Concentration inequalities

Homework assignment #2

Due date: Wednesday, December 23, 2015

Problem 1. Assume that the random variables $X_1, \ldots, X_n$ are independent and $\{-1, 1\}$-valued with $\Pr(X_i = 1) = p_i$ and suppose that $f\colon \{-1,1\}^n \to \mathbb{R}$ has the bounded differences property with constants $c_1, \ldots, c_n$. Show that if $Z = f(X_1, \ldots, X_n)$, then

$$\mathrm{Var}(Z) \leqslant \sum_{i=1}^{n} c_i^2 p_i (1 - p_i).$$

Problem 2. Denote by $\mathcal{F}$ the class of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ that are Lipschitz with respect to the ℓ^1-distance, that is, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$|f(x) - f(y)| \leqslant \sum_{i=1}^{n} |x_i - y_i|.$$

Let $X = (X_1, \ldots, X_n)$ be a vector of independent random variables with finite variance(s). Use the Efron–Stein inequality to show that the maximal value of $\mathrm{Var}(f(X))$ over $f \in \mathcal{F}$ is attained by the function $f(x) = x_1 + \ldots + x_n$.

Problem 3. Let Z be the number of triangles in the random graph $G(n,p)$. Calculate the variance of Z and compare it with the result obtained using the Efron–Stein inequality.

Problem 4. Let P be a probability distribution on a countable set $\mathcal{X}$ and let $\mathcal{Y}$ be a finite set. A *uniquely decodable encoding* of $\mathcal{X}$ using alphabet $\mathcal{Y}$ is a mapping φ from $\mathcal{X}$ to the set $\mathcal{Y}^* = \bigcup_{n>0} \mathcal{Y}^n$ of sequences of elements of $\mathcal{Y}$ of finite length with the following property: For any two sequences $x_1, \ldots, x_k$ and $x'_1, \ldots, x'_\ell$ of elements of $\mathcal{X}$, if the concatenations of $\varphi(x_1), \ldots, \varphi(x_k)$ and $\varphi(x'_1), \ldots, \varphi(x'_\ell)$ are equal, then $k = \ell$ and $x_i = x'_i$ for all $i \in [k]$. If $x \in \mathcal{X}$, then $\varphi(x)$ is the *codeword* associated with x and $|\varphi(x)|$ denotes the length of the codeword.

(a) Prove that for any uniquely decodable encoding φ of $\mathcal{X}$ using an alphabet $\mathcal{Y}$,

$$\sum_{x \in \mathcal{X}} |\mathcal{Y}|^{-|\varphi(x)|} \leqslant 1.$$

(b) Use (a) to prove that Shannon's entropy with base $|\mathcal{Y}|$ is a lower bound on the average codeword length under P. In other words, if $X \in \mathcal{X}$ is distributed according to P, then

$$\mathbb{E}\big[|\varphi(X)|\big] \geqslant \frac{H(X)}{\log |\mathcal{Y}|}.$$

(c) Let $\ell\colon \mathcal{X} \to \{1, 2, \ldots\}$ be such that

$$\sum_{x \in \mathcal{X}} |\mathcal{Y}|^{-\ell(x)} \leqslant 1.$$

Prove that there exists a uniquely decodable encoding $\varphi\colon \mathcal{X} \to \mathcal{Y}^*$ such that $|\varphi(x)| = \ell(x)$ for all $x \in \mathcal{X}$.

(d) Suppose that $X \in \mathcal{X}$ is distributed accoring to P. Use (c) to prove that there exists a uniquely decodable encoding φ such that

$$\mathbb{E}\big[|\varphi(X)|\big] \leqslant \frac{H(X)}{\log |\mathcal{Y}|} + 1.$$

Problem 5. Prove the following statements using Han's inequality:

(a) Let A be a finite subset of $\mathbb{Z}^d$ and for each $i \in [d]$, let A_i denote the canonical projection of A along the i-th coordinate. Show that

$$|A|^{d-1} \leqslant \prod_{i=1}^{d} |A_i|.$$

(b) Deduce from (a) the following version of the *Loomis–Whitney inequality*. Suppose that $C \subseteq \mathbb{R}^d$ is a bounded convex body and for each $i \in [d]$, let $C_i \subseteq \mathbb{R}^{d-1}$ denote the canonical projection of C alonth the i-th coordinate. Show that

$$\mathrm{vol}(C) \leqslant \left(\prod_{i=1}^{d} \mathrm{vol}(C_i)\right)^{\frac{1}{d-1}}.$$

(c) Let A denote a finite subset of $\mathbb{Z}^d$ and let $B \subseteq \mathbb{Z}^d$ be the canonical basis of $\mathbb{R}^d$. Let ∂A be the edge boundary of the set A in the nearest-neighbor graph on $\mathbb{Z}^d$, that is, let

$$\partial A = \big\{\{x, x+\varepsilon b\} \colon x \in A, b \in B, \varepsilon \in \{-1, 1\}, x + \varepsilon b \notin A\big\}.$$

Use (a) to show that

$$|\partial A| \geqslant 2d|A|^{\frac{d-1}{d}}.$$