [23]; the hypergraphs $C_3^{(2k)}$ (the $2k$-uniform hypergraphs on the vertex set $\{1, \ldots, 3k\}$ with edge set $\{1 \ldots 2k, (k+1) \ldots 3k, 1 \ldots k(2k+1) \ldots 3k\}$), proved by Keevash and Sudakov [19]; and the expanded cliques (the k-uniform hypergraphs obtained from K_ℓ by adding to each edge $\ell - 2$ new vertices), proved by Pikhurko [22]. Among these hypergraphs, the Fano plane, F^3, F^4, $C_3^{(2k)}$, and the expanded cliques are strictly balanced and therefore, the best-possible sparse random analogues of the respective stability theorems follow from the methods of Conlon and Gowers. For example, the following statement is an immediate corollary of the results proved in [4] and [12, 18].

Theorem 1.7 ([4]). *For every positive δ, there exist positive constants C and ε such that if $p_n \geq Cn^{-2/3}$, then a.a.s. every subhypergraph of $G^{(3)}(n, p_n)$ with at least $\left(\frac{3}{4} - \varepsilon\right) \binom{n}{3} p_n$ edges that does not contain the Fano plane may be made bipartite by removing from it at most $\delta n^3 p_n$ edges.*

Our methods imply analogous statements for the other hypergraphs mentioned above. We remark that such statements can also be deduced from the arguments used in [4], under the somewhat stronger assumption that the edge probability p_n exceeds the respective threshold by a polylogarithmic factor. For example, the following theorem can be easily deduced from our main result and the results of [17].

Theorem 1.8. *For every positive δ, there exist positive constants C and ε such that if $p_n \geq Cn^{-1}$, then a.a.s. the following holds: For every subhypergraph of $G^{(3)}(n, p_n)$ with at least $\left(\frac{2}{9} - \varepsilon\right) \binom{n}{3} p_n$ edges that does not contain the 3-book of 2 pages, there exists a partition of $[n]$ into sets V_1, V_2, and V_3 such that all but at most $\delta n^3 p_n$ edges have one point in each V_i.*

We remark that, as in Theorem 1.5, the "asymptotically almost surely" in the statement of Theorem 1.8 can be replaced by "with probability at least $1 - \exp(-\Omega(n^3 p_n))$". Similarly, the sparse random version of a stability result for a k-uniform hypergraph obtained using our methods will hold with probability at least $1 - \exp(-\Omega(n^k p_n))$.

1.1.3 Sum-free sets

A *Schur triple* in an Abelian group G is any triple $(x, y, z) \in G^3$ that satisfies the equation $x + y = z$. A set $A \subseteq G$ is called *sum-free* if A^3 contains no Schur triples or, in other words, if $(A + A) \cap A = \emptyset$. Here is an important definition in the study of sum-free sets: We say that a finite Abelian group G is of type I if $|G|$ has a prime divisor q with $q \equiv 2 \pmod 3$ and it is of type I(q) if q is the smallest such prime. For a set $B \subseteq G$, let $\mu(B)$ be the density of the largest sum-free subset of B (so that this subset has $\mu(B)|B|$ elements). Diananda and Yap [5] showed that $\mu(G) = \frac{1}{3} + \frac{1}{3q}$ for all G of type I(q) and characterised all sum-free subsets of G with $\mu(G)|G|$ elements. The results of Conlon and Gowers, and Schacht yield the following statement.

Theorem 1.9 ([4, 26]). *Let q be a prime with $q \equiv 2 \pmod 3$ and let (G_n) be a sequence of type I(q) groups satisfying $|G_n| = n$. Then for every positive ε, there exist positive constants c and C such that*

$$\lim_{n \to \infty} P\left(\mu((G_n)_{p_n}) \leq \left(\frac{1}{3} + \frac{1}{3q} + \varepsilon\right) n p_n \right) = \begin{cases} 1, & \text{if } p_n \geq Cn^{-1/2}, \\ 0, & \text{if } p_n \leq cn^{-1/2}. \end{cases}$$

It was proved by Green and Ruzsa [15] that the property of being sum-free in a group of type I exhibits very strong stability.

Theorem 1.10 ([15]). *Let G be an Abelian group of type $I(q)$. If A is a sum-free subset of G and*

$$|A| \geq \left(\mu(G) - \frac{1}{3q^2 + 3q} \right) |G|,$$

then A is contained in some sum-free set A' of maximum size.

As a last application of our main result, we will give a much more transparent proof of the following sparse random analogue of Theorem 1.10, originally derived from the transference theorem of Conlon and Gowers [4] by Balogh, Morris, and Samotij [3], with an improved probability estimate.

Theorem 1.11 ([3, 4]). *Let q be a prime with $q \equiv 2 \pmod{3}$ and let (G_n) be a sequence of type $I(q)$ groups satisfying $|G_n| = n$. Then for every positive δ, there exist positive constants ε and C such that if $p_n \geq Cn^{-1/2}$, then with probability at least $1 - \exp(-\Omega(np_n))$, for every sum-free subset $A \subseteq (G_n)_{p_n}$ with at least $(\mu(G_n) - \varepsilon)np_n$ elements, there exists a sum-free set $A' \subseteq G_n$ of maximum size such that $|A \setminus A'| \leq \delta np_n$.*

1.2 Notation

Given a (hyper)graph H, we denote its vertex and edge sets by $V(H)$ and $E(H)$, and the cardinalities of these two sets by $v(H)$ and $e(H)$, respectively. As one often identifies the hypergraph H with its edge set $E(H)$, sometimes instead of $e(H)$ or $|E(H)|$, we will simply write $|H|$. For a set $U \subseteq V(H)$, we write $H[U]$ to denote the subhypergraph of H induced by U, i.e., the hypergraph on the vertex set U whose edges are all the edges of H that are fully contained in U. Given a vertex $v \in V(H)$, we let $\deg(v, U)$ denote the degree of v in $H[U]$, i.e., the number of edges of $H[U]$ that contain v. Finally, we will denote by $[n]$ the set $\{1, \ldots, n\}$ of the first n positive integers.

Since throughout the paper, we will deal with many sequences indexed by (subsets of) the natural numbers, in order to unclutter the notation and, hopefully, improve readability, we use the (somewhat informal) notational convention that the sequences are denoted by boldface letters, e.g., $\mathbf{p}$ stands for (p_n), that is, the sequence $p \colon \mathbb{N} \to [0,1]$ indexed by the set of natural numbers[4]. The only exception is that, due to typesetting limitations, the sequence $(\mathcal{B}_n)$ will be denoted by $\mathfrak{B}$.

1.3 Outline

The remainder of this paper is organised as follows. In Section 2, we state a few auxiliary results that will be used in our proofs. In Section 3, we state the main result of this paper, Theorem 3.4, which we then prove in Section 4. Finally, in Section 5, we use Theorem 3.4 to deduce Theorems 1.5, 1.8, and 1.11.

