Additive combinatorics

Homework assignment #2

Problem 1. Prove that every finite abelian group of order n contains a sum-free set with at least $2n/7$ elements.

Problem 2. Let G be a finite abelian group whose order is divisible by a prime p satisfying $p \equiv 2 \pmod 3$. Let $p = 3k+2$ be the smallest such prime.

(a) Prove that the largest size of a sum-free subset of G is $\frac{k+1}{3k+2} \cdot |G|$.

(b) Prove that there is a positive δ_k that depends only on k such that every sum-free set with at least $\left(\frac{k+1}{3k+2} - \delta_k\right) \cdot |G|$ elements is contained in some sum-free set of maximum size.

Problem 3. Solve the following two problems about Sidon sets:

(a) Construct an infinite Sidon set $A \subseteq \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n^{1/3}} > 0.$$

(b) Let $A \subseteq \mathbb{N}$ be finite. Show that if $B \subseteq A$ is a Sidon set, then $|B| \leqslant \sqrt{2|A+A|}$.

Problem 4. Suppose that $k \geqslant 3$ and let $a_1, \ldots, a_k$ be *nonzero* integers. Consider the equation

$$a_1x_1 + \ldots + a_kx_k = 0. \tag{1}$$

Let us say that a solution $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ to (1) is *generic* if $x_1, \ldots, x_k$ are pairwise distinct. Define $f\colon \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \max\bigl\{|A|\colon A \subseteq \{1, \ldots, n\} \text{ and } A^k \text{ contains no generic solutions to (1)}\bigr\}.$$

(a) Suppose that $a_1 + \ldots + a_k \neq 0$. Show that $f(n) = \Omega(n)$.

(b) Suppose that $a_1 + \ldots + a_k = 0$. Show that $f(n) = o(n)$.

(c) Suppose that $a_1 + \ldots + a_k = 0$ and $a_1, \ldots, a_{k-1} > 0$. Show that $f(n) = n^{1-o(1)}$.