Edge Disjoint Hamilton Cycles

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1 Introduction

In the late 70s, it was shown by Komlós and Szemerédi ([7]) that for $p = \frac{\ln n + \ln\ln n + c}{n}$, the limit probability for $G(n,p)$ to contain a Hamilton cycle equals the limit probability for $G(n,p)$ to have minimum degree at least 2. A few years later, Ajtai, Komlós and Szemerédi ([1]) have shown a hitting time version of this in the $G(n,m)$ model.

Say a graph G has property $\mathcal{H}$ if it contains $\lfloor \delta(G)/2 \rfloor$ edge disjoint Hamilton cycles, plus a further edge disjoint near perfect matching in the case $\delta(G)$ is odd. Is it true that for every $0 \leq p \leq 1$ the random graph $G(n,p)$ has property $\mathcal{H}$ with high probability? This is clear whenever $\delta(G) = 0$. In the early 80s, Bollobás and Frieze ([3]) have proved that conjecture for $\delta(G) = O(1)$. In this talk I plan to prove the result for $p(n) \leq (1+o(1)) \ln n/n$. This is a result of Frieze and Krivelevich from '08 ([4]).

Remark 1. *The conjecture is nowadays known to be true for every p. It was proved for the range $\ln^{50} n/n \leq p \leq 1 - \ln^9 n/n^{1/4}$ by Knox, Kühn and Osthus in '13 ([6]), in a rather technically complicated paper. Later, Krivelevich and Samotij ([8]) have closed the gap for the sparse case, and Kühn and Osthus ([9]) have closed the gap for the dense case.*

This is the main result we intend to prove:

Theorem 2. *Let $p = p(n) \leq (1+o(1)) \ln n/n$. Then **whp** $G(n,p)$ has property $\mathcal{H}$.*

Remark 3. *In this talk I will not consider the extra near perfect matching, expected in the case where $\delta(G)$ is odd. This adds some technicality, but nothing really different.*

2 Preliminaries

2.1 Probability

Theorem 4 (Chernoff bounds, [5], Theorem 2.1)**.** *Let $X \sim Bin(n,p)$, $\mu = np$, $a \geq 0$. Then the following inequality holds:*

$$\mathbb{P}(X \leq \mu - a) \leq exp\left(-\frac{a^2}{2\mu}\right).$$

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Definition 5. *A* monotone increasing graph property P *is a set of graphs which is closed upwards; that is, if* $G \in P$ *and* $G \subseteq H$ *then* $H \in P$*. Similarly, a* monotone decreasing graph property Q *is a set of graphs which is closed downwards; that is, if* $G \in Q$ *and* $H \subseteq G$ *then* $H \in Q$*.*

Theorem 6 (The FKG inequality for monotone graph properties, [2], Theorem 6.3.3)**.** *Let* P_1, P_2 *be two monotone increasing graph properties and* Q_1, Q_2 *be two monotone decreasing graph properties. Let* $G \sim G(n,p)$*. Then:*

$$\begin{aligned} \mathbb{P}(G \in P_1 \cap P_2) &\geq \mathbb{P}(G \in P_1)\,\mathbb{P}(G \in P_2), \\ \mathbb{P}(G \in Q_1 \cap Q_2) &\geq \mathbb{P}(G \in Q_1)\,\mathbb{P}(G \in Q_2), \\ \mathbb{P}(G \in P_1 \cap Q_1) &\leq \mathbb{P}(G \in P_1)\,\mathbb{P}(G \in Q_1). \end{aligned}$$

2.2 Sprinkle sprinkle

In the proof, we will use several standard techniques/tricks. The first trick is the trick of "sprinkling" random edges. Formally, we'd like to present G as a union of G_0, which is very similar to G, and some random leftovers, R. This can be achieved easily by taking p_0 and ρ so that $1-p = (1-p_0)(1-\rho)$ and letting $\rho = o(1/n)$, thus decomposing $G \sim G(n,p)$ to $G = G_0 \cup R$ where $G_0 \sim G(n,p_0)$ and $R \sim G(n,\rho)$. In which sense are G and G_0 similar? In the following:

Claim 7. *Fix* G_0*, let* $R \sim G(n,\rho)$ *and* $G = G_0 \cup R$*; then* ***whp*** $\delta(G_0) = \delta(G)$*.*

