1 Introduction

In the late 70s, it was shown by Komlós and Szemerédi ([7]) that for \( p = \frac{\ln n + \ln \ln n + c}{n} \), the limit probability for \( G(n, p) \) to contain a Hamilton cycle equals the limit probability for \( G(n, p) \) to have minimum degree at least 2. A few years later, Ajtai, Komlós and Szemerédi ([1]) have shown a hitting time version of this in the \( G(n, m) \) model.

Say a graph \( G \) has property \( \mathcal{H} \) if it contains \( \lceil \frac{\delta(G)}{2} \rceil \) edge disjoint Hamilton cycles, plus a further edge disjoint near perfect matching in the case \( \delta(G) \) is odd. Is it true that for every \( 0 \leq p \leq 1 \) the random graph \( G(n, p) \) has property \( \mathcal{H} \) with high probability? This is clear whenever \( \delta(G) = 0 \). In the early 80s, Bollabás and Frieze ([3]) have proved that conjecture for \( \delta(G) = O(1) \). In this talk I plan to prove the result for \( p(n) \leq (1 + o(1)) \ln n/n \). This is a result of Frieze and Krivelevich from '08 ([4]).

Remark 1. The conjecture is nowadays known to be true for every \( p \). It was proved for the range \( \ln^{50} n/n \leq p \leq 1 - \ln^9 n/n^{1/4} \) by Knox, Kühn and Osthus in '13 ([6]), in a rather technically complicated paper. Later, Krivelevich and Samotij ([8]) have closed the gap for the sparse case, and Kühn and Osthus ([9]) have closed the gap for the dense case.

This is the main result we intend to prove:

**Theorem 2.** Let \( p = p(n) \leq (1 + o(1)) \ln n/n \). Then \textbf{whp} \( G(n, p) \) has property \( \mathcal{H} \).

Remark 3. In this talk I will not consider the extra near perfect matching, expected in the case where \( \delta(G) \) is odd. This adds some technicality, but nothing really different.

2 Preliminaries

2.1 Probability

**Theorem 4** (Chernoff bounds, [5], Theorem 2.1). Let \( X \sim \text{Bin}(n, p) \), \( \mu = np \), \( a \geq 0 \). Then the following inequality holds:

\[
P(X \leq \mu - a) \leq \exp \left( -\frac{a^2}{2\mu} \right).
\]

\(^{0}\text{Git revision 94a18287}\)
Thus, letting $D$ from this assumption it follows that $p = (1 - p)(1 - \rho)$ and letting $\rho = o(1/n)$, thus decomposing $G \sim G(n, p)$ to $G = G_0 \cup R$ where $G_0 \sim G(n, p_0)$ and $R \sim G(n, \rho)$. In which sense are $G$ and $G_0$ similar? In the following:

**Claim 7.** Fix $G_0$, let $R \sim G(n, \rho)$ and $G = G_0 \cup R$; then \textbf{whp} $\delta(G_0) = \delta(G)$.

**Proof.** Clearly, $\delta(G_0) \leq \delta(G)$, as $G$ contains all the edges of $G_0$ and more. Now, let $v \in G_0$ with $d_{G_0}(v) = \delta(G_0)$. As $\rho = o(1/n)$, $d_R(v) = 0$ \textbf{whp} (a standard first moment argument), implying

$$\delta(G) \leq d_G(v) = d_{G_0}(v) = \delta(G_0).$$

From now on, write $\delta_0 = \delta(G_0)$. It follows that it is enough to prove that $G$ contains (\textbf{whp}) $\lfloor \delta_0/2 \rfloor$ edge disjoint Hamilton cycles and an edge disjoint near perfect matching if $\delta_0$ is odd. We also assume that $p = (1 + o(1))\ln n/n$, as otherwise $\delta_0 \leq 1$ and there’s nothing new to prove. We also note that from this assumption it follows that $\delta(G) = o(\ln n)$; this will follow from the following claims. Let $D_k$ be the random variable counting the number of vertices in $G(n, p)$ with degree exactly $k$. Clearly,

$$D_k = \sum_{v \in [n]} D_k(v),$$

where $D_k(v)$ is the indicator of the event that $v$ is of degree $k$. Note that

$$\mathbb{E}(D_k) = \mathbb{E}(D_k(v)) = n\mathbb{P}(d(v) = k) = n\binom{n-1}{k}p^k(1-p)^{n-1-k}. $$

