Graph Theory

List of theorems whose proofs one should know during the exam

1) A graph is bipartite \iff it contains no odd cycles.

2) The following statements about an \( n \)-vertex graph \( G \) are equivalent:
   (i) \( G \) is a tree, that is, \( G \) is acyclic and connected,
   (ii) \( G \) is connected and has \( n - 1 \) edges,
   (iii) \( G \) is acyclic and has \( n - 1 \) edges,
   (iv) \( G \) contains a unique path between every pair of its vertices.

3) Every graph \( G \) contains each tree with \( \delta(G) \) edges as a subgraph.

4) Mader’s theorem: A graph with avg degree \( \geq 4k \) contains a \((k + 1)\)-connected subgraph.

5) Whitney’s theorem: \( \kappa(G) \leq \kappa'(G) \leq \delta(G) \) for every graph \( G \).

6) Euler’s theorem: A connected graph contains an Eulerian circuit \iff all its vertices have even degrees.

7) Dirac’s theorem: Every \( n \)-vertex graph \( G \) with \( \delta(G) \geq n/2 \) is Hamiltonian.

8) The Chvátal–Erdős Theorem: Every graph \( G \) with at least three vertices that satisfies \( \kappa(G) \geq \alpha(G) \) is Hamiltonian.

9) Hall’s marriage theorem.

10) Petersen’s theorem: Each regular graph with positive even degree has a 2-factor.

11) König’s theorem: In every bipartite graph, the largest size of a matching equals the smallest size of a vertex cover.

12) Petersen’s theorem: Every 3-regular graph without cut-edges has a perfect matching.

13) Dirac’s theorem: Every \((k + 1)\)-colour-critical graph is \( k \)-edge-connected.

14) König’s line colouring theorem: If \( G \) is bipartite, then \( \chi'(G) = \Delta(G) \).

15) The Erdős–Szekeres Theorem: \( R(s,t) \leq \binom{s+t-2}{s-1} \) for all positive integers \( s \) and \( t \).

16) Mantel’s theorem: \( \text{ex}(n,K_3) = \lceil n/2 \rceil \lfloor n/2 \rfloor \).

17) Euler’s formula for connected plane graphs.

18) Heawood’s theorem: Every planar graph is five-colourable.