Proof. Clearly, $\delta(G_0) \leq \delta(G)$, as G contains all the edges of G_0 and more. Now, let $v \in G_0$ with $d_{G_0}(v) = \delta(G_0)$. As $\rho = o(1/n)$, $d_R(v) = 0$ **whp** (a standard first moment argument), implying

$$\delta(G) \leq d_G(v) = d_{G_0}(v) = \delta(G_0).$$

□

From now on, write $\delta_0 = \delta(G_0)$. It follows that it is enough to prove that G contains (**whp**) $\lfloor \delta_0/2 \rfloor$ edge disjoint Hamilton cycles and an edge disjoint near perfect matching if δ_0 is odd. We also assume that $p = (1+o(1))\ln n/n$, as otherwise $\delta_0 \leq 1$ and there's nothing new to prove. We also note that from this assumption it follows that $\delta(G) = o(\ln n)$; this will follow from the following claims. Let D_k be the random variable counting the number of vertices in $G(n,p)$ with degree exactly k. Clearly, $D_k = \sum_{v \in [n]} D_k(v)$, where $D_k(v)$ is the indicator of the event that v is of degree k. Note that

$$\mathbb{E}(D_k) = \sum_{v \in [n]} \mathbb{E}(D_k(v)) = n\mathbb{P}(d(v) = k) = n\binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

Thus, letting $k = \delta \ln n$ and $p = (1+\varepsilon)\ln n/n$ for $\varepsilon = o(1)$,

$$\begin{aligned} \mathbb{E}(D_k) &= n\binom{n-1}{\delta \ln n} p^{\delta \ln n} (1-p)^{n-1-\delta \ln n} \\ &\geq n\left(\frac{(n-1)p}{\delta \ln n}\right)^{\delta \ln n} e^{-np} \\ &\geq n^{-\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)^{\delta \ln n} \geq n^{\delta \ln(1/\delta) - \varepsilon} = \omega(1), \end{aligned}$$

if we take $\delta = \delta(n)$ to be large enough, say, $\delta = \varepsilon$.

Claim 8. *For $k = O(\ln n)$, if $\mathbb{E}(D_k) = \omega(1)$ then $Var(D_k) = o\left(\mathbb{E}^2(D_k)\right)$.*

Proof. Let $u \neq v$ be two vertices. Note that

$$\frac{\binom{n}{k-1}p^{k-1}}{\binom{n}{k}p^k} = \frac{k}{np}(1+o(1)) = \frac{k}{\ln n}(1+o(1)) = O(1),$$

thus

$$\begin{aligned}
\mathrm{Cov}\left(D_k(u), D_k(v)\right) &= \mathbb{P}\left(d(u) = k = d(v) \mid u \sim v\right)\mathbb{P}\left(u \sim v\right) \\
&\quad + \mathbb{P}\left(d(u) = k = d(v) \mid u \nsim v\right)\mathbb{P}\left(u \nsim v\right) - \mathbb{P}^2\left(d(u) = k\right) \\
&= \left(\binom{n-2}{k-1}p^{k-1}(1-p)^{n-1-k}\right)^2 p \\
&\quad + \left(\binom{n-2}{k}p^k(1-p)^{n-2-k}\right)^2(1-p) - \left(\binom{n-1}{k}p^k\,(1-p)^{n-1-k}\right)^2 \\
&= O\left(\mathbb{E}^2(D_k)pn^{-2}\right) + O\left(\mathbb{E}^2(D_k)n^{-2}\cdot\left(\frac{1}{1-p} - 1\right)\right) \\
&= o\left(\mathbb{E}^2(D_k)n^{-2}\right).
\end{aligned}$$

It follows that

$$\mathrm{Var}\left(D_k\right) \leq \mathbb{E}\left(D_k\right) + \sum_{u \neq v}\mathrm{Cov}\left(D_k(u), D_k(v)\right) = o\left(\mathbb{E}^2(D_k)\right).$$

□

For technical reasons, we'll define a very particular ρ so that $\rho = o(1/n)$ will hold. Set $d_0 = \min\{k \mid \mathbb{E}(D_k) \geq 1\}$.

Claim 9. $d_0 = o(\ln n)$.

