Problem 1. Let $G$ be a bipartite graph with bipartition $V(G) = A \cup B$. Define

$$\delta(A) = \max_{S \subseteq A} (|S| - |N(S)|).$$

Prove that the maximum size of a matching in $G$ is $|A| - \delta(A)$.

Problem 2. An $n \times n$ Latin square (resp. $r \times n$ Latin rectangle) is an $n \times n$ (resp. $r \times n$) matrix with entries in $\{1, \ldots, n\}$ such that no two entries in the same row or column are the same. Prove that if $r < n$, then every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

Problem 3. Let $v$ be a vertex of a connected graph $G$ and, for $r \geq 0$, let $G_r$ be the subgraph of $G$ induced by the vertices at distance exactly $r$ from $v$. Show that

$$\chi(G) \leq \max \{\chi(G_r) + \chi(G_{r+1}) : r \geq 0\}.$$

Problem 4. Let $G$ be a graph on $n$ vertices. Prove that $\chi(G) \cdot \chi(\overline{G}) \geq n$.

Problem 5. Suppose that a graph $G$ is a union of $k$ trees. Prove that $\chi(G) \leq 2k$.

Problem 6. Suppose that every pair of odd cycles in a graph $G$ has a common vertex. Show that $\chi(G) \leq 5$.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Let $G$ be a connected graph with an even number of edges. Use Tutte’s theorem to prove that the set of edges of $G$ can be partitioned into pairwise disjoint paths of length 2.

Exercise 2. Prove that every bipartite graph with maximum degree $\Delta$ is a subgraph of some $\Delta$-regular bipartite graph. Use this fact to give another proof of König’s theorem.

Exercise 3. A square matrix $A = (a_{ij})$ of nonnegative real numbers is called doubly stochastic if the entries of each row and each column sum up to 1, that is, for every $i$ and $j$,

$$\sum_i a_{ij} = \sum_j a_{ij} = 1.$$

A doubly stochastic matrix with all entries in $\{0, 1\}$ is called a permutation matrix. Prove the Birkhoff–von Neumann theorem, which states that every doubly stochastic matrix is a convex combination of permutation matrices.