2 Preliminaries

2.1 Bounding large deviations

In the proof of Lemma 3.6, we will often use the following standard estimates for tail probabilities of the binomial distribution, see, e.g., [2, Appendix A].

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[4]Since in order to aid readability, in the proofs of our theorems we will often drop the subscript n, we need a way to distinguish between the nth element of the sequence, abbreviated by p, and the sequence p itself.

Lemma 2.1 (Chernoff's inequality)**.** *Let n be a positive integer, let $p \in [0,1]$, and let $X \sim \mathrm{Bin}(n,p)$. For every positive a,*

$$P(X < np - a) < \exp\left(-\frac{a^2}{2np}\right) \quad and \quad P(X > np + a) < \exp\left(-\frac{a^2}{2np} + \frac{a^3}{2(np)^2}\right)$$

In particular, if $a \leq np/2$, then

$$P(X > np + a) < \exp\left(-\frac{a^2}{4np}\right).$$

We will also need the following approximate concentration result for $(K, \mathbf{p})$-bounded hypergraphs. The definition of $(K, \mathbf{p})$-boundedness and $\deg_i$ are given in Section 3, in Definition 3.3 and in (4), respectively.

Lemma 2.2 ([24, 26])**.** *Let $\mathbf{p}$ be a sequence of probabilities, let K be a positive constant, and suppose that $\mathbf{H}$ is a sequence of k-uniform hypergraphs that is $(K, \mathbf{p})$-bounded and satisfies $|V(H_n)| \to \infty$ as $n \to \infty$. Then for every $i \in \{0, \ldots, k-1\}$ and every positive η, there exist positive b and N such that for every n with $n \geq N$ and every $q \in [0,1]$ with $q \geq p_n$, with probability at least $1 - \exp(-bq|V(H_n)|)$, there exists a set $X \subseteq V(H_n)_q$ with $|X| \leq \eta q|V(H_n)|$ such that*

$$\sum_{v \in V(H_n)} \deg_i^2\left(v, V(H_n)_q \setminus X\right) \leq 4^k k^2 K q^{2i} \frac{|H_n|^2}{|V(H_n)|}.$$

We remark that [24, 26] stated Lemma 2.2 with the assumption $i \in \{1, \ldots, k-1\}$; in the case $i = 0$, the assertion of the lemma holds trivially (one can take $X = \emptyset$) with probability 1.

2.2 Stability theorems and removal lemmas

The proof of Theorem 1.5 will rely on the following classical result known as the *graph removal lemma*, originally proved in the case $H = K_3$ by Ruzsa and Szemerédi [25].

Theorem 2.3. *For an arbitrary graph H and any positive constant δ, there exists a positive constant ε such that every graph on n vertices with at most $\varepsilon n^{v(H)}$ copies of H can be made H-free by removing from it at most δn^2 edges.*

The proof of Theorem 1.8 will rely on the aforementioned stability result for the book hypergraph and a version of Theorem 2.3 for hypergraphs.

Theorem 2.4 ([17])**.** *For every positive constant δ, there exists a positive constant ε such that the following holds: For every 3-uniform hypergraph with at least $\left(\frac{2}{9} - \varepsilon\right)\binom{n}{3}$ edges that does not contain the 3-book of 2 pages, there exists a partition of $[n]$ into sets V_1, V_2, and V_3 such that all but at most δn^3 edges have one point in each V_i.*

Theorem 2.5 ([13, 21, 29])**.** *For an arbitrary k-uniform hypergraph H and any positive constant δ, there exists a positive constant ε such that every k-uniform hypergraph on n vertices with at most $\varepsilon n^{v(H)}$ copies of H may be made H-free by removing from it at most δn^k edges.*

Finally, the proof of Theorem 1.11 will use the following corollary of Theorem 1.10 and the so-called *removal lemma for Abelian groups* proved by Green [14].

Corollary 2.6 ([3])**.** *Let q be a prime with $q \equiv 2 \pmod 3$, let G be a group of type $I(q)$, and let ε be a constant satisfying $0 < \varepsilon < 1/(9q^2 + 9q)$. Then every $A \subseteq G$ with $|A| \geq (\mu(G) - \varepsilon)|G|$ either contains at least $\varepsilon^3|G|^2/27$ Schur triples or satisfies $|A \setminus A'| \leq \varepsilon|G|$ for some sum-free set A' of maximum size.*

3 Main results

Following [26], we will phrase the main result in the language of sequences **H** of uniform hyper-graphs. In the setting of Theorem 1.5, we have $V(H_n) = E(K_n)$ and the edges of H_n are edge sets of copies of a fixed graph H in K_n. Similarly, in Theorem 1.8, H_n represents copies of the book hypergraph in the complete 3-uniform hypergraph on n vertices. In the setting of Theorem 1.11, the vertex set of H_n will be the set of elements of some Abelian group G_n of order n, whereas the edges of H_n will be triples $\{x, y, z\}$ satisfying $x + y = z$. Since we are heavily borrowing from the paper of Schacht [26], our presentation and notation closely follow [26].

In order to transfer an extremal result from the deterministic to the probabilistic setting, both the result of Conlon and Gowers [4] and the one of Schacht [26] require a more robust version of this extremal result. One needs to assume that every sufficiently dense substructure (e.g., sufficiently large subgraph of the complete graph) not only contains one copy of the forbidden configuration (e.g., a copy of a fixed graph H), but also that the number of copies of the forbidden configuration in this substructure is of the same order of magnitude as the total number of copies of this configuration in the full structure. Note that in many natural settings, such a property does hold (e.g., by the supersaturation theorem of Erdős and Simonovits [7]). The following definition makes this condition rigorous.

Definition 3.1 ([26])**.** Let **H** be a sequence of k-uniform hypergraphs and let α be a nonnegative real. We say that **H** is α-*dense* if for every positive δ, there exist positive ε and N such that for every n with $n \geq N$ and every $U \subseteq V(H_n)$ with $|U| \geq (\alpha + \delta)|V(H_n)|$, we have $|H_n[U]| \geq \varepsilon|H_n|$.

Similarly as in [4], in order to transfer a stability result from the deterministic to the probabilistic setting, we will need a robust version of this stability result. Here, we need to assume that every sufficiently dense substructure is either close to some special substructure (e.g., a $(\chi(H)-1)$-partite graph) or it contains many copies of the forbidden configuration. Again, note that in many natural settings, such a property does hold (e.g., as a consequence of the Erdős-Simonovits stability theorem, Theorem 1.2, and the removal lemma for graphs, Theorem 2.3; see the proof of Theorem 1.5 in Section 5). The following definition makes this condition rigorous.

Definition 3.2 ([1])**.** Let **H** be a sequence of k-uniform hypergraphs, let α be a positive real, and let $\mathfrak{B}$ be a sequence of sets with $\mathcal{B}_n \subseteq \mathcal{P}(V(H_n))$. We say that **H** is $(\alpha, \mathfrak{B})$-*stable* if for every positive δ, there exist positive ε and N such that for every n with $n \geq N$ and every $U \subseteq V(H_n)$ with $|U| \geq (\alpha - \varepsilon)|V(H_n)|$, we have either $|H_n[U]| \geq \varepsilon|H_n|$ or $|U \setminus B| \leq \delta|V(H_n)|$ for some $B \in \mathcal{B}_n$.