Thus, letting $k = \delta \ln n$ and $p = (1 + \varepsilon)\ln n/n$ for $\varepsilon = o(1)$,

$$\mathbb{E}(D_k) = n\binom{n-1}{\delta \ln n}p^{\delta \ln n}(1-p)^{n-1-\delta \ln n}$$

$$\geq n\binom{(n-1)p}{\delta \ln n}^{\delta \ln n}e^{-np}$$

$$\geq n^{-\varepsilon}\left(1 + \varepsilon\right)^{-\delta \ln n} \geq n^{\delta \ln (1/\delta) - \varepsilon} = o(1).$$
If we take $\delta = \delta(n)$ to be large enough, say, $\delta = \varepsilon$.

**Claim 8.** For $k = O(\ln n)$, if $\mathbb{E}(D_k) = \omega(1)$ then $\text{Var}(D_k) = o\left(\mathbb{E}^2(D_k)\right)$.

**Proof.** Let $u \neq v$ be two vertices. Note that

$$\left(\frac{n}{k}\right)^{k-1} \frac{p^k}{k^{k-1}} = \frac{k}{np}(1 + o(1)) = \frac{k}{\ln n}(1 + o(1)) = O(1),$$

thus

$$\text{Cov}(D_k(u), D_k(v)) = \mathbb{P}(d(u) = k = d(v) \mid u \sim v) \mathbb{P}(u \sim v) + \mathbb{P}(d(u) = k = d(v) \mid u \sim v) \mathbb{P}(u \sim v) - \mathbb{P}^2(d(u) = k)$$

$$= \left(\left(\frac{n-2}{k-1}\right)^{k-1}(1-p)^{n-1-k}\right)^2 p$$

$$+ \left(\left(\frac{n}{k}\right)^{k}(1-p)^{n-2-k}\right)^2 (1-p) - \left(\left(\frac{n-1}{k}\right)^{k}(1-p)^{n-1-k}\right)^2$$

$$= O\left(\mathbb{E}^2(D_k)pn^{-2}\right) + O\left(\mathbb{E}^2(D_k)n^{-2} \cdot \left(\frac{1}{1-p} - 1\right)\right)$$

$$= o\left(\mathbb{E}^2(D_k)pn^{-2}\right).$$

It follows that

$$\text{Var}(D_k) \leq \mathbb{E}(D_k) + \sum_{u \neq v} \text{Cov}(D_k(u), D_k(v)) = o\left(\mathbb{E}^2(D_k)\right).$$

\[\square\]

For technical reasons, we’ll define a very particular $\rho$ so that $\rho = o(1/n)$ will hold. Set $d_0 = \min \{k \mid \mathbb{E}(D_k) \geq 1\}$.

**Claim 9.** $d_0 = o(\ln n)$.

**Proof.** As we’ve seen, for $k = \delta \ln n$, $\delta = o(1)$, $\mathbb{E}(D_k) \to \infty$, and $d_0 < k$, so $d_0 = o(\ln n)$.

\[\square\]

Note that $d_0$ approximates $\delta(G)$; formally,

**Claim 10.** whp, $|\delta(G) - d_0| \leq 2$.

**Proof.** Note that

$$\frac{\mathbb{E}(D_{k+1})}{\mathbb{E}(D_k)} = \frac{n(n-1)p^{k+1}(1-p)^{n-1-k}}{n(n-1)p^{k}(1-p)^{n-1-k}} = \frac{(n-1-k)p}{(k+1)(1-p)}.$$