Proof. As we've seen, for $k = \delta \ln n$, $\delta = o(1)$, $\mathbb{E}(D_k) \to \infty$, and $d_0 < k$, so $d_0 = o(\ln n)$. □

Note that d_0 approximates $\delta(G)$; formally,

Claim 10. ***whp**, $|\delta(G) - d_0| \leq 2$.*

Proof. Note that

$$\frac{\mathbb{E}\left(D_{k+1}\right)}{\mathbb{E}\left(D_k\right)} = \frac{n\binom{n-1}{k+1}p^{k+1}(1-p)^{n-2-k}}{n\binom{n-1}{k}p^k(1-p)^{n-1-k}} = \frac{(n-1-k)p}{(k+1)(1-p)}.$$

As we've seen, $d_0 = o(\ln n)$. Thus it follows that for $b \geq 1$,

$$\mathbb{E}\left(D_{d_0-b-1}\right) = \mathbb{E}\left(D_{d_0-b}\right)\cdot\frac{(d_0-b)(1-p)}{(n-1-(d_0-b-1))p} < \frac{d_0}{\frac{1}{2}np} = \varepsilon' = o(1),$$

and by a Markov's inequality and the union bound,

$$\mathbb{P}\left(\exists b \geq 1, D_{d_0-b-1} > 0\right) \leq \sum_{b=1}^{d_0-1} (\varepsilon')^b \leq \frac{\varepsilon'}{1-\varepsilon'} = o(1).$$

In addition,

$$\mathbb{E}\left(D_{d_0+1}\right) = \mathbb{E}\left(D_{d_0}\right) \cdot \frac{(n-1-d_0)p}{(d_0+1)(1-p)} \geq \frac{\frac{1}{2}np}{d_0} = \omega(1),$$

and by Chebyshev's inequality and the previous claim,

$$\mathbb{P}\left(D_{d_0+1} = 0\right) \leq \mathbb{P}\left(\left|D_{d_0+1} - \mathbb{E}\left(D_{d_0+1}\right)\right| \geq 1\right) \leq \frac{\mathrm{Var}\left(D_{d_0+1}\right)}{\mathbb{E}^2\left(D_{d_0+1}\right)} = o(1).$$

Therefore, **whp** there is no vertex with degree at most $d_0 - 2$ and there is a vertex with degree d_0+1, thus $|\delta(G) - d_0| \leq 2$. □

Corollary 11. ***whp**, $\delta(G) = o(\ln n)$.* □

We then define

$$\rho = \frac{2001(d_0 + \ln\ln n)}{n \ln n},$$

and observe that $\rho = o(1/n)$ (again, since $d_0 = o(\ln n)$), and that $np_0 = np(1+o(1))$.

2.3 Properties of random graphs

In this section we give a list of properties, each occuring **whp**, in the random graph $G_0 \sim G(n, p_0)$.

Define the set SMALL:

$$\mathrm{SMALL} = \left\{ v \in V(G) \mid d_{G_0(v)} \leq 0.1 \ln n \right\}.$$

Lemma 12. *The random graph $G_0 \sim G(n, p_0)$ with p_0 defined earlier, has **whp** the following properties:*

(P1) *There is no non-empty path of length at most 4 in G_0 such that both of its (possibly identical) endpoints lie in SMALL.*

(P2) *Every set $U \subseteq V(G)$ with $|U| \leq 100n/\ln n$ spans at most $|U|(\ln n)^{1/2}$ edges in G_0.*

(P3) *For every two disjoint sets $U, W \subseteq V(G)$ with $|U| \leq 100n/\ln n$, $|W| \leq |U| \ln n/10000$,*

$$|E_{G_0}(U, W)| < 0.09|U| \ln n.$$

(P4) *For every two disjoint sets $U, W \subseteq V(G)$ with $|U| \geq 100n/\ln n$, $|W| \geq n/4$,*

$$|E_{G_0}(U, W)| \geq 0.1|U| \ln n.$$

Proof of (P1). Fix a vertex v. Note that

$$\begin{aligned}\mathbb{P}(v \in \mathrm{SMALL}) &= \sum_{k=0}^{0.1\ln n} \mathbb{P}(\mathrm{Bin}(n-1, p_0) = k) \\ &\le 0.1 \ln n \binom{n-1}{0.1\ln n} p_0^{0.1\ln n}(1-p_0)^{n-1-0.1\ln n} \\ &\le 0.1\ln n \left(\frac{10enp}{\ln n}\right)^{0.1\ln n} e^{-p_0(n-1-0.1\ln n)} \\ &\le 28^{0.1\ln n} e^{-(1-o(1))\ln n} < n^{-0.6}.\end{aligned}$$