The second condition in Theorem 3.4, which one may view as a measure of uniformity of the distribution of copies of the forbidden configuration in the full structure, imposes a lower bound on the probability for which we can transfer our stability (extremal) result to the random setting. Before we state this condition (Definition 3.3), we need to introduce some notation. For a hypergraph H, a vertex $v \in V(H)$, and a set $U \subseteq V(H)$, let $\deg_i(v, U)$ denote the number of edges of H containing v and at least i vertices in $U \setminus \{v\}$. More precisely, let

$$\deg_i(v, U) = \left| \left\{ e \in H \colon v \in e \text{ and } \left| e \cap (U \setminus \{v\}) \right| \geq i \right\} \right|. \tag{4}$$

For $q \in [0, 1]$, we let $\mu_i(H, q)$ denote the expected value of the sum of squares of such degrees over all $v \in V(H)$ with U replaced by the q-random subset of $V(H)$, namely,

$$\mu_i(H, q) = \mathbb{E}\left[\sum_{v \in V} \deg_i^2(v, V_q) \right],$$

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where $V = V(H)$.

Definition 3.3 ([26]). Let $\mathbf{H}$ be a sequence of k-uniform hypergraphs, let $\mathbf{p}$ be a sequence of probabilities, and let K be a positive constant. We say that $\mathbf{H}$ is $(K, \mathbf{p})$-*bounded* if for every $i \in \{0, \dots, k-1\}$, there exists an N such that for every n with $n \geq N$ and every $q \in [0, 1]$ with $q \geq p_n$, we have

$$\mu_i(H_n, q) \leq Kq^{2i} \frac{|H_n|^2}{|V(H_n)|}.$$

Finally, we are ready to state our main result, a stability version of [26, Theorem 3.3].

Theorem 3.4. *Let $\mathbf{H}$ be a sequence of k-uniform hypergraphs, let α be a positive real, let $\mathfrak{B}$ be a sequence of sets with $\mathcal{B}_n \subseteq \mathcal{P}(V(H_n))$, and suppose that $\mathbf{H}$ is $(\alpha, \mathfrak{B})$-stable. Furthermore, let K be a positive real and let $\mathbf{p}$ be a sequence of probabilities such that $p_n^k |H_n| \to \infty$ as $n \to \infty$, $\mathbf{H}$ is $(K, \mathbf{p})$-bounded, and $|\mathcal{B}_n| = \exp(o(p_n |V(H_n)|))$. Then for every positive δ, there exist positive ξ, b, C, and N such that for every n with $n \geq N$ and every q satisfying $Cp_n \leq q \leq 1$, the following holds with probability at least $1 - \exp(-bq|V(H_n)|)$: Every subset $W \subseteq V(H_n)_q$ with $|W| \geq (\alpha - \xi)q|V(H_n)|$ that satisfies $|W \setminus B| \geq \delta q |V(H_n)|$ for every $B \in \mathcal{B}_n$ satisfies $|H[W]| \geq \xi q^k |H_n| > 0$.*

Remark 3.5. Note that unlike [26, Theorem 3.3], the statement of Theorem 3.4 no longer contains the somewhat artificial assumption that $q \leq 1/\omega_n$ for some sequence ω satisfying $\omega_n \to 0$ as $n \to \infty$. This is due to our refined treatment of multiple exposure in the proof of Lemma 3.6, see Section 4.3.1 and the discussion at the beginning of Section 4.

Similarly as in [26], Theorem 3.4 will be derived from a stronger statement, Lemma 3.6 below, which will be proved by induction. Before we state it, we need a few more definitions. For a k-uniform hypergraph H, sets W and U with $W \subseteq U \subseteq V(H)$, and an integer $i \in \{0, \dots, k\}$, we let $E_U^i(W)$ denote the edges of $H[U]$ that have at least i vertices in W, namely,

$$E_U^i(W) = \big\{ e \in H[U] : |e \cap W| \geq i \big\}.$$

Note that for every $U \subseteq V(H)$ and every $W \subseteq U$,

$$E_U^0(W) = H[U] \quad \text{and} \quad E_U^k(W) = H[W].$$

Lemma 3.6. *Let $\mathbf{H}$ be a sequence of k-uniform hypergraphs, let α be a positive real, let $\mathfrak{B}$ be a sequence of sets with $\mathcal{B}_n \subseteq \mathcal{P}(V(H_n))$, and suppose that $\mathbf{H}$ is $(\alpha, \mathfrak{B})$-stable. Furthermore, let K be a positive real and let $\mathbf{p}$ be a sequence of probabilities such that $p_n^k |H_n| \to \infty$ as $n \to \infty$, $|\mathcal{B}_n| = \exp\big(o(p_n |V(H_n)|)\big)$, and $\mathbf{H}$ is $(K, \mathbf{p})$-bounded. Then for every $i \in \{0, \dots, k\}$ and every positive δ, there exist positive ξ, b, C, and N such that for all $\beta, \gamma \in (0, 1]$ with $\beta\gamma \geq \alpha - \xi$, every n with $n \geq N$, and every q satisfying $Cp_n \leq q \leq 1$, the following holds:*
For every $U \subseteq V(H_n)$ with $|U| \geq \beta|V(H_n)|$, with probability at least $1 - \exp(-bq|V(H_n)|)$, the random set U_q has the following property: Every subset $W \subseteq U_q$ with $|W| \geq \gamma q|U|$ that satisfies $|W \setminus B| \geq \delta q |V(H_n)|$ for every $B \in \mathcal{B}_n$ satisfies $|E_U^i(W)| \geq \xi q^i |H_n|$.

4 Proof of Lemma 3.6

The proof of Lemma 3.6 follows very closely the proof of [26, Lemma 3.4]. For easier comparison, our notation mirrors (with few minor changes) the notation used in [26]. The proof goes by

