Graph Theory

Homework assignment #1

Due date: Sunday, November 15, 2015

Problem 1. Prove that for each $n \geqslant 1$, the number of graphs with vertex set $\{1, \ldots, n\}$ and all degrees even is $2^{\binom{n-1}{2}}$.

Problem 2. Suppose that $n \geqslant 8$. Prove that every n-vertex graph graph with at least $6n - 20$ edges contains a subgraph with minimum degree at least 7.

Problem 3. Let G be a graph with n vertices. Prove that G contains a cycle with a chord (an edge connecting nonconsecutive vertices of the cycle) if either

(a) $\delta(G) \geqslant 3$ or

(b) $|E(G)| \geqslant 2n - 3$ and $n \geqslant 4$.

Problem 4. Prove that every graph G with m edges admits a bipartition $V(G) = V_1 \cup V_2$ such that the number of edges of G crossing between V_1 and V_2 is at least $m/2$.

Problem 5. Let $d_1, \ldots, d_n$ be positive integers. Prove that there exists a tree with degrees $d_1, \ldots, d_n$ if and only if

$$d_1 + \ldots + d_n = 2n - 2.$$

Problem 6. Prove that if $T_1, \ldots, T_k$ are pairwise intersecting subtrees of a tree T, then T has a vertex that belongs to each of $T_1, \ldots, T_k$.

Problem 7. Prove that every graph G contains each tree with $\delta(G)$ edges as a subgraph.

Problem 8. Compute the number of spanning trees of the complete bipartite graph $K_{m,n}$.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Show that a graph is bipartite if and only if it contains no odd cycles. In particular, all trees are bipartite.

Exercise 2. Suppose that $m \leqslant n$, let A be an $m \times n$ matrix and let B be an $n \times n$ matrix. Prove, using the Lindström–Gessel–Viennot lemma, the Cauchy–Binet formula:

$$\det AB = \sum_{J \in \binom{[n]}{m}} \det A_J \cdot \det B_J,$$

where A_J is the $m \times m$ submatrix of A consisting of the columns indexed by J and B_J is the $m \times m$ submatrix of B consisting of the rows indexed by J.

Exercise 3. Show that the block graph of a connected graph is a tree.