As we’ve seen, $d_0 = o(\ln n)$. Thus it follows that for $b \geq 1$,

$$\mathbb{E}(D_{d_0-b-1}) = \mathbb{E}(D_{d_0-b}) \cdot \frac{(d_0-b)(1-p)}{(n-1-(d_0-b-1)p)} < \frac{d_0}{\mathbb{E}np} = \varepsilon' = o(1),$$

where $\varepsilon' = o(1)$.
and by a Markov’s inequality and the union bound,
\[ P(\exists b \geq 1, D_{d_0-b-1} > 0) \leq \sum_{b=1}^{d_0-1} (\varepsilon')^b \leq \frac{\varepsilon'}{1-\varepsilon'} = o(1). \]

In addition,
\[ E(D_{d_0+1}) = E(D_{d_0}) \cdot \frac{(n-1-d_0)p}{(d_0+1)(1-p)} \geq \frac{1}{2}np = \omega(1), \]
and by Chebyshev’s inequality and the previous claim,
\[ P(D_{d_0+1} = 0) \leq P(|D_{d_0+1} - E(D_{d_0+1})| \geq 1) \leq \frac{\text{Var}(D_{d_0+1})}{E^2(D_{d_0+1})} = o(1). \]

Therefore, \text{whp} there is no vertex with degree at most $d_0 - 2$ and there is a vertex with degree $d_0 + 1$, thus $|\delta(G) - d_0| \leq 2$. \hfill \Box

**Corollary 11.** \text{whp}, $\delta(G) = o(\ln n)$. \hfill \Box

We then define
\[ \rho = \frac{2001(d_0 + \ln \ln n)}{n \ln n}, \]
and observe that $\rho = o(1/n)$ (again, since $d_0 = o(\ln n)$), and that $np_0 = np(1 + o(1))$.

### 2.3 Properties of random graphs

In this section we give a list of properties, each occurring \text{whp}, in the random graph $G_0 \sim G(n,p_0)$.

Define the set SMALL:
\[ \text{SMALL} = \{ v \in V(G) \mid d_{G_0(v)} \leq 0.1 \ln n \}. \]

**Lemma 12.** The random graph $G_0 \sim G(n,p_0)$ with $p_0$ defined earlier, has \text{whp} the following properties:

(P1) There is no non-empty path of length at most 4 in $G_0$ such that both of its (possibly identical) endpoints lie in SMALL.

(P2) Every set $U \subseteq V(G)$ with $|U| \leq 100n/\ln n$ spans at most $|U|(|\ln n|^{1/2}$ edges in $G_0$.

(P3) For every two disjoint sets $U,W \subseteq V(G)$ with $|U| \leq 100n/\ln n$, $|W| \leq |U|\ln n/10000$,
\[ |E_{G_0}(U,W)| < 0.09|U|\ln n. \]

(P4) For every two disjoint sets $U,W \subseteq V(G)$ with $|U| \geq 100n/\ln n$, $|W| \geq n/4$,\n\[ |E_{G_0}(U,W)| \geq 0.1|U|\ln n. \]
Proof of (P1). Fix a vertex $v$. Note that
\[
\mathbb{P}(v \in \text{SMALL}) = \sum_{k=0}^{0.1 \ln n} \mathbb{P}(\text{Bin}(n-1, p_0) = k) \\
\leq 0.1 \ln n \left( \frac{n-1}{0.1 \ln n} \right) p_0^{0.1 \ln n} (1-p_0)^{n-1-0.1 \ln n} \\
\leq 0.1 \ln n \left( \frac{10e^{\ln p}}{\ln n} \right)^{0.1 \ln n} e^{-p_0(n-1-0.1 \ln n)} \\
\leq 28^{0.1 \ln n} e^{-(1-o(1)) \ln n} < n^{-0.6}.
\]