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induction on i. Similarly as in [26], the base of the induction (Section 4.2), which can be viewed as a justification of our choice of the definition of $(\alpha, \mathfrak{B})$-stability (Definition 3.2), follows very easily. The proof of the induction step (Section 4.3) is much more involved. As in [26], we construct the elements of $E_U^{i+1}(W)$ in stages and hence we expose the random set U_q in several rounds, letting $U_q = U_{q_1} \cup \ldots \cup U_{q_R}$ for appropriately chosen R and $q_1, \ldots, q_R$. Here comes the main new obstacle. Unlike the extremal setting considered in [26], the most important property of the sets $W \subseteq U_q$ that we have to consider, i.e., $|W \setminus B| \geq \delta q |V(H_n)|$ for every $B \in \mathcal{B}_n$, no longer implies the corresponding property, $|(W \cap U_{q_s}) \setminus B| \geq \delta' q_s |V(H_n)|$ for every $B \in \mathcal{B}_n$, in the sets U_{q_s}. The solution to this problem (Section 4.3.1), which is the main novelty in our approach, is analysing in more detail the relations between the probability space of the random sets U_q and the richer space of the sequences of random sets $U_{q_1}, \ldots, U_{q_R}$. Even though the crucial property of the set W mentioned above does not imply the analogous property relative to the sets U_{q_s} in every sequence $U_{q_1}, \ldots, U_{q_R}$, this does happen in a *typical* representation of U_q as $U_{q_1} \cup \ldots \cup U_{q_R}$, see Claim 1. This observation allows us to replace our setting of a single random set U_q to the setting of sequences of independent random sets $U_{q_1}, \ldots, U_{q_R}$, which, as already proved by [26], is much more convenient to work in. Moreover, the more rigorous treatment of the equivalence between these two settings allows us to remove the somewhat artificial assumption $q \leq 1/\omega_n$ that was necessary in the approach taken in [26]. The rest is as in the proof of [26, Lemma 3.4]. In each of the R rounds, we either construct sufficiently many elements of $E_U^{i+1}(W)$ or, appealing to the inductive assumption, we exhibit more than $\frac{1}{R}|V(H_n)|$ new 'rich' vertices in U that complete 'many' elements of $E_U^i(W)$ to elements of $E_U^{i+1}(W)$, see Claim 2. In subsequent rounds, including any of these 'rich' vertices in the set W will give rise to 'many' elements of $E_U^{i+1}(W)$. Since the latter (exhibiting many new 'rich' vertices) can happen at most $R-1$ times (U contains at most $|V(H_n)|$ vertices), Lemma 3.6 will follow.

4.1 Setup

Let $\mathbf{H}$ be a sequence of k-uniform hypergraphs, let $\mathfrak{B}$ be a sequence of sets with $\mathcal{B}_n \subseteq \mathcal{P}(V(H_n))$, let $\mathbf{p}$ be a sequence of probabilities, and let α and K be positive constants such that $\mathbf{H}$ is $(\alpha, \mathfrak{B})$-stable and $(K, \mathbf{p})$-bounded, $|\mathcal{B}_n| = \exp(o(p_n |V(H_n)|))$, and $p_n^k |H_n| \to \infty$ as $n \to \infty$. Note for future reference that since trivially $|H_n| \leq |V(H_n)|^k$, the last assumption implies that

$$p_n |V(H_n)| \to \infty \text{ as } n \to \infty. \tag{5}$$

Finally, let δ be a positive constant. We prove Lemma 3.6 by induction on i.

4.2 Induction base $(i = 0)$

The base of induction follows quite easily from the $(\alpha, \mathfrak{B})$-stability of $\mathbf{H}$. Let $\xi = \varepsilon_{3.2}(\delta/2)$ and assume that $n \geq N$, where N is sufficiently large; in particular, $N \geq N_{3.2}(\delta/2)$. Moreover, let $b = \delta/32$ and $C = 1$. For the sake of clarity of the presentation, let $H = H_n$, let $V = V(H_n)$, and let $\mathcal{B} = \mathcal{B}_n$. Let $\beta, \gamma \in (0, 1]$ satisfy $\beta\gamma \geq \alpha - \xi$, let q satisfy $q \geq Cp_n$, and fix some $U \subseteq V$ with $|U| \geq \beta|V|$. Since H is $(\alpha, \mathfrak{B})$-stable and $|U| \geq (\alpha - \xi)|V|$, if $|U \setminus B| > (\delta/2)|V|$ for every $B \in \mathcal{B}$, then

$$|E_U^0(W)| = |H[U]| \geq \xi|H|,$$

as claimed, so we may assume that $|U \setminus B| \leq (\delta/2)|V|$ for some $B \in \mathcal{B}$. Observe that for every $W \subseteq U_q$, one clearly has $|W \setminus B| \leq |U_q \setminus B|$. Hence, by Chernoff's inequality, with probability at least $1 - \exp(-bq|V|)$, the set U_q (vacuously) has the claimed property, as it satisfies $|U_q \setminus B| < \delta q |V|$.

4.3 Induction step $(i \to i+1)$

Let ξ', b', C', and N' be the constants whose existence is asserted by the inductive assumption with parameters i and $\delta/4$, i.e., let

$$\xi' = \xi_{3.6}(i, \delta/4), \quad b' = b_{3.6}(i, \delta/4), \quad C' = C_{3.6}(i, \delta/4), \quad \text{and} \quad N' = N_{3.6}(i, \delta/4).$$

We also let $\eta = \min\{\xi'/4, \delta/8\}$ and let $\hat{b} = b_{2.2}(\eta)$. Throughout the proof, we will assume that $n \geq N$, where N is sufficiently large; in particular, $N \geq \max\{N', N_{2.2}(\eta)\}$. Similarly as before, for the sake of clarity of the presentation, we let $H = H_n$, $V = V(H_n)$, $\mathcal{B} = \mathcal{B}_n$, and $p = p_n$. We first define some constants. We set

$$R = \left\lceil \frac{4^{k+1} k^2 K}{(\xi')^2} + 1 \right\rceil \tag{6}$$

and let

$$\xi = \frac{(\xi')^2}{8k(RL^R)^{i+1}}, \quad b = \min\left\{ \frac{(\xi')^2}{16^2}, \frac{b^*}{40RL^R} \right\}, \quad \text{and} \quad C = RL^R C',$$

where

$$b^* = \min\left\{ \frac{\beta(\xi')^2}{16^2}, \frac{b'}{4}, \frac{\hat{b}}{4} \right\} \quad \text{and} \quad L = \frac{3}{b^*}.$$

Finally, let $\beta, \gamma \in (0, 1]$ satisfy $\beta\gamma \geq \alpha - \xi$, let q satisfy $q \geq Cp_n$, and fix some $U \subseteq V$ with $|U| \geq \beta|V|$. Note that WLOG we may assume that $|U| = \beta|V|$ and that $\xi' \leq \delta/2$.

4.3.1 Multiple exposure trick

Let $\mathcal{S}$ denote the event that the random set U_q possesses the postulated stability property:

$\mathcal{S}$: Every subset $W \subseteq U_q$ with $|W| \geq \gamma q|U|$ that satisfies $|W \setminus B| \geq \delta q|V|$ for every $B \in \mathcal{B}$ satisfies $|E_U^{i+1}(W)| \geq \xi q^{i+1}|H|$.