Now fix $u \neq v$. The probability that u, v are connected by a path of length ℓ in G_0 is at most $n^{\ell-1}p_0^{\ell} = ((1+o(1))\ln n)^{\ell} n^{-1}$ (choosing $\ell - 1$ inner vertices and for each such choice requiring ℓ edges). Furthermore, as there's exactly one edge of K_n connecting u with v, conditioning on the event "$u \in \mathrm{SMALL}$" cannot increase the probability of "$v \in \mathrm{SMALL}$" by too much:

$$\begin{aligned}\mathbb{P}(u, v \in \mathrm{SMALL}) &\le \mathbb{P}(v \in \mathrm{SMALL} \mid u \in \mathrm{SMALL})\,\mathbb{P}(u \in \mathrm{SMALL}) \\ &\le \mathbb{P}(v \in \mathrm{SMALL} \mid \{u, v\} \notin E)\,\mathbb{P}(u \in \mathrm{SMALL}) \\ &\le \mathbb{P}(v \in \mathrm{SMALL})\,\mathbb{P}(u \in \mathrm{SMALL}) \cdot \frac{1}{1-p}.\end{aligned}$$

Note also that "$u, v \in \mathrm{SMALL}$" is a monotone decreasing event and "$d(u, v) \le 4$" is a monotone increasing event. Thus, according to the FKG inequality,

$$\mathbb{P}(u, v \in \mathrm{SMALL} \wedge d(u, v) \le 4) \le \mathbb{P}(u, v \in \mathrm{SMALL}) \cdot \mathbb{P}(d(u, v) \le 4).$$

Therefore,

$$\begin{aligned}\mathbb{P}(u, v \in \mathrm{SMALL} \wedge d(u, v) \le 4) &\le \mathbb{P}(u, v \in \mathrm{SMALL}) \cdot \mathbb{P}(d(u, v) \le 4) \\ &\le \mathbb{P}(v \in \mathrm{SMALL})\,\mathbb{P}(u \in \mathrm{SMALL})\,\mathbb{P}(d(u, v) \le 4)\,(1+o(1)) \\ &\le n^{-0.6} \cdot n^{-0.6} \cdot \frac{\ln^4 n}{n} \cdot (1+o(1)) < n^{-2.1}.\end{aligned}$$

Applying the union bound over all possible pairs of u, v we establish (P1). □

Proof of (P2). For a given $U \subseteq [n]$ with $|U| = u \le 100n/\ln n$, let A_U be the event by which $|E(U)| \ge u \ln^{1/2} n$. By the union bound,

$$\begin{aligned}\mathbb{P}(\exists U,\ |U| = u \le 100n/\ln n,\ A_U) &\le \sum_{u=1}^{100n/\ln n} \binom{n}{u} \binom{\binom{u}{2}}{u\ln^{1/2} n} p^{u\ln^{1/2} n} \\ &\le \sum_{u=1}^{100n/\ln n} \left(\frac{en}{u}\left(\frac{eup}{2\ln^{1/2} n}\right)^{\ln^{1/2} n}\right)^{u} \\ &\le \sum_{u=1}^{100n/\ln n} \left(\frac{en}{u}\left(\frac{2u\ln^{1/2} n}{n}\right)^{\ln^{1/2} n}\right)^{u}.\end{aligned}$$

We now separate the sum into two:

$$\sum_{u=1}^{\ln n} \left(\frac{en}{u} \left(\frac{2u \ln^{1/2} n}{n} \right)^{\ln^{1/2} n} \right)^u \leq \ln n \cdot en \left(\frac{2 \ln^{3/2} n}{n} \right)^{\ln^{1/2} n} = o(1),$$

and

$$\begin{aligned} \sum_{u=\ln n}^{100n/\ln n} \left(\frac{en}{u} \left(\frac{2u \ln^{1/2} n}{n} \right)^{\ln^{1/2} n} \right)^u &\leq n \left(e \left(\frac{u}{n} \right)^{\ln^{1/2} n - 1} \left(2 \ln^{1/2} n \right)^{\ln^{1/2} n} \right)^u \\ &\leq n \left(e \left(\frac{1}{\ln n} \right)^{\ln^{1/2} n - 1} \left(2 \ln^{1/2} n \right)^{\ln^{1/2} n} \right)^u = o(1). \end{aligned}$$