Now fix $u \neq v$. The probability that $u,v$ are connected by a path of length $\ell$ in $G_0$ is at most $n^{\ell-1}p_0^\ell = ((1+o(1)) \ln n)^{\ell-1}$ (choosing $\ell-1$ inner vertices and for each such choice requiring $\ell$ edges). Furthermore, as there’s exactly one edge of $K_n$ connecting $u$ with $v$, conditioning on the event “$u \in \text{SMALL}$” cannot increase the probability of “$v \in \text{SMALL}$” by too much:
\[
\mathbb{P}(u,v \in \text{SMALL}) \leq \mathbb{P}(v \in \text{SMALL} | u \in \text{SMALL}) \mathbb{P}(u \in \text{SMALL}) \\
\leq \mathbb{P}(v \in \text{SMALL} | \{u,v\} \notin E) \mathbb{P}(u \in \text{SMALL}) \\
\leq \mathbb{P}(v \in \text{SMALL}) \mathbb{P}(u \in \text{SMALL}) \cdot \frac{1}{1-p}.
\]

Note also that “$u,v \in \text{SMALL}$” is a monotone decreasing event and “$d(u,v) \leq 4$” is a monotone increasing event. Thus, according to the FKG inequality,
\[
\mathbb{P}(u,v \in \text{SMALL} \land d(u,v) \leq 4) \leq \mathbb{P}(u,v \in \text{SMALL}) \cdot \mathbb{P}(d(u,v) \leq 4).
\]

Therefore,
\[
\mathbb{P}(u,v \in \text{SMALL} \land d(u,v) \leq 4) \leq \mathbb{P}(u,v \in \text{SMALL}) \cdot \mathbb{P}(d(u,v) \leq 4) \\
\leq \mathbb{P}(v \in \text{SMALL}) \mathbb{P}(u \in \text{SMALL}) \mathbb{P}(d(u,v) \leq 4) (1+o(1)) \\
\leq n^{-0.6} \cdot n^{-0.6} \cdot \frac{\ln^4 n}{n} (1+o(1)) < n^{-2.1}.
\]

Applying the union bound over all possible pairs of $u,v$ we establish (P1). \hfill $\square$

Proof of (P2). For a given $U \subseteq [n]$ with $|U| = u \leq 100n/\ln n$, let $A_U$ be the event by which $|E(U)| \geq u \ln^{1/2} n$. By the union bound,
\[
\mathbb{P}(\exists U, |U| = u \leq 100n/\ln n, A_U) \leq \sum_{u=1}^{100n/\ln n} \binom{n}{u} \binom{u}{2} p^{u \ln^{1/2} n} \\
\leq \sum_{u=1}^{100n/\ln n} \left( \frac{en}{u} \left( \frac{eup}{2 \ln^{1/2} n} \right)^{\ln^{1/2} n} \right)^u \\
\leq \sum_{u=1}^{100n/\ln n} \left( \frac{en}{u} \left( \frac{2u \ln^{1/2} n}{n} \right)^{\ln^{1/2} n} \right)^u.
\]
We now separate the sum into two:

\[
\sum_{u=1}^{\ln n} \left( \frac{en}{u} \left( \frac{2u \ln^{1/2} n}{n} \right)^{\ln^{1/2} n} \right)^u \leq \ln n \cdot en \left( \frac{2 \ln^{3/2} n}{n} \right)^{\ln^{1/2} n} = o(1),
\]

and

\[
\sum_{u=1}^{100n/\ln n} \left( \frac{en}{u} \left( \frac{2u \ln^{1/2} n}{n} \right)^{\ln^{1/2} n} \right)^u \leq n \left( \frac{e}{n} \right)^{\ln^{1/2} n - 1} \left( 2 \ln^{1/2} n \right)^{\ln^{1/2} n} = o(1).
\]

Proof of (P3). For a given \( U \subseteq \{n\} \) with \( |U| = u \leq 100n/\ln n \) and \( W \subseteq \{n\} \) with \( |W| \leq u' = \frac{u \ln n}{10000} \), let \( A_{U,W} \) be the event by which \( |E(U,W)| \geq 0.09u \ln n \). By the union bound,