In order to estimate the probability of $\mathcal{S}$, we will consider a richer probability space that is in a natural correspondence with the space $\mathcal{P}(U)$ of all subsets of U equipped with the obvious probability measure P, i.e., the distribution of the random variable U_q. To this end, let $q_1, \ldots, q_R \in [0, 1]$ be the unique sequence of numbers that satisfies

$$1 - q = \prod_{s=1}^{R} (1 - q_s) \quad \text{and} \quad q_{s+1} = Lq_s \text{ for every } s \in [R-1], \tag{7}$$

and observe that

$$\sum_{s=1}^{R} q_s \geq q \quad \text{and consequently} \quad q_s \geq q_1 \geq \frac{q}{RL^R} \text{ for every } s \in [R]. \tag{8}$$

The richer probability space will be the space $\mathcal{P}(U)^R$ equipped with the product measure P^* that is the distribution of the sequence $(U_{q_1}, \ldots, U_{q_R})$ of independent random variables, where for each s, the variable U_{q_s} is a q_s-random subset of U. Crucially, observe that due to our choice of $q_1, \ldots, q_s$, see (7), the natural mapping

$$\varphi \colon \mathcal{P}(U)^R \to \mathcal{P}(U) \quad \text{defined by} \quad \varphi(U_1, \ldots, U_R) = U_1 \cup \ldots \cup U_R$$

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is measure preserving, i.e., for every $U_0 \subseteq U$,

$$P(U_q = U_0) = P^*\big((U_{q_1}, \ldots, U_{q_R}) \in \varphi^{-1}(U_0)\big).$$

In other words, the variables U_q and $U_{q_1} \cup \ldots \cup U_{q_R}$ have the same distribution. Finally, let $\delta^* = \delta/2$, let $\gamma^* = \gamma - \xi'/4$, and consider the following event in the space $\mathcal{P}(U)^R$:

$\mathcal{S}^*$: For every $W_1 \subseteq U_{q_1}, \ldots, W_R \subseteq U_{q_R}$ such that $|W_s| \geq \gamma^* q_s |U|$ and $|W_s \setminus B| \geq \delta^* q_s |V|$ for every $s \in [R]$ and every $B \in \mathcal{B}$, we have $|E_U^{i+1}(W_1 \cup \ldots \cup W_R)| \geq \xi q^{i+1} |H|$.

There are two reasons why we consider the probability space $\mathcal{P}(U)^R$. The first reason is that the probability of $\mathcal{S}^*$ is much easier to estimate than the probability of $\mathcal{S}$. The second reason is that a lower bound on $P^*(\mathcal{S}^*)$ implies a (marginally weaker) lower bound on $P(\mathcal{S})$, which we show below.

Claim 1. $1 - P(\mathcal{S}) \leq 2 \cdot \big(1 - P^*(\mathcal{S}^*)\big).$

Proof. Note that in order to prove the claim, it suffices to show that

$$P^*\big(\mathcal{S}^* \mid (U_{q_1}, \ldots, U_{q_R}) \in \varphi^{-1}(\hat{U})\big) = P^*\big(\mathcal{S}^* \mid U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}\big) \leq 1/2 \qquad (9)$$

for every $\hat{U}$ that does not satisfy $\mathcal{S}$. Indeed, assuming that (9) holds for every $\hat{U}$ which does not satisfy $\mathcal{S}$, we have

$$
\begin{aligned}
1 - P(\mathcal{S}) = \sum_{\hat{U} \notin \mathcal{S}} P(U_q = \hat{U}) &= \sum_{\hat{U} \notin \mathcal{S}} P^*(U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}) \\
&\leq \sum_{\hat{U} \notin \mathcal{S}} 2\left(1 - P^*\big(\mathcal{S}^* \mid U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}\big)\right) \cdot P^*(U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}) \\
&\leq 2 \sum_{\hat{U}} \left(1 - P^*\big(\mathcal{S}^* \mid U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}\big)\right) P^*(U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}) = 2(1 - P^*(\mathcal{S}^*)).
\end{aligned}
$$

Consider an arbitrary $\hat{U} \subseteq U$ that does not satisfy $\mathcal{S}$. By the definition of $\mathcal{S}$, there exists a set $W \subseteq \hat{U}$ with $|W| \geq \gamma q |U|$ that satisfies $|W \setminus B| \geq \delta q |V|$ for every $B \in \mathcal{B}$ and $|E_U^{i+1}(W)| < \xi q^{i+1} |H|$. Consider the event $\varphi^{-1}(\hat{U})$, i.e., the event $U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}$. Now, for each $s \in [R]$, let $W_s = W \cap U_{q_s}$. Since clearly $E_U^{i+1}(W) = E_U^{i+1}(W_1 \cup \ldots \cup W_R)$, it suffices to show that with probability at least $1/2$, we have $|W_s| \geq \gamma^* q_s |U|$ and $|W_s \setminus B| \geq \delta^* q_s |V|$ for every $s \in [R]$ and $B \in \mathcal{B}$.

To this end, observe that conditioned on the event $U_{q_1} \cup \ldots \cup U_{q_R} = \hat{U}$, for each $s \in [R]$, the variable U_{q_s} has the same distribution as $\hat{U}_{q_s'}$, where $q_s' = q_s/q$ (although $U_{q_1}, \ldots, U_{q_R}$ are no longer independent). Recalling the definitions of γ^* and δ^*, it now follows from Chernoff's inequality that for fixed $s \in [R]$ and $B \in \mathcal{B}$, both the probability that $|W_s| < \gamma^* q_s |U| = \gamma^* q_s' q |U|$ and the probability that $|W_s \setminus B| < \delta^* q_s |V| = \delta^* q_s' q |V|$ are at most $\exp(-c q_s |V|)$, where c is some positive constant depending only on α, δ, and ξ'. Since $q_s \geq q/(RL^R) \geq p$ for every $s \in [R]$, see (8), $|\mathcal{B}_n| = \exp(o(p_n |V(H_n)|))$, and (5), then the claimed estimate follows from the union bound, provided that n is sufficiently large. $\qquad \square$

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4.3.2 Estimating the probability of $\mathcal{S}^*$

In the remainder of the proof, we will work in the space $\mathcal{P}(U)^R$ and estimate the probability of the event $\mathcal{S}^*$, that is, $P^*(\mathcal{S}^*)$. Let $U_{q_1}, \ldots, U_{q_R}$ be independent random subsets of U. Given $W_1 \subseteq U_{q_1}, \ldots, W_R \subseteq U_{q_R}$, let

$$W(s) = (W_1, \ldots, W_s) \quad \text{and} \quad U(s) = (U_{q_1}, \ldots, U_{q_s}).$$

We consider the set Z_s of 'rich' vertices that extend many sets in $E_U^i(W_s)$ defined by

$$Z_s = \left\{ u \in U \colon \deg_i\left(u, W_s, U\right) \geq \frac{\xi'}{2} q_s^i \frac{|H|}{|V|} \right\},$$

where

$$\deg_i\left(u, W_s, U\right) = \left| \left\{ e \in H \colon \left| e \cap (W_s \setminus \{u\}) \right| \geq i \text{ and } e \subseteq U \right\} \right|,$$

and let

$$Z(s) = Z_1 \cup \ldots \cup Z_s.$$

Now comes the key step in the proof. We show that with very high probability, for every $s \in [R]$, regardless of what happens in rounds $1, \ldots, s-1$, either $E_U^{i+1}(W_1 \cup \ldots \cup W_s)$ is large or the set $Z(s)$ of 'rich' elements grows by more than $|V|/R$.