$\square$

Proof of (P3). For a given $U \subseteq [n]$ with $|U| = u \leq 100n/\ln n$ and $W \subseteq [n]$ with $|W| \leq u' = \frac{u \ln n}{10000}$, let $A_{U,W}$ be the event by which $|E(U,W)| \geq 0.09u \ln n$. By the union bound,

$$\begin{aligned} \mathbb{P}\left(\exists U, W,\ A_{U,W}\right) &\leq \sum_{u=1}^{100n/\ln n} \sum_{w=1}^{u \ln n/10000} \binom{n}{u} \binom{n}{w} \binom{uw}{0.09u \ln n} p^{0.09u \ln n} \\ &\leq \sum_{u=1}^{100n/\ln n} u' \binom{n}{u} \binom{n}{u'} \binom{uu'}{0.09u \ln n} p^{0.09u \ln n} \\ &\leq \sum_{u=1}^{100n/\ln n} u' \left(\left(\frac{en}{u} \right) \left(\frac{en}{u'} \right)^{\ln n/10000} \left(\frac{eu'p}{0.09 \ln n} \right)^{0.09 \ln n} \right)^u \\ &\leq \sum_{u=1}^{100n/\ln n} u' \left(n e^{\ln n/10000} \left(\frac{e}{0.09} \right)^{0.09 \ln n} \left(\frac{n}{u'} \right)^{\frac{\ln n}{10000} - 0.09 \ln n} \right)^u \\ &\leq \sum_{u=1}^{100n/\ln n} u' \left(n^2 \left(\frac{n}{u'} \right)^{-0.08 \ln n} \right)^u \\ &\leq \sum_{u=1}^{100n/\ln n} u' \left(n^{2 - 0.08n/u'} \right)^u = o(1), \end{aligned}$$

as $0.08n/u' \geq 8$. $\square$

Proof of (P4). Fix U, W, $|U| \geq 100n/\ln n$, $|W| \geq n/4$. Note that the number of edges between U, W in G_0 is binomially distributed with $|U||W|$ trials and success probability p_0, hence

$$\mathbb{E}\left(|E_{G_0}(U,W)|\right) \geq (1 + o(1))|U| \ln n/4.$$

By Chernoff bounds (Theorem 4),

$$\begin{aligned}
\mathbb{P}\left(|E_{G_0}(U,W)| \le 0.1|U|\ln n\right) &\le \mathbb{P}\left(|E_{G_0}(U,W)| \le 0.25|U|\ln n - 0.15|U|\ln n\right) \\
&\le \exp\left(-\frac{(0.15|U|\ln n)^2}{2\cdot 0.25|U|\ln n}\right) \\
&< \exp\left(-2\cdot 0.15^2|U|\ln n\right) < \exp\left(-4n\right).
\end{aligned}$$

Now, the number of pairs U, W is at most 4^n, union bound gives that the probability that such a pair exists is at most $4^n e^{-4n} = o(1)$. □

2.4 Expanders, rotations and boosters

One of the key concepts in many connectivity and Hamiltonicity related problems is that of an expander.

Definition 13. *For every $c > 0$ and every positive integer R we say that a graph $G = (V, E)$ is an (R, c)-*expander *if every subset of vertices $U \subseteq V$ of cardinality $|U| \le R$ satisfies $|N_G(U)| \ge c|U|$.*

Claim 14. *Let G be a $(k, 2)$-expander on n vertices, with $k > \frac{n}{4}$. Then, G is connected.*

Proof. Since every set of cardinality at most $n/4$ expands, every connected component must be of cardinality at least $3n/4$, and there's room for only 1 such component. □

Our approach will consist of that concept, bundled with the so-called rotation-extension technique, introduced by Pósa in '76 ([10]). Here we will cover the technique, including a key lemma.

Given a path $P = (v_0, \ldots, v_m)$, we can *extend* it by adding v_{m+1} which is not part of the path but is a neighbour of v_m, or we can *rotate* it by finding a neighbour v_i of v_m inside the path, adding the edge $\{v_m, v_i\}$ and erasing the edge $\{v_i, v_{i+1}\}$ $(1 \le i < m)$.