\[
\mathbb{P}(\exists U, W, A_{U,W}) \leq \sum_{u=1}^{100n/\ln n} \sum_{u'=1}^{100n/\ln n} \left( \frac{n}{u} \right) \left( \frac{n}{u'} \right) \left( \frac{uw}{0.09u \ln n} \right)^p 0.09u \ln n \leq \sum_{u=1}^{100n/\ln n} u' \left( \frac{en}{u'} \right) \left( \frac{en}{u} \right)^{\ln n/10000} \left( -\frac{eu'p}{0.09 \ln n} \right)^{0.09 \ln n} u \leq \sum_{u=1}^{100n/\ln n} u' \left( \frac{en}{u'} \right)^{\ln n/10000} \left( \frac{e}{0.09} \right)^{0.09 \ln n} \left( \frac{n}{u'} \right)^{\ln n/10000} - 0.09 \ln n \leq \sum_{u=1}^{100n/\ln n} u' \left( n^2 \left( \frac{n}{u'} \right)^{-0.08 \ln n} \right) u \leq \sum_{u=1}^{100n/\ln n} u' \left( n^2 - 0.08n/u' \right) u = o(1),
\]

as \( 0.08n/u' \geq 8 \). 

Proof of (P4). Fix \( U, W, |U| \geq 100n/\ln n, |W| \geq n/4 \). Note that the number of edges between \( U, W \) in \( G_0 \) is binomially distributed with \( |U||W| \) trials and success probability \( p_0 \), hence

\[
\mathbb{E}(|E_{G_0}(U,W)|) \geq (1 + o(1))|U| \ln n/4.
\]

6
By Chernoff bounds (Theorem 4),

\[ P(\left| \mathbb{E}_0(1, \mathbb{U}, \mathbb{W}) \right| \leq 0.1|\mathbb{U}| \ln n) \leq P(\left| \mathbb{E}_0(1, \mathbb{U}, \mathbb{W}) \right| \leq 0.25|\mathbb{U}| \ln n - 0.15|\mathbb{U}| \ln n) \]
\[ \leq \exp \left( -\frac{0.15|\mathbb{U}| \ln n}{2 \cdot 0.25|\mathbb{U}| \ln n} \right) \]
\[ < \exp (-2 \cdot 0.15^2 |\mathbb{U}| \ln n) < \exp (-4n) . \]

Now, the number of pairs \( \mathbb{U}, \mathbb{W} \) is at most \( 4^n \), union bound gives that the probability that such a pair exists is at most \( 4^n e^{-4n} = o(1) \). \qed

2.4 Expanders, rotations and boosters

One of the key concepts in many connectivity and Hamiltonicity related problems is that of an expander.

**Definition 13.** For every \( c > 0 \) and every positive integer \( R \) we say that a graph \( \mathbb{G} = (\mathbb{V}, \mathbb{E}) \) is an \( (R,c) \)-expander if every subset of vertices \( \mathbb{U} \subseteq \mathbb{V} \) of cardinality \( |\mathbb{U}| \leq R \) satisfies \( |N_G(\mathbb{U})| \geq c|\mathbb{U}| \).

**Claim 14.** Let \( \mathbb{G} \) be a \( (k,2) \)-expander on \( n \) vertices, with \( k > \frac{n}{4} \). Then, \( \mathbb{G} \) is connected.

**Proof.** Since every set of cardinality at most \( n/4 \) expands, every connected component must be of cardinality at least \( 3n/4 \), and there’s room for only 1 such component. \qed

Our approach will consist of that concept, bundled with the so-called rotation-extension technique, introduced by Pósa in ’76 ([10]). Here we will cover the technique, including a key lemma.

Given a path \( \mathbb{P} = (v_0, \ldots, v_m) \), we can extend it by adding \( v_{m+1} \) which is not part of the path but is a neighbour of \( v_m \), or we can rotate it by finding a neighbour \( v_i \) of \( v_m \) inside the path, adding the edge \( \{v_m, v_i\} \) and erasing the edge \( \{v_i, v_{i+1}\} \) (\( 1 \leq i < m \)).