Claim 2. *For every $s \in [R]$ and every choice of $W(s-1) \in \mathcal{P}(U)^{s-1}$, let $\mathcal{S}^*_{W(s-1)}$ denote the event that U_{q_s} has the following property: For every $W_s \subseteq U_{q_s}$ with $|W_s| \geq \gamma^* q_s |U|$ that satisfies $|W_s \setminus B| \geq \delta^* q_s |V|$ for every $B \in \mathcal{B}_n$, either*

$$|E_U^{i+1}(W_1 \cup \ldots \cup W_s)| \geq \xi q^{i+1} |H| \tag{10}$$

or

$$|Z(s) \setminus Z(s-1)| \geq \frac{(\xi')^2}{4^{k+1} k^2 K} |V|. \tag{11}$$

Then for every $\hat{U} \in \mathcal{P}(U)^{s-1}$,

$$P^*\left(\mathcal{S}^*_{W(s-1)} \mid U(s-1) = \hat{U}\right) \geq 1 - \exp(-2b^* q_s |V|),$$

where $P^*\left(\mathcal{S}^*_{W(0)} \mid U(0) = \hat{U}\right) = P^*\left(\mathcal{S}^*_{W(0)}\right)$.

4.3.3 Deducing Lemma 3.6 from Claims 1 and 2

For every $t \in [R]$, let $\mathcal{A}_t$ denote the event that $|U_{q_t}| \leq 2q_t|V|$ and let $\mathcal{A}(s) = \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_s$. Observe that by (7),

$$\sum_{t=1}^{s-1} 2q_t|V| \leq 3q_{s-1}|V| \leq \frac{3q_s|V|}{L}. \tag{12}$$

By Chernoff's inequality, (5), and (8),

$$P^*\left(\neg \mathcal{A}(s-1)\right) \leq \sum_{t=1}^{s-1} \exp\left(-\frac{q_t|V|}{16}\right) \leq \exp\left(-\frac{q_1|V|}{20}\right).$$

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Now, for every $s \in [R]$, let $\mathcal{S}_s^*$ denote the event that $\mathcal{A}(s-1)$ holds and $S_{W(s-1)}^*$ holds for all $W(s-1) \subseteq U(s-1)$[5]. Observe that if $\mathcal{S}_s^*$ holds for all $s \in [R]$, then $\mathcal{S}^*$ must hold since (11) in Claim 2 can occur at most $R-1$ times, see (6). Let

$$\hat{\mathcal{U}} = \left\{ \hat{U} \in \mathcal{P}(U)^{s-1} \colon |\hat{U}_t| \leq 2q_t|V| \text{ for all } t \in [s-1] \right\}$$

and note that

$$P^*\big(\neg \mathcal{S}_s^*\big) \leq P^*\big(\neg \mathcal{A}(s-1)\big) + P^*\big(\neg \mathcal{S}_s^* \wedge \mathcal{A}(s-1)\big) \tag{13}$$
$$= P^*\big(\neg \mathcal{A}(s-1)\big) + \sum_{\hat{U} \in \hat{\mathcal{U}}} P^*\big(\neg \mathcal{S}_s^* \wedge U(s-1) = \hat{U}\big).$$

Now, by Claim 2, for every $\hat{U} \in \hat{\mathcal{U}}$,

$$P^*\big(\neg \mathcal{S}_s^* \wedge U(s-1) = \hat{U}\big) = P^*\big(\neg \mathcal{S}_s^* \mid U(s-1) = \hat{U}\big) \cdot P^*\big(U(s-1) = \hat{U}\big) \tag{14}$$
$$\leq \sum_{W(s-1) \subseteq \hat{U}} P^*\big(\neg \mathcal{S}_{W(s-1)}^* \mid U(s-1) = \hat{U}\big) \cdot P^*\big(U(s-1) = \hat{U}\big)$$
$$\leq 2^{\sum_{t=1}^{s-1} 2q_t|V|} \cdot \exp(-2b^* q_s|V|) \cdot P^*\big(U(s-1) = \hat{U}\big).$$

Since clearly $\sum_{\hat{U} \in \hat{\mathcal{U}}} P^*(U(s-1) = \hat{U}) \leq 1$, it follows from (12), (13), and (14) that

$$P^*\big(\neg S^*\big) \leq \sum_{s=1}^{R} P^*\big(\neg S_s^*\big) \leq R \exp\left(-\frac{q_1|V|}{20}\right) + \sum_{s=1}^{R} 2^{\frac{3q_s|V|}{L}} \exp(-2b^* q_s|V|) \tag{15}$$
$$\leq R \exp\left(-\frac{q_1|V|}{20}\right) + R \exp(-b^* q_1|V|) \leq \frac{1}{2} \exp(-bq|V|).$$

Now, Lemma 3.6 easily follows from (15) and Claim 1.

4.3.4 Proof of Claim 2

Let $s \in [R]$, condition on the event $U(s-1) = \hat{U}$ for some $\hat{U} \in \mathcal{P}(U)^{s-1}$ and assume that $W(s-1)$ is given. Note that this uniquely defines $Z(s-1)$. Also, observe that it follows from the definition of $Z(s-1)$ and (8) that

$$|E_U^{i+1}(W_1 \cup \ldots \cup W_s)| \geq \frac{1}{k} \sum_{w \in W_s} \deg_i\big(w, W_1 \cup \ldots \cup W_{s-1}, U\big) \tag{16}$$
$$\geq \frac{1}{k} \cdot |W_s \cap Z(s-1)| \cdot \frac{\xi'}{2} q_1^i \frac{|H|}{|V|}$$
$$\geq \frac{|W_s \cap Z(s-1)|}{q_s} \cdot \frac{\xi'}{2} \frac{q^{i+1}}{k(RL^R)^{i+1}} \frac{|H|}{|V|} = \frac{|W_s \cap Z(s-1)|}{(\xi'/4)q_s|V|} \cdot \xi q^{i+1}|H|,$$

hence it will be enough if we show that

$$|W_s \cap Z(s-1)| \geq \frac{\xi'}{4} q_s|V|. \tag{17}$$

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[5]We write $W(s-1) \subseteq U(s-1)$ to denote the fact that $W_t \subseteq U_t$ for all $t \in [s-1]$

We consider two cases, depending on the cardinality of $Z(s-1)$.

Case 1. $|U \setminus Z(s-1)| < (\gamma^* - \xi'/2)|U|$.

By Chernoff's inequality, with probability at least $1 - \exp(-2b^* q_s |V|)$, the set U_{q_s} satisfies $|U_{q_s} \setminus Z(s-1)| \leq (\gamma^* - \xi'/4)q_s|U|$. Consequently, for every $W_s \subseteq U$ with $|W_s| \geq \gamma^* q_s |U|$, we have

$$|W_s \cap Z(s-1)| \geq |W_s| - |U_{q_s} \setminus Z(s-1)| \geq \frac{\xi'}{4} q_s |U|,$$

which, by (16), proves (10), see (17).