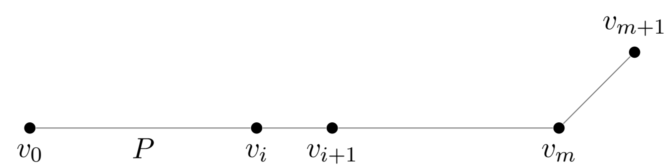

Figure 1: Pósa extension

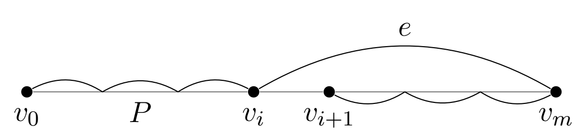

Figure 2: Pósa rotation

Lemma 15. *Let G be a graph, P a path of maximal length in G, $\mathcal{P}$ the set of all (rooted) paths obtained by P be a sequence of rotations, U the set of endpoints of these paths, N^- and N^+ the sets of vertices immediately preceeding and following the vertices of U along P, respectively. Then, $N(U) \subseteq N^- \cup N^+$.*

Proof. Denote $P = (v_0, \ldots, v_m)$. Let $u \in U$, $v \notin (U \cup N^- \cup N^+)$, and let P_u be a rotation of P ending at u. If $v \notin P$ then $\{u, v\} \notin E$, otherwise we could have extended P_u and get a longer path, contradicting our assumption.

Thus, $v \in P$. Let v^-, v^+ be its two possible neighbours in P. Suppose $\{u, v\} \in E$. Then, we can rotate P_u to get P_w ending at w, where w is a neighbour of v. If w is v^- or v^+, we get a contradiction, as this puts v in N^- or N^+. Thus, one of the edges in P between v and v^-, v^+ broke during a rotation. Let's look at when it has happened; then, if $\{v^-, v\}$ broke, that rotation has made v an end vertex, and if $\{v, v^+\}$ broke, that rotation has put $v \in N^-, N^+$. Thus, $\{u, v\} \notin E$. □

Corollary 16.

$$|N(U)| \leq |N^- \cup N^+| \leq 2|U| - 1.$$

□

Corollary 17. *Let G be a connected non-Hamiltonian $(k, 2)$-expander. Then G contains a path of (edge) length $3k - 1$.*

Proof. Let P be a path of maximal length m (counting in edges) in G. Recall that $|N(U)| \leq 2|U| - 1$, that is, U does not expand, hence $|U| > k$. Let $U' \subseteq U$ with $|U'| = k$. Since P is maximal, $N(U') \subseteq V(P)$, thus $|V(P)| \geq 3k$, hence P is of length at least $3k - 1$. □

In order to utilize that lemma for our needs, we introduce the notion of a booster:

Definition 18. *Given a graph G, a non-edge $e = \{u, v\}$ of G is called a* booster *if adding e to G creates a graph G', which is either Hamiltonian or whose maximum path is longer than that of G.*

Note that technically every non-edge of a Hamiltonian graph G is a booster by definition.

Boosters advance a graph towards Hamiltonicity when added; adding sequentially n boosters clearly brings any graph on n vertices to Hamiltonicity.

Corollary 19. *Let G be a connected non-Hamiltonian $(k, 2)$-expander. Then G has at least $\frac{(k+1)^2}{2}$ boosters.*

Proof. Let P be a path of maximal length m (counting in edges) in G. Again, $|U| > k$. We now seek of $\frac{(k+1)^2}{2}$ non-edges which, when added, create a cycle of length $m + 1$.

Fix a set $u_1, \ldots, u_{k+1}$ of end vertices. For each, let P_i be the rotation of P ending at u_i. For such i, fix u_i as a starting vertex, and let $\mathcal{P}_i$ be the set of rotations of P_i. Let U_i be the set of endpoints retrieved that way. As before, $|U_i| > k$. Let $u_1^{(i)}, \ldots, u_{k+1}^{(i)}$ be a set of such end vertices.

Note that for every $i, j \in [k + 1]$, $u_i, u_j^{(i)}$ are not connected, as if they were, we would have a cycle of length $m + 1$, and either end up with a Hamilton cycle, or, if $m + 1 < n$, since G is connected, get a longer path. As each non-edge was counted at most twice that way, we have at least $(k + 1)^2/2$ such non-edges, each is a booster. □

3 The proof

The outline of the proof is as follows: we split the graph R into $\lceil \delta_0/2 \rceil$ identically distributed random graphs R_i. We start with G_0, finding enough boosters in R_1 to get a Hamilton cycle, deleting its edges and end up in G_1, and continuing so: given G_{i-1} ($1 \le i \le \lceil \delta_0/2 \rceil$), we find boosters in R_i to get a Hamilton cycle H_i, and by deleting it we get G_i. During the process, we'll keep the following attributes of G_i:

(I1) $\delta(G_i) \ge 2$

(I2) G_i is a $(n/3 - cn/\ln n, 2)$-expander (that will follow from (P1)-(P4))

(I3) G_i is connected

(I4) G_i has a path of length at least $n - cn/\ln n$

(I5) G_i has quadratic number of boosters.