![Figure 1: Pósa extension](image1)

![Figure 2: Pósa rotation](image2)

**Lemma 15.** Let \( G \) be a graph, \( \mathbb{P} \) a path of maximal length in \( G \), \( \mathcal{P} \) the set of all (rooted) paths obtained by \( \mathbb{P} \) be a sequence of rotations, \( \mathbb{U} \) the set of endpoints of these paths, \( N^- \) and \( N^+ \) the sets of vertices immediately preceding and following the vertices of \( \mathbb{U} \) along \( \mathbb{P} \), respectively. Then, \( N(\mathbb{U}) \subseteq N^- \cup N^+ \).
Proof. Denote $P = (v_0, \ldots, v_m)$. Let $u \in U$, $v \notin (U \cup N^- \cup N^+)$, and let $P_u$ be a rotation of $P$ ending at $u$. If $v \notin P$ then $\{u, v\} \notin E$, otherwise we could have extended $P_u$ and get a longer path, contradicting our assumption.

Thus, $v \in P$. Let $v^-, v^+$ be its two possible neighbours in $P$. Suppose $\{u, v\} \in E$. Then, we can rotate $P_u$ to get $P_w$ ending at $w$, where $w$ is a neighbour of $v$. If $w$ is $v^-$ or $v^+$, we get a contradiction, as this puts $v$ in $N^-$ or $N^+$. Thus, one of the edges in $P$ between $v$ and $v^-, v^+$ broke during a rotation. Let’s look at when it has happened; then, if $\{v^-, v\}$ broke, that rotation has made $v$ an end vertex, and if $\{v, v^+\}$ broke, that rotation has put $v \in N^-, N^+$. Thus, $\{u, v\} \notin E$. \hfill \Box

Corollary 16.

$$\left| N(U) \right| \leq |N^- \cup N^+| \leq 2|U| - 1.$$ \hfill \Box

Corollary 17. Let $G$ be a connected non-Hamiltonian $(k, 2)$-expander. Then $G$ contains a path of (edge) length $3k - 1$.

Proof. Let $P$ be a path of maximal length $m$ (counting in edges) in $G$. Recall that $|N(U)| \leq 2|U| - 1$, that is, $U$ does not expand, hence $|U| > k$. Let $U' \subseteq U$ with $|U'| = k$. Since $P$ is maximal, $N(U') \subseteq V(P)$, thus $|V(P)| \geq 3k$, hence $P$ is of length at least $3k - 1$. \hfill \Box

In order to utilize that lemma for our needs, we introduce the notion of a booster:

Definition 18. Given a graph $G$, a non-edge $e = \{u, v\}$ of $G$ is called a booster if adding $e$ to $G$ creates a graph $G'$, which is either Hamiltonian or whose maximum path is longer than that of $G$.

Note that technically every non-edge of a Hamiltonian graph $G$ is a booster by definition.

Boosters advance a graph towards Hamiltonicity when added; adding sequentially $n$ boosters clearly brings any graph on $n$ vertices to Hamiltonicity.

Corollary 19. Let $G$ be a connected non-Hamiltonian $(k, 2)$-expander. Then $G$ has at least $\frac{(k+1)^2}{2}$ boosters.

Proof. Let $P$ be a path of maximal length $m$ (counting in edges) in $G$. Again, $|U| > k$. We now seek of $\frac{(k+1)^2}{2}$ non-edges which, when added, create a cycle of length $m + 1$.

Fix a set $u_1, \ldots, u_{k+1}$ of end vertices. For each, let $P_i$ be the rotation of $P$ ending at $u_i$. For such $i$, fix $u_i$ as a starting vertex, and let $P_i$ be the set of rotations of $P_i$. Let $U_i$ be the set of endpoints retrieved that way. As before, $|U_i| > k$. Let $u_1^{(i)}, \ldots, u_{k+1}^{(i)}$ be a set of such end vertices.

