Additive combinatorics

Homework assignment #1

Problem 1. Show that every 2-colouring of $\mathbb{Z}_{2n+1}$ contains at least n^2+n+1 monochromatic solutions to the equation $x+y=2z$. In other words, prove that, for every partition $\mathbb{Z}_{2n+1} = R \cup B$, there are at least n^2+n+1 ordered triples $(x,y,z) \in R^3 \cup B^3$ that satisfy $x+y=2z$. Conclude that if $n \geqslant 2$, then in every 2-colouring of $\mathbb{Z}_{2n+1}$, one of the colour classes must contain a genuine 3-term AP.

Problem 2. Consider the following greedy construction of a subset of $\mathbb{N} = \{0,1,\ldots\}$ without a 3-term AP. Let $A_0 = \{0\}$ and for every $n \in \mathbb{N}$, let $A_{n+1} = A_n \cup \{n+1\}$ if $n+1$ does not form a 3-term AP with two elements of A_n and $A_{n+1} = A_n$ otherwise; in particular, $A_{10} = \{0,1,3,4,9,10\}$. Determine

$$\lim_{n\to\infty} \frac{\log |A_n|}{\log n}.$$

Problem 3. Derive Szemerédi's theorem from the r-dimensional 'corners theorem' of Solymosi: For every $\delta > 0$ and $r \geqslant 2$, there is an n_0 such that every subset of $\{1,\ldots,n\}^r$ with at least δn^r elements contains the $r+1$ points

$$(x_1,\ldots,x_r),\ (x_1+d,x_2,\ldots,x_r),\ \ldots,\ (x_1,\ldots,x_{r-1},x_r+d)$$

for some $x_1,\ldots,x_r \in \{1,\ldots,n\}$ and nonzero d, provided that $n \geqslant n_0$.

Problem 4. Let $\mathbb{F}$ be a finite field and suppose that $A \subseteq \mathbb{F} \setminus \{0\}$ satisfies $|A| > |\mathbb{F}|^{3/4}$. Prove that each element of $\mathbb{F}$ can be written as $a_1a_2 + a_3a_4 + a_5a_6$ for some $a_1,\ldots,a_6 \in A$.

To this end, consider the function $f\colon \mathbb{F} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{|A|} \sum_{a\in A} \mathbf{1}[xa^{-1} \in A]$$

and observe (show) that it is sufficient to prove that $f * f * f(x) > 0$ for all $x \in \mathbb{F}$.

Hint: Start by showing that $|\hat{f}(\xi)| \leqslant |\mathbb{F}|^{-1/2}$ for every nonprincipal character ξ.