If δ_0 is odd, we'll need a final stage to create a near perfect matching.

3.1 Formal argument

We may assume that $\delta_0 \ge 2$, otherwise there's nothing new to prove. For $1 \le i \le \lceil \delta_0/2 \rceil$ define ρ_i by

$$1 - \rho = (1 - \rho_i)^{\lceil \delta_0/2 \rceil}.$$

Observe that

$$1 - \rho = (1 - \rho_i)^{\lceil \delta_0/2 \rceil} \ge 1 - \rho_i \lceil \delta_0/2 \rceil,$$

and thus

$$\rho_i \ge \frac{\rho}{\lceil \delta_0/2 \rceil} = \frac{2001(d_0 + \ln\ln n)}{\lceil \delta_0/2 \rceil \, n \ln n} \ge \frac{4000}{n \ln n}.$$

Now let

$$R = \bigcup_{i=1}^{\lceil \delta_0/2 \rceil} R_i,$$

where $R_i \sim G(n, \rho_i)$, and let G_i be the graph obtained from $G_0 \cup \bigcup_{j=1}^{i} R_i$ after having deleted the first i Hamilton cycles, assuming that the previous $i-1$ stages were indeed successful. Let $i < \lfloor \delta_0/2 \rfloor$. To see (I1), note that every vertex had its degree in G_0 reduced by at most $2i$ in G_i. Thus,

$$\delta(G_i) \ge \delta_0 - 2i \ge \delta_0 - 2(\lfloor \delta_0/2 \rfloor - 1) \ge 2.$$

To see (I2), we now show that G_i is a $(k, 2)$-expander for $k = n/3 - 100n/(3\ln n)$. For that, let X be a vertex set with t vertices. Consider the following two cases:

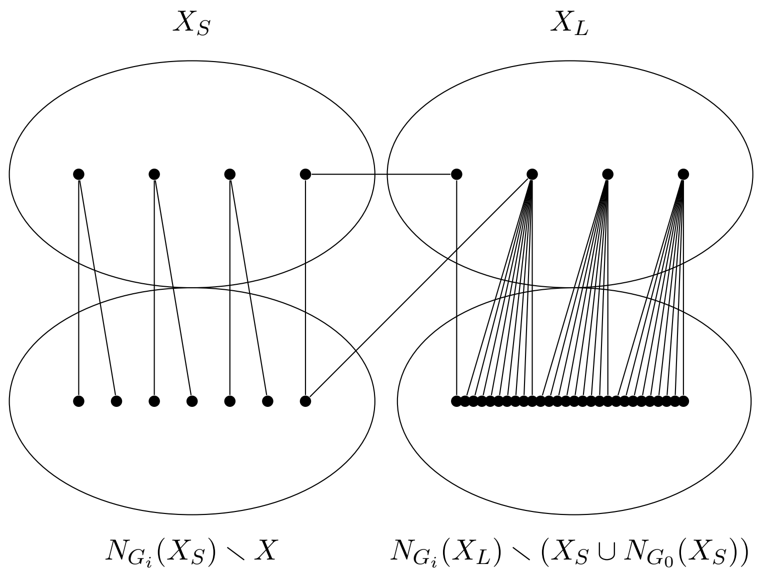

Figure 3: Case I

Case 1 ($t \leq 100n/\ln n$): Denote $X_S = X \cap \text{SMALL}$ and $X_L = X \setminus X_S$. Denote $t_S = |X_S|$ and $t_L = |X_L|$. Observe that $|N_{G_i}(X_S, V \setminus X)| \geq 2t_S - t_L$; indeed, by property (P1), every vertex in X_S has at least 2 unique neighbours in $V \setminus X_S$, and at most t_L of these $2t_S$ neighbours lie in X_L. By property (P2), X_L spans at most $t_L(\ln n)^{1/2}$ edges in G_0. Recall that the minimal degree of X_L in G_0 is at least $0.1 \ln n$; thus, at least $0.09t_L \ln n$ edges leave X_L in G_0. But then by (P3), the set of neighbours of X_L must be of cardinality greater than $t_L \ln n/10000$. By (P1) again, at most t_L of these fall into $X_S \cup N_{G_0}(X_S)$. As in G_i each vertex has lost at most δ_0 neighbours comparing to what it had in G_0, we have that in G_i, the set of neighbours of X_L outside $X_S \cup N_{G_0}(X_S)$ is at least $t_L(\ln n/10000 - 1 - \delta_0) \geq t_L \ln n/100000$. Altogether,