Note that for every $i, j \in [k+1]$, $u_i, u_j^{(i)}$ are not connected, as if they were, we would have a cycle of length $m + 1$, and either end up with a Hamilton cycle, or, if $m + 1 < n$, since $G$ is connected, get a longer path. As each non-edge was counted at most twice that way, we have at least $(k+1)^2/2$ such non-edges, each is a booster. \hfill \Box
3 The proof

The outline of the proof is as follows: we split the graph $R$ into $\lceil \delta_0/2 \rceil$ identically distributed random graphs $R_i$. We start with $G_0$, finding enough boosters in $R_1$ to get a Hamilton cycle, deleting its edges and end up in $G_1$, and continuing so: given $G_{i-1}$ ($1 \leq i \leq \lceil \delta_0/2 \rceil$), we find boosters in $R_i$ to get a Hamilton cycle $H_i$, and by deleting it we get $G_i$. During the process, we’ll keep the following attributes of $G_i$:

(I1) $\delta(G_i) \geq 2$

(I2) $G_i$ is a $(n/3 - cn/\ln n, 2)$-expander (that will follow from (P1)-(P4))

(I3) $G_i$ is connected

(I4) $G_i$ has a path of length at least $n - cn/\ln n$

(I5) $G_i$ has quadratic number of boosters.

If $\delta_0$ is odd, we’ll need a final stage to create a near perfect matching.

3.1 Formal argument

We may assume that $\delta_0 \geq 2$, otherwise there’s nothing new to prove. For $1 \leq i \leq \lceil \delta_0/2 \rceil$ define $\rho_i$ by

$$1 - \rho = (1 - \rho_i)^{\lceil \delta_0/2 \rceil}.$$ 

Observe that

$$1 - \rho = (1 - \rho_i)^{\lceil \delta_0/2 \rceil} \geq 1 - \rho_i \lceil \delta_0/2 \rceil,$$

and thus

$$\rho_i \geq \frac{\rho}{\lceil \delta_0/2 \rceil} = \frac{2001(d_0 + \ln \ln n)}{\lceil \delta_0/2 \rceil n \ln n} \geq \frac{4000}{n \ln n}.$$ 

Now let

$$R = \bigcup_{i=1}^{\lceil \delta_0/2 \rceil} R_i,$$

where $R_i \sim G(n, \rho_i)$, and let $G_i$ be the graph obtained from $G_0 \cup \bigcup_{i=1}^{\lceil \delta_0/2 \rceil} R_i$ after having deleted the first $i$ Hamilton cycles, assuming that the previous $i-1$ stages were indeed successful. Let $i < \lceil \delta_0/2 \rceil$. To see (I1), note that every vertex had its degree in $G_0$ reduced by at most $2i$ in $G_i$. Thus,

$$\delta(G_i) \geq \delta_0 - 2i \geq \delta_0 - 2(\lceil \delta_0/2 \rceil - 1) \geq 2.$$ 

To see (I2), we now show that $G_i$ is a $(k, 2)$-expander for $k = n/3 - 100n/(3 \ln n)$. For that, let $X$ be a vertex set with $t$ vertices. Consider the following two cases:
Case 1 \((t \leq 100n/\ln n)\): Denote \(X_S = X \cap \text{SMALL}\) and \(X_L = X \setminus X_S\). Denote \(t_S = |X_S|\) and \(t_L = |X_L|\). Observe that \(|N_G(X_S, V \setminus X)| \geq 2t_S - t_L\); indeed, by property (P1), every vertex in \(X_S\) has at least 2 unique neighbors in \(V \setminus X_S\), and at most \(t_L\) of these 2\(t_S\) neighbors lie in \(X_L\). By property (P2), \(X_L\) spans at most \(t_L(\ln n)^{1/2}\) edges in \(G_0\). Recall that the minimal degree of \(X_L\) in \(G_0\) is at least \(0.1\ln n\); thus, at least \(0.09t_L\ln n\) edges leave \(X_L\) in \(G_0\). But then by (P3), the set of neighbors of \(X_L\) must be of cardinality greater than \(t_L\ln n/10000\). By (P1) again, at most \(t_L\) of these fall into \(X_S \cup N_{G_0}(X_S)\). As in \(G_i\) each vertex has lost at most \(\delta_0\) neighbors comparing to what it had in \(G_0\), we have that in \(G_i\), the set of neighbors of \(X_L\) outside \(X_S \cup N_{G_0}(X_S)\) is at least \(t_L\ln n/100000 - 1 - \delta_0\). Altogether, \(|N_G(X)| \geq 2t_S - t_L + t_L\ln n/100000 \geq 2t_S + 2t_L = 2t\), as claimed.

