Additive combinatorics

Homework assignment #3

Due date: Monday, June 26, 2017

Problem 1. Solve the following two problems about Sidon sets:

(a) Construct an infinite Sidon set $A \subseteq \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{n^{1/3}} > 0.$$

(b) Let $A \subseteq \mathbb{N}$ be finite. Show that if $B \subseteq A$ is a Sidon set, then $|B| \leqslant \sqrt{2|A+A|}$.

Problem 2. Let A, B, and C be finite nonempty sets of positive real numbers. Show that $|A + B \cdot C| \geqslant c\sqrt{|A| \cdot |B| \cdot |C|}$ for some positive constant c.

Problem 3. Let A be a finite set of integers. Without invoking Freiman's theorem, show that $|A + A| \leqslant 2|A| - 1$ if and only if A is an arithmetic progression.

Problem 4. Let A and B be nonempty finite set of elements of an abelian group.

(a) Show that $|A + B| = |A|$ if and only if $|A - B| = |A|$.

(b) Show that $|A + B| = |A| \cdot |B|$ if and only if $|A - B| = |A| \cdot |B|$.

Problem 5. Let A be a nonempty finite set of elements of an abelian group.

(a) Show that $|A + A| \leqslant |A - A|^{3/2}$.

(b) Show that $|A - A| \leqslant |A + A|^{3/2}$.

Problem 6. Let G be an abelian group whose each element has order at most r. Let A be a finite set of elements of G and let H be the subgroup of G generated by A. Suppose that there exists a set B of elements of G with $|B| = |A|$ such that

$$|A + B| \leqslant c|A|$$

for some $c \geqslant 1$. Show that $|H| \leqslant K|A|$ for some constant K that depends only on c and r.