$$|N_{G_i}(X)| \geq 2t_S - t_L + t_L \ln n/100000 \geq 2t_S + 2t_L = 2t,$$

as claimed.

Case 2 ($100n/\ln n \leq t \leq n/3 - 100n/(3\ln n)$): Assume to the contrary that $|N_{G_i}(X)| < 2t$. In that case we can find a vertex set Y disjoint to $X \cup N_{G_i}(X)$ of cardinality $n - 3t$. Thus, in G_0 there were at most $2\lfloor \delta_0/2 \rfloor \min\{t, n-3t\}$ edges between X and Y. If $t \leq n/4$ then $n - 3t \geq n/4$ and by (P4) we should have had $|E_{G_0}(X, Y)| \geq 0.1t \ln n \gg \delta_0 t$. If $t \geq n/4$ then $n - 3t \geq n - 3(n/3 - 100n/(3\ln n)) = 100n/\ln n$, and again by (P4) we should have had $|E_{G_0}(Y, X)| \geq 0.1|Y| \ln n \gg \delta_0 |Y|$.

The proof of Theorem 2 will follow from:

Lemma 20. *Let $G = (V, E)$ be a $(n/3 - k, 2)$-expander on n vertices, where $k = o(n)$. Let R be a random graph $G(n, p)$ with $p = 120k/n^2$. Then, $\mathbb{P}(G \cup R \text{ is not Hamiltonian}) < exp(-\Omega(k))$.*

Proof. Note that the following properties hold for G:

(I3) G is connected (due to Claim 14)

(I4) G has a path of length at least $n - 3k - 1$ (due to Corollary 17)

(I5) If a supergraph of G is non-Hamiltonian it has at least $n^2/20$ boosters (due the Corollary 19, and since $(n/3 - k + 1)^2/2 > n^2/20$).

We split the random graph R into $6k$ identically distributed graphs

$$R = \bigcup_{i=1}^{6k} R_i$$

where $R_i \sim G(n, p_i)$ and $p_i \geq p/(6k) = 20/n^2$. Set $G_0 = G$ and for $i \in [6k]$ let

$$G_i = G \cup \bigcup_{j=1}^{i} R_j.$$

At stage i we add to G_{i-1} the next random graph R_i. We call a stage *successful* if the maximal length of a path in G_i is longer than that of G_{i-1}, or if G_i is Hamiltonian. Clearly, if at least $3k+1$ stages are successful then the final graph G_{6k} is Hamiltonian (due to (I5)). Observe that for stage i to be successful, if G_{i-1} is not yet Hamiltonian, it is enough for the random graph R_i to hit one of the boosters of G_{i-1}. Thus, stage i is unsuccessful with probability at most $(1-p_i)^{n^2/20} < 1/e$. Thus, the number of successful stages S is a random variable which stochastically domainates $\text{Bin}(6k, 1 - 1/e)$. Therefore, putting $c = 1 - 1/e$ and using Chernoff bounds (Theorem 4),

$$\mathbb{P}(G \cup R \text{ is not Hamiltonian}) \leq \mathbb{P}(S \leq 3k) \leq \exp\left(-\frac{(6c-3)^2k^2}{2 \cdot 6ck}\right) < \exp(-\Omega(k)).$$

□

Proof of Theorem 2. Suppose we have G_{i-1} for $1 \leq i \leq \lceil \delta_0/2 \rceil$. We have shown that G_{i-1} is a $(n/3 - k, 2)$-expander for $k = 100n/(3 \ln n) = o(n)$. R_i is a random graph with probability $\rho_i \geq 4000/(n \ln n) = 120k/n^2$. Therefore by the above lemma, $G_{i-1} \cup R_i$ is not Hamiltonian with probability at most $\exp(-\Omega(k))$. Union bound over all $\lfloor \delta_0/2 \rfloor$ steps yields the desired result. □

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