Case 2 \((100n/\ln n \leq t \leq n/3-100n/(3\ln n))\): Assume to the contrary that \(|N_{G_i}(X)| < 2t\). In that case we can find a vertex set \(Y\) disjoint to \(X \cup N_{G_i}(X)\) of cardinality \(n - 3t\). Thus, in \(G_0\) there were at most \(2|\delta_0/2|\) min \(\{t, n - 3t\}\) edges between \(X\) and \(Y\). If \(t \leq n/4\) then \(n - 3t \geq n/4\) and by (P4) we should have had \(|E_{G_0}(X, Y)| \geq 0.1t\ln n \gg \delta_0t\). If \(t \geq n/4\) then \(n - 3t \geq n - 3(n/3 - 100n/(3\ln n)) = 100n/\ln n\), and again by (P4) we should have had \(|E_{G_0}(Y, X)| \geq 0.1|Y|\ln n \gg \delta_0|Y|\).

The proof of Theorem 2 will follow from:

**Lemma 20.** Let \(G = (V, E)\) be a \((n/3 - k, 2)\)-expander on \(n\) vertices, where \(k = o(n)\). Let \(R\) be a random graph \(G(n, p)\) with \(p = 120k/n^2\). Then, \(\mathbb{P}(G \cup R\text{ is not Hamiltonian}) < \exp(-\Omega(k))\).

**Proof.** Note that the following properties hold for \(G\):

(I3) \(G\) is connected (due to Claim 14)
\textbf{(I4)} $G$ has a path of length at least $n - 3k - 1$ (due to Corollary 17)

\textbf{(I5)} If a supergraph of $G$ is non-Hamiltonian it has at least $n^2/20$ boosters (due the Corollary 19, and since $(n/3 - k + 1)^2/2 > n^2/20$).

We split the random graph $R$ into $6k$ identically distributed graphs

$$R = \bigcup_{i=1}^{6k} R_i$$

where $R_i \sim G(n, p_i)$ and $p_i \geq p/(6k) = 20/n^2$. Set $G_0 = G$ and for $i \in [6k]$ let

$$G_i = G \cup \bigcup_{j=1}^{i} R_j.$$  

At stage $i$ we add to $G_{i-1}$ the next random graph $R_i$. We call a stage \textit{successful} if the maximal length of a path in $G_i$ is longer than that of $G_{i-1}$, or if $G_i$ is Hamiltonian. Clearly, if at least $3k + 1$ stages are successful then the final graph $G_{6k}$ is Hamiltonian (due to (I5)). Observe that for stage $i$ to be successful, if $G_{i-1}$ is not yet Hamiltonian, it is enough for the random graph $R_i$ to hit one of the boosters of $G_{i-1}$. Thus, stage $i$ is unsuccessful with probability at most $(1 - p_i)^{n^2/20} < 1/e$. Thus, the number of successful stages $S$ is a random variable which stochastically domainates Bin $(6k, 1 - 1/e)$. Therefore, putting $c = 1 - 1/e$ and using Chernoff bounds (Theorem 4),

$$P(G \cup R \text{ is not Hamiltonian}) \leq P(S \leq 3k) \leq \exp \left( -\frac{(6c - 3)^2k^2}{2 \cdot 6ck} \right) < \exp(-\Omega(k)).$$

\textbf{Proof of Theorem 2.} Suppose we have $G_{i-1}$ for $1 \leq i \leq \lceil \delta_0/2 \rceil$. We have shown that $G_{i-1}$ is a $(n/3 - k, 2)$-expander for $k = 100n/(3 \ln n) = o(n)$. $R_i$ is a random graph with probability $\rho_i \geq 4000/(n \ln n) = 120k/n^2$. Therefore by the above lemma, $G_{i-1} \cup R_i$ is not Hamiltonian with probability at most $\exp(-\Omega(k))$. Union bound over all $\lceil \delta_0/2 \rceil$ steps yields the desired result. \hfill \square

\textbf{References}