Case 2. $|U \setminus Z(s-1)| \geq (\gamma^* - \xi'/2)|U|$.

In this case, we will apply the inductive assumption (Lemma 3.6) to the set $U \setminus Z(s-1)$. First, observe that if $|W_s \cap Z(s-1)| \geq (\xi'/4)q_s|V|$, then this, by (16), proves (10), see (17). Hence, from now on we may assume that the inverse inequality holds, i.e., that

$$|W_s \cap Z(s-1)| < \frac{\xi'}{4} q_s |V|. \tag{18}$$

Let

$$U' = U \setminus Z(s-1), \quad \beta' = \frac{|U'|}{|V|}, \quad \text{and} \quad \gamma' = \left(\gamma^* - \frac{\xi'}{2}\right)\frac{|U|}{|U'|}.$$

Clearly, $\beta', \gamma' \in (0,1]$ and

$$\beta'\gamma' = \left(\gamma^* - \frac{\xi'}{2}\right) \cdot \frac{|U|}{|V|} = \left(\gamma^* - \frac{\xi'}{2}\right)\beta = \left(\gamma - \frac{3}{4}\xi'\right)\beta \geq \beta\gamma - \frac{3}{4}\xi' \geq \alpha - \xi - \frac{3}{4}\xi' \geq \alpha - \xi'.$$

Note that by (8) and our assumption on q and C, we have $q_s \geq \frac{q}{RL^R} \geq \frac{Cp}{RL^R} \geq C'p$ and hence by the inductive assumption (Lemma 3.6) applied to U', with probability at least $1 - \exp(-b'q_s|V|)$, every subset $W' \subseteq U'_{q_s}$ with $|W'| \geq \gamma'q_s|U'|$ such that $|W' \setminus B| \geq (\delta/4)q_s|V|$ for every $B \in \mathcal{B}$ satisfies $|E^i_{U'}(W')| \geq \xi'q_s^i|H|$. Moreover, it follows from Lemma 2.2 that with probability at least $1 - \exp(-\hat{b}q_s|V|)$, there exists a set $X \subseteq U'_{q_s}$ satisfying

$$|X| \leq \eta q_s |V| = \min\left\{\frac{\xi'}{4}, \frac{\delta^*}{4}\right\} q_s |V|. \tag{19}$$

and

$$\sum_{u \in U'} \deg_i^2\left(u, U'_{q_s} \setminus X\right) \leq 4^k k^2 K q_s^{2i} \frac{|H|^2}{|V|}. \tag{20}$$

Consider the set $W' \subseteq U'$ defined by $W' = W_s \setminus (X \cup Z(s-1))$. It follows from (18) and (19) that

$$|W'| \geq |W_s| - |W_s \cap Z(s-1)| - |X| \geq \left(\gamma^* - \frac{\xi'}{4} - \eta\right)q_s|V| \geq \gamma'q_s|U'|.$$

and that for every $B \in \mathcal{B}$,

$$|W' \setminus B| \geq |W_s \setminus B| - |W_s \cap Z(s-1)| - |X| \geq \left(\delta^* - \frac{\xi'}{4} - \eta\right)q_s|V| \geq \frac{\delta}{4}q_s|V|.$$

From the inductive assumption (which holds for U'_{q_s} with probability at least $1 - \exp(-b'q_s|V|)$), we infer that

$$\sum_{u \in U'} \deg_i\left(u, W', U'\right) \geq |E^i_{U'}(W')| \geq \xi'q_s^i|H|. \tag{21}$$

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Let

$$Z'_s = \left\{ u \in U' \colon \deg_i \left(u, W', U' \right) \geq \frac{\xi'}{2} q_s^i \frac{|H|}{|V|} \right\}$$

and note that, by definition, $Z'_s \subseteq Z_s$ and, by (21),

$$\sum_{u \in Z'_s} \deg_i \left(u, W', U' \right) \geq \sum_{u \in U'} \deg_i \left(u, W', U' \right) - |U' \setminus Z'_s| \cdot \frac{\xi'}{2} q_s^i \frac{|H|}{|V|} \geq \frac{\xi'}{2} q_s^i |H|. \tag{22}$$

It follows form (20), (22), and the Cauchy-Schwarz inequality that

$$4^k k^2 K q_s^{2i} \frac{|H|^2}{|V|} \geq \sum_{u \in U'} \deg_i^2(u, U') \geq \sum_{u \in Z'_s} \deg_i^2 \left(u, W', U' \right)$$

$$\geq \frac{1}{|Z'_s|} \left(\sum_{u \in Z'_s} \deg_i \left(u, W', U' \right) \right)^2 \geq \frac{1}{|Z'_s|} \left(\frac{\xi' q_s^i |H|}{2} \right)^2$$

and consequently,

$$|Z'_s| \geq \frac{(\xi')^2}{4^{k+1} k^2 K} |V|.$$

Since $Z'_s \subseteq U' = U \setminus Z(s-1)$, the sets Z'_s and $Z(s-1)$ are disjoint. Therefore, (11) holds with probability at least

$$1 - \exp(-b' q_s |V|) - \exp(-\hat{b} q |V|),$$

which, by (5), is at least $1 - \exp(-2b^* q_s |V|)$. This concludes the proof of Claim 2 and consequently, the proof of Lemma 3.6.

5   Proofs of the new results

In this section, we prove Theorems 1.5 and 1.11. The derivation of Theorem 1.8 from Theorems 2.4, 2.5, and 3.4 and the calculations done in [26] (proving that the appropriate sequence of hypergraphs is $(K, \mathbf{p})$-bounded) does not differ much from the proof of Theorem 1.5 given below and hence we shall leave it to the reader.

Proof of Theorem 1.5. Let H be a graph with at least one vertex contained in at least two edges. We want to apply Theorem 3.4. To this end, consider the sequence $\mathbf{H}$ of $e(H)$-uniform hypergraphs with $V(H_n) = E(K_n)$ and $E(H_n)$ consisting of edge sets of all copies of H in K_n. Moreover, let $p_n = n^{-1/m_2(H)}$ and let $\mathcal{B}_n$ be the family of edge sets of all complete $(\chi(H) - 1)$-partite graphs on the vertex set $[n]$. Observe that in order to complete the proof, it suffices to verify that the assumptions of Theorem 3.4 are satisfied. Since H contains a vertex with degree at least 2, we have that $m_2(H) \geq 1$ and hence

$$p_n^{e(H)} |H_n| \geq p_n^{e(H)} \binom{n}{v(H)} = \Omega\big(p_n^{e(H)} n^{v(H)}\big) = \Omega\big(p_n n^2\big) = \Omega(n).$$

