Graph Theory
Homework assignment #1
Due date: Sunday, November 19, 2017

Problem 1. Prove that for each $n \geq 1$, the number of graphs with vertex set $\{1, \ldots, n\}$ and all degrees even is $2^{\binom{n-1}{2}}$.

Problem 2. Suppose that $n \geq 8$. Prove that every $n$-vertex graph with at least $6n - 20$ edges contains a subgraph with minimum degree at least 7.

Problem 3. Prove that every graph $G$ with $m$ edges admits a bipartition $V(G) = V_1 \cup V_2$ such that the number of edges of $G$ crossing between $V_1$ and $V_2$ is at least $m/2$.

Problem 4. Let $G$ be a connected graph of order $n$ and suppose that $k \in \{1, \ldots, n\}$. Show that $G$ contains a connected subgraph of order $k$.

Problem 5. Let $G$ be a graph with $\delta(G) \geq 2$. Show that there is a connected graph with the same vertex set and the same degree sequence. More precisely, show that there is a connected graph $H$ with $V(H) = V(G)$ such that $\deg_G(v) = \deg_H(v)$ for all $v \in V(G)$.

Problem 6. Let $d_1, \ldots, d_n$ be positive integers. Prove that there exists a tree with degrees $d_1, \ldots, d_n$ if and only if

$$d_1 + \ldots + d_n = 2n - 2.$$

Problem 7. Prove that every graph $G$ contains each tree with $\delta(G)$ edges as a subgraph.

Problem 8. Compute the number of spanning trees of the complete bipartite graph $K_{m,n}$.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Show that every graph with at least two vertices has two vertices of equal degree.

Exercise 2. Show that every tree with maximum degree $\Delta \geq 1$ has at least $\Delta$ leaves.

Exercise 3. Show that a graph is bipartite if and only if it contains no odd cycles. In particular, all trees are bipartite.