Concentration inequalities

Homework assignment #1

Due date: Wednesday, November 25, 2015

Problem 1. Let $\mathbb{M}Z$ be a median of the square-integrable random variable Z. (That is, $Z \geqslant \mathbb{M}Z$ and $Z \leqslant \mathbb{M}Z$ both hold with probability at least 1/2.) Show that

$$\left|\mathbb{M}Z - \mathbb{E}Z\right| \leqslant \sqrt{\mathrm{Var}(Z)}.$$

Problem 2. Show that if Y is a nonnegative random variable, then for any $a \in (0, 1)$,

$$\Pr(Y \geqslant a\mathbb{E}Y) \geqslant (1-a)^2 \frac{(\mathbb{E}Y)^2}{\mathbb{E}\left[Y^2\right]}.$$

Problem 3. Show that moment bounds for tail probabilities are always better than Cramér–Chernoff bounds. Let Y be a nonnegative random variable and let $t > 0$. Prove that

$$\min_q \mathbb{E}\left[Y^q\right] t^{-q} \leqslant \inf_{\lambda > 0} \mathbb{E}\left[e^{\lambda(Y-t)}\right].$$

Problem 4. Establish the following upper bounds on the lower tail of the binomial distribution.

(a) Let B be binomially distributed with parameters (n, p). Show that for $0 < a < p$,

$$\Pr(B \leqslant an) \leqslant \left(\left(\frac{p}{a}\right)^a \left(\frac{1-p}{1-a}\right)^{1-a}\right)^n.$$

(b) Let k and n be positive integers with $1 \leqslant k \leqslant n$. Use (a) to derive the inequality

$$\sum_{j=0}^{k} \binom{n}{j} \leqslant \left(\frac{en}{k}\right)^k.$$

Problem 5. Assume that X is a centered sub-Gaussian random variable with variance factor v, that is,

$$\log \mathbb{E}\left[e^{\lambda X}\right] \leqslant \frac{\lambda^2 v}{2} \quad \text{for every } \lambda \in \mathbb{R}.$$

Prove that $\mathrm{Var}(X) \leqslant v$.

Problem 6. Let X be a nonnegative random variable with finite second moment. Show that for any $\lambda > 0$,

$$\mathbb{E}\left[e^{-\lambda(X - \mathbb{E}X)}\right] \leqslant e^{\lambda^2 \mathbb{E}[X^2]/2}.$$

In particular, if $X_1, \ldots, X_n$ are independent nonnegative random variables, then for any $t > 0$,

$$\Pr\left(\sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \leqslant -t\right) \leqslant \exp\left(-\frac{t^2}{2v}\right),$$

where $v = \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right]$.