Moreover, it was proved in [26] that the sequence $\mathbf{H}$ is $(K, \mathbf{p})$-bounded for some sufficiently large constant K. Finally, note that if $\chi(H) > 2$, then H contains an odd cycle (of length at most $v(H)$) and hence $m_2(H) \geq 1 + 1/(v(H) - 2) > 1$. It follows that regardless of $\chi(H)$,

$$|B_n| = \big(\chi(H) - 1\big)^n = \exp\big(o(p_n n^2)\big) = \exp\big(o(p_n |V(H_n)|)\big).$$

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Crucially, we need to verify that the sequence $\mathbf{H}$ is $\left(1 - \frac{1}{\chi(H)-1}, \mathfrak{B}\right)$-stable. For that, we appeal to the original stability theorem of Erdős and Simonovits (Theorem 1.2) and to the graph removal lemma (Theorem 2.3). Fix a positive δ, let $\delta'' = \delta/5$, $\varepsilon' = \varepsilon_{1.2}(\delta'')$, $\delta' = \min\{\delta'', \varepsilon'/2\}$, and let $\varepsilon = \min\{\varepsilon'/2, \varepsilon_{2.3}(\delta')\}$. Let G be a subgraph of K_n with at least $\left(1 - \frac{1}{\chi(H)-1} - \varepsilon\right)\binom{n}{2}$ edges that cannot be made $(\chi(H) - 1)$-partite by removing from it $\delta\binom{n}{2}$ edges. We claim that it contains at least $\varepsilon n^{v(H)}$ copies of H. If it did not, then by Theorem 2.3, removing at most $\delta' n^2$ edges from G would make it into an H-free graph G'. Since such G' would still have at least $\mathrm{ex}(n, H) - (\varepsilon + \delta')n^2$ edges, by Theorem 1.2 it could be made $(\chi(H) - 1)$-partite by removing from it some further $\delta'' n^2$ edges. Hence, G could be made bipartite by removing at most $2\delta'' n^2$ edges, which is fewer than $\delta\binom{n}{2}$ edges, contradicting our assumption. $\qquad\square$

Proof of Theorem 1.11. Let q be a prime with $q \equiv 2 \pmod 3$ and let $\mathbf{G}$ be a sequence of type $I(q)$ groups satisfying $|G_n| = n$. We want to apply Theorem 3.4. To this end, consider the sequence $\mathbf{H}$ of 3-uniform hypergraphs with $V(H_n) = G_n$ and $E(H_n)$ consisting of all triples $\{x, y, z\}$ satisfying $x + y = z$ and note that $|H_n| = \Omega(n^2)$. Moreover, let $p_n = n^{-1/2}$ and let $\mathcal{B}_n$ be the family of all maximum-size sum-free subsets of G_n. In order to complete the proof, it suffices to show that the assumptions of Theorem 3.4 are satisfied. First, note that $p_n^3 |H_n| = \Omega(n^{1/2})$. Since for each Schur triple $\{x, y, z\} \in H_n$, there are only constantly many Schur triples $\{x', y', z'\} \in H_n$ intersecting $\{x, y, z\}$ in more than one element, an easy computation (see [26]) shows that $\mathbf{H}$ is $(K, \mathbf{p})$-bounded for sufficiently large constant K. Finally, since $|\mathcal{B}_n| \leq n$ (see, e.g., [3, Corollary 3.4]), we have that $|\mathcal{B}_n| = \exp\left(o(n^{1/2})\right) = \exp\left(o(p_n |V(H_n)|)\right)$. Crucially, we need to verify that $\mathbf{H}$ is $\left(\frac{1}{3} + \frac{1}{3q}, \mathfrak{B}\right)$-stable. For that, we simply appeal to Corollary 2.6. $\qquad\square$

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References

[1] N. Alon, J. Balogh, R. Morris, and W. Samotij, *Counting sum-free sets in Abelian groups*, Israel J. Math., to appear.

[2] N. Alon and J. H. Spencer, *The probabilistic method*, third ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., Hoboken, NJ, 2008, With an appendix on the life and work of Paul Erdős.

[3] J. Balogh, R. Morris, and W. Samotij, *Random sum-free subsets of Abelian groups*, arXiv:1103.2041v1 [math.CO].

[4] D. Conlon and T. Gowers, *Combinatorial theorems in sparse random sets*, arXiv:1011.4310v1 [math.CO].

[5] P. H. Diananda and H. P. Yap, *Maximal sum-free sets of elements of finite groups*, Proc. Japan Acad. **45** (1969), 1–5.

[6] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar. **1** (1966), 51–57.

[7] ———, *Supersaturated graphs and hypergraphs*, Combinatorica **3** (1983), 181–192.

[8] P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091.

[9] Z. Füredi, D. Mubayi, and O. Pikhurko, *Quadruple systems with independent neighborhoods*, J. Combin. Theory Ser. A **115** (2008), 1552–1560.

[10] Z. Füredi, O. Pikhurko, and M. Simonovits, *On triple systems with independent neighbourhoods*, Combin. Probab. Comput. **14** (2005), 795–813.

[11] ———, *4-books of three pages*, J. Combin. Theory Ser. A **113** (2006), 882–891.

[12] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Probab. Comput. **14** (2005), 467–484.

[13] W. T. Gowers, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), 143–184.

[14] B. Green, *A Szemerédi-type regularity lemma in abelian groups, with applications*, Geom. Funct. Anal. **15** (2005), 340–376.

[15] B. Green and I. Z. Ruzsa, *Sum-free sets in abelian groups*, Israel J. Math. **147** (2005), 157–188.

[16] P. Haxell, Y. Kohayakawa, and T. Łuczak, *Turán's extremal problem in random graphs: forbidding even cycles*, J. Combin. Theory Ser. B **64** (1995), 273–287.

[17] P. Keevash and D. Mubayi, *Stability theorems for cancellative hypergraphs*, J. Combin. Theory Ser. B **92** (2004), 163–175.

[18] P. Keevash and B. Sudakov, *The Turán number of the Fano plane*, Combinatorica **25** (2005), 561–574.

[19] Peter Keevash and Benny Sudakov, *On a hypergraph Turán problem of Frankl*, Combinatorica **25** (2005), 673–706.

[20] Y. Kohayakawa, Y. Łuczak, and V. Rödl, *On K_4-free subgraphs of random graphs*, Combinatorica **17** (1997), 173–213.

[21] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular k-uniform hypergraphs*, Random Structures Algorithms **28** (2006), 113–179.

[22] O. Pikhurko, *Exact computation of the hypergraph Turán function for expanded complete 2-graphs*, J. Combin. Theory Ser. B, to appear.

[23] ———, *An exact Turán result for the generalized triangle*, Combinatorica **28** (2008), 187–208.

[24] V. Rödl and A. Ruciński, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), 917–942.

[25] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, Colloq. Math. Soc. János Bolyai, vol. 18, North-Holland, Amsterdam, 1978, pp. 939–945.

[26] M. Schacht, *Extremal results for random discrete structures*, submitted.

[27] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 279–319.

[28] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

[29] T. Tao, *A variant of the hypergraph removal lemma*, J. Combin. Theory Ser. A **113** (2006), 1257–1280.